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AN EMBEDDING THEOREM FOR FUNCTION SPACES

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Let G be an open set in E_n, and let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ in the norm

$$||u||_m = \left\{\int_G \sum_{|\alpha| \leqq m} |D^\alpha u(x)|^2 dx\right\}^{1/2} .$$

We show that the embedding maps $H_0^{m+1}(G) \subset H_0^m(G)$ are completely continuous if G is "narrow at infinity" and satisfies an additional regularity condition. This generalizes the classical case of bounded sets G.

As an application, the resolvent operator R_λ, associated with a uniformly strongly elliptic differential operator A with zero boundary conditions is completely continuous in $\mathscr{L}_2(G)$ provided G satisfies the same conditions. This generalizes a theorem of A. M. Molcanov.

Let G be an open set in Euclidean n-space E_n. Following standard usage, we denote by $C_0^\infty(G)$ the space of infinitely differentiable complex valued functions having compact support in G. Let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ relative to the norm

$$||f||_m = \left\{\int_G \sum_{|\alpha| \leqq m} |D^\alpha f(x)|^2 dx\right\}^{1/2} .$$

(See (3) below for notations.) It is an important and well-known result of functional analysis that each embedding

$$H_0^{m+1}(G) \subset H_0^m(G) , \qquad m = 0, 1, 2, \cdots$$

is completely continuous provided G is a bounded set. In this paper we show that this assumption can be relaxed; it turns out that a certain condition on G called "narrowness at infinity" (see Definition 2), which is obviously necessary, is also sufficient for complete continuity of the embeddings, provided G also satisfies a certain regularity condition. This result could be anticipated on the basis of theorems of F. Rellich [4] and A. M. Molcanov [3] concerning discreteness of the spectrum for the Laplace operator (with zero boundary conditions) on G.

DEFINITION 1. For an arbitrary open set $G \subset E_n$, with boundary ∂G, define

$$(1) \qquad \rho(G) = \sup_{x \in G} \operatorname{dist}(x, \partial G) .$$

Clearly $\rho(G)$ is the supremum of the radii of spheres inscribable in G.

DEFINITION 2. The open set G is said to be "narrow at infinity" if

$$(2) \qquad \lim_{R\to\infty} \rho(G_R) = 0 , \qquad \text{where } G_R = G \cap \{x : |x| > R\} .$$

Evidently G is narrow at infinity if and only if it does not contain infinitely many disjoint spherical balls of equal positive radius. Our main result concerns such sets G, but we also require the following regularity condition:

1. Corresponding to each $R \geqq 0$ there exist positive numbers $d(R)$ and $\delta(R)$ satisfying
   (a) $d(R) + \delta(R) \to 0$ as $R \to \infty$
   (b) $d(R)/\delta(R) \leqq M < \infty$ for all R
   (c) for each $x \in G_R$ there exists a point y such that $|x - y| < d(R)$ and $G \cap \{z : |z - y| < \delta(R)\} = \varnothing$.

Note that Condition 1 clearly implies that G is narrow at infinity.

We use the following standard notations.

$$(3) \qquad \begin{cases} D_i = \dfrac{\partial}{\partial x_i} , & i = 1, 2, \cdots, n ; \\ D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} & \text{for } \alpha = (\alpha_1, \cdots, \alpha_n) ; \\ |\alpha| = \sum \alpha_i . \end{cases}$$

The following theorem is a generalization of Poincaré's inequality, cf. Agmon [1]. Although the proof is similar to that of Agmon, we give it here for the sake of completeness.

THEOREM 1. *Let G be an open set in E_n satisfying the Condition 1. Then there exists a constant c such that*

$$(4) \qquad \int_{G_R} |f(x)|^2 \, dx \leqq c(d(R))^2 \int_G \sum_i |D_i f(x)|^2 \, dx$$

for all $f \in H_0^1(G)$. Moreover if G satisfies only Condition 1(c) *for $R = 0$, then the inequality* (4) *is valid for $R = 0$.*

Proof. Assume that G satisfies Condition 1. Let $R > 0$ be fixed, and write $d = d(R)$, $\delta = \delta(R)$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of integers, let $Q_\alpha = \{x \in E_n : n^{-1/2} d\alpha_k \leqq x_k \leqq n^{-1/2} d(\alpha_k + 1), k = 1, \cdots, n\}$. Then $E_n = \bigcup_\alpha Q_\alpha$.

Now let $\varphi \in C_0^\infty(G)$ and let $x \in G_R \cap Q_\alpha$; let y satisfy 1(c). Note that $Q_\alpha \subset \{z : |z - y| < 2d\}$. Let $S = \{z : |z - y| < \delta\}$ and integrate $|\varphi|^2$ over $Q_\alpha - S$:

$$\int_{Q_\alpha - S} |\varphi|^2 dx \leqq \int_{\delta \leqq |x-y| \leqq 2d} |\varphi|^2 dx$$
$$= \int_\Sigma \int_\delta^{2d} |\varphi(r, \sigma)|^2 r^{n-1} dr d\sigma ,$$

where Σ is the unit sphere centred at y. If $\delta \leqq r \leqq 2d$, we have by Schwarz's inequality

$$|\varphi(r, \sigma)|^2 r^{n-1} = \left| \int_\delta^r \varphi_r(t, \sigma) dt \right|^2 r^{n-1}$$
$$\leqq (2d)^n \int_\delta^{2d} |\varphi_r(t, \sigma)|^2 dt$$
$$\leqq (2d)^n \delta^{1-n} \int_\delta^{2d} |\varphi_r(t, \sigma)|^2 t^{n-1} dt .$$

Therefore, integrating over $\delta \leqq |x - y| \leqq 2d$, we obtain

$$\int_{Q_\alpha - S} |\varphi|^2 dx \leqq (2d)^{n+1} \delta^{1-n} \int_{\delta \leqq |x-y| \leqq 2d} \sum_i |D_i \varphi|^2 dx$$
$$\leqq (2d)^{n+1} \delta^{1-n} \int_{Q'_\alpha} \sum_i |D_i \varphi|^2 dx ,$$

where Q'_α is the union of all cubes Q_β which meet the set $\delta \leqq |x - y| \leqq 2d$. There is a number N, depending only on n, such that any $N + 1$ of the sets Q'_α have empty intersection. Summation of the above inequality over the set A of all indices α for which Q_α meets G_R therefore yields

$$\int_{G_R} |\varphi|^2 dx \leqq \int_{\cup_{\alpha \in A}(Q_\alpha - S)} |\varphi|^2 dx$$
$$\leqq \sum_{\alpha \in A} (2d)^{n+1} \delta^{1-n} \int_{Q'_\alpha} \sum_i |D_i \varphi|^2 dx$$
$$\leqq N \cdot 2^{n+1} M^{n-1} (d(R))^2 \int_G \sum_i |D_i \varphi|^2 dx ,$$

where M is as in 1(b). This proves inequality (4) for $\varphi \in C_0^\infty(G)$; the extension to $H_0^1(G)$ is trivial.

The second assertion of the theorem is now obvious.

COROLLARY. *Let G be an open set in E_n, satisfying the condition 1(c) for $R = 0$, and consider the norm $|\ \ |_m$ defined in $H_0^m(G)$ by*

$$|f|_m^2 = \int_G \sum_{|\alpha| = m} |D^\alpha f(x)|^2 dx .$$

Then the norms $|\ \ |_m$ and $\|\ \ \|_m$ are equivalent in $H_0^m(G)$. On the other hand these norms are not equivalent for any open set G for which $\rho(G) = +\infty$.

Proof. Applying the second assertion of the theorem to the k-th order derivatives of $f \in H_0^m(G)$ $(k < m)$, we get $|f|_k \leqq \text{const.}\, |f|_{k+1}$ and hence $\|f\|_m^2 = \sum_0^m |f|_k^2 \leqq \text{const.}\, |f|_m^2$. Since obviously $|f|_m \leqq \|f\|_m$, this proves the first assertion. For the second assertion, note that G must contain spheres of arbitrarily large radius if $\rho(G) = \infty$. Thus for example $H_0^1(G)$ will contain suitable translates of the functions $g_\alpha(x) = g(\alpha^{-1}x)$ for arbitrarily large values of α, where $g(x) \not\equiv 0$ is chosen as some function in $C_0^\infty(\{x : |x| < 1\})$. Since $|g_\alpha|_0 = \text{const.}\, \alpha\, |g_\alpha|_1$, an inequality of the form

$$\|g_\alpha\|_1^2 = |g_\alpha|_0^2 + |g_\alpha|_1^2 \leqq \text{const.}\, |g_\alpha|_1^2$$

is precluded. This argument clearly extends to $H_0^m(G)$.

We next introduce some useful notation. If R is a positive real number, set

$$B_R^n = \{x \in E_n : |x| < R\}\ ;$$
$$G_R' = G \cap B_R^n \text{ if } G \text{ is an open set in } E_n\ .$$

DEFINITION 3. Let G be an open set in E_n and let $R > 0$. Denote by $C_0^\infty(G, R)$ the space of all C^∞ functions on E_n whose support is a compact subset of $G \cap \bar{B}_R^n$. We define $H^m(G, R)$ to be the completion of $C_0^\infty(G, R)$ with respect to the norm $\|\ \|_m$.

DEFINITION 4. We say that a sequence $\{x_n\}$ in a Hilbert space H is *compact* if every subsequence of $\{x_n\}$ has a subsequence converging in H.

Thus a linear operator $T: H_1 \to H_2$ (H_2 a separable Hilbert space) is completely continuous if and only if it maps bounded sequences into compact sequences.

THEOREM 2. *If G is an arbitrary open set in E_n then the embeddings*

$$H^{m+1}(G, R) \subset H^m(G, R)\ , \qquad m = 0, 1, 2, \cdots$$

are completely continuous.

Proof. This follows easily from the complete continuity of the embeddings $H^{m+1}(B_R^n) \subset H^m(B_R^n) = H^m(E_n, R)$ [2, Ch. XIV]. For let $f \in H^m(G, R)$ and let $\{f_k\}$ be a sequence in $C_0^\infty(G, R)$ with $\|f_k - f\|_m \to 0$. Extending f_k to be zero outside its support, we get $f_k \to \hat{f}$ in $H^m(B_R^n)$ where $\hat{f}$ is obtained by extending f to be zero in $B_R^n - \bar{G}_R'$. Now if $\{\varphi_j\}$ is a bounded sequence in $H^{m+1}(G, R)$, then $\{\hat{\varphi}_j\}$ is bounded in

$H^{m+1}(B_R^n)$ and hence compact in $H^m(B_R^n)$, and therefore $\{\varphi_j\}$ itself is compact in $H^m(G, R)$.

The following criterion for compactness is well-known.

LEMMA. *Let $\{f_k\}$ be a bounded sequence in $\mathscr{L}_2(G)$, where $G \subset E_n$. Suppose that*

(a) *$\{f_k \mid G'\}$ is compact for every bounded subset G' of G, and*

(b) *given $\varepsilon > 0$, there exists $R > 0$ such that for all k,*

$$\int_{G_R} |f_k(x)|^2 \, dx < \varepsilon \, .$$

Then $\{f_k\}$ is compact in $\mathscr{L}_2(G)$.

THEOREM 3. *Let G be an open set in E_n, satisfying the Condition 1. Then G is narrow at infinity and each of the embedding maps*

$$H_0^{m+1}(G) \subset H_0^m(G) \, , \qquad m = 0, 1, 2, \cdots$$

is completely continuous. On the other hand if $G \subset E_n$ is not narrow at infinity, then the indicated embeddings are not completely continuous.

Proof. First, if G is not narrow at infinity, it must contain an infinite denumerable family $\{U_j\}$ of nonintersecting spherical balls of equal positive radius. Let f_1 be an arbitrary nonzero function in $C_0^\infty(U_1)$, and let f_j be constructed for $j = 2, 3, \cdots$ by translating f_1 to have support contained in U_j. Then we have

$$(f_j, f_k)_m = c_m \delta_{k,j}$$

where $(\ , \ )_m$ is the natural inner product in $H_0^m(G)$ and c_m is a nonzero constant depending only on m and f_1. Consequently none of the embeddings can be completely continuous.

To prove that if G satisfies Condition 1 then the embeddings are completely continuous, it suffices by the standard inductive argument to consider the case $m = 0$. Thus suppose $\{f_k\}$ is a sequence in $H_0^1(G)$ with $\|f_k\|_1 \leqq 1$. If G' is a bounded subset of G, then $G' \subset G'_R$ for some R, and by Theorem 2 the sequence $\{f_k \mid G'_R\}$ is compact in $\mathscr{L}_2(G_R)$, and *a fortiori* $\{f_k \mid G'\}$ is compact in $\mathscr{L}_2(G')$. Thus (a) of the Lemma is satisfied; to verify (b) we merely have to apply the inequality (4) to f_k:

$$\int_{G_R} |f_k(x)|^2 \, dx \leqq c(d(R))^2 \, \|f_k\|_1^2 \leqq c(d(R))^2 \, .$$

By hypothesis the right hand side approaches zero as $R \to \infty$.

Functions in $H_0^m(G)$ vanish in some sense on ∂G. This property is essential for the embedding theorem in the case of unbounded sets G, as is indicated in the following theorem. Here $H^m(G)$ is the Hilbert space of functions which together with their (distribution) derivatives of all orders $\leqq m$ are in $\mathscr{L}_2(G)$.

THEOREM 4. *Let G be an open set in E_n, contained in a cylinder of finite $n-1$ dimensional cross-section. If G has infinite n dimensional volume, then the embedding $H^1(G) \subset \mathscr{L}_2(G)$ is not completely continuous.*

Proof. Assume that the x_1-axis is the centre of the cylinder containing G, and let C denote the $n-1$ dimensional volume of the section of the cylinder by the hyperplane $x_1 = 0$. We may also suppose that $\mu_n(G \cap \{x : x_1 > 0\}) = \infty$; then for fixed a, $\mu_n(G \cap \{x : a \leqq x_1 \leqq b\})$ is a continuous increasing function of $b \geqq a$, with range the half-line $[0, \infty)$.

For $x \in E_n$ define the function $f_1(x)$ as follows.

$$f_1(x) = \begin{cases} x_1 & \text{if } 0 \leqq x_1 \leqq 1 \\ 1 & \text{if } 1 \leqq x_1 \leqq b_1 \\ 1 + b_1 - x_1 & \text{if } b_1 \leqq x_1 \leqq b_1 + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu_n(G \cap \{x : 1 \leqq x_1 \leqq b_1\}) = 1$. Similarly define $f_2(x)$ to have support in the strip $b_1 + 1 \leqq x_1 \leqq b_2 + 1$, where $\mu_n(G \cap \{x : b_1 + 1 \leqq x_1 \leqq b_2\}) = 1$, and so on. Then $f_k \perp f_j$ $(j \neq k)$ and

$$1 \leqq \| f_k \|_0^2 \leqq 1 + 2C .$$

Moreover

$$\begin{aligned} \| f_k \|_1^2 &= \| f_k \|_0^2 + \int_G \sum_i | D_i f_k(x) |^2 \, dx \\ &\leqq \| f_k \|_0^2 + 2C \leqq 1 + 4C . \end{aligned}$$

Thus the sequence $\{f_k\}$ is bounded in $H^1(G)$ but not compact in $\mathscr{L}_2(G)$, so that the embedding $H^1(G) \subset H^0(G) = \mathscr{L}_2(G)$ is not completely continuous.

As an application of Theorem 3, consider a given differential operator $a(x, D)$ of order $2m$:

$$a(x, D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha .$$

We assume that the coefficients are infinitely differentiable, bounded complex functions on an open set G in E_n. Let $a(x, D)$ be *uniformly strongly elliptic* in the following sense:

$$(-1)^m \operatorname{Re}(a_0(x, \xi)) \geqq \text{const.} \, |\xi|^{2m}, \; x \in G, \xi \in E_n ,$$

where $a_0(x, \xi)$ is the characteristic form,

$$a_0(x, \xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha .$$

Under certain additional conditions on the coefficients $a_\alpha(x)$ and on the set G, it is known that the following inequalities are valid (cf. [1]).

$$(5) \qquad |(a(x, D)\varphi, \psi)| \leqq \text{const.} \, \|\varphi\|_m \|\psi\|_m, \; \varphi, \psi \in C_0^\infty(G) ;$$

and "Gårding's inequality"

$$(6) \qquad \operatorname{Re}(a(x, D)\varphi, \varphi) \geqq c_1 \|\varphi\|_m^2 - c_2 \|\varphi\|_0^2, \; \varphi \in C_0^\infty(G) ,$$

where $c_1 > 0$ and c_2 are constants. For the purpose of the following theorem we use these inequalities as hypotheses. Theorem 5 was obtained in the case of the Laplacian operator in a smoothly bounded domain G by A. M. Molcanov [3].

THEOREM 5. *Let G be an open set in E_n, satisfying the hypotheses of Theorem 3. Let $a(x, D)$ be a uniformly strongly elliptic differential operator with coefficients defined in G, and suppose that the inequalities* (5) *and* (6) *are satisfied. Define the operator T in $\mathscr{L}_2(G)$ by*

$$\mathscr{D}(T) = H_0^m(G) \cap \{f \in \mathscr{L}_2(G) : a(x, D)f \in \mathscr{L}_2(G)\}$$
$$Tf = a(x, D)f , \qquad f \in \mathscr{D}(T) .$$

Then T is a closed linear operator; the spectrum $\sigma(T)$ is discrete and has no finite limit points; for $\lambda \notin \sigma(T)$, the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ is completely continuous.

Proof. We have worded the theorem to agree with Corollary 14.6.11 of [2]; in fact the proof is the same. At the suggestion of the referee, however, we include an outline here.

If λ is a given complex number with $\operatorname{Re}\lambda > c_2$, we have by (5) and (6)

$$(7) \qquad |((a + \lambda)\varphi, \psi)| \leqq k_1 \|\varphi\|_m \|\psi\|_m, \; \varphi, \psi \in C_0^\infty(G) ;$$

$$(8) \qquad \operatorname{Re}((a + \lambda)\varphi, \varphi) \geqq k_2 \|\varphi\|_m^2, \qquad \varphi \in C_0^\infty(G) .$$

Hence $((a+\lambda)\varphi, \psi)$ can be extended to a continuous bilinear form $B[\varphi, \psi]$ on $H_0^m(G)$, satisfying (7) and (8). By the Lax-Milgram lemma (cf. [1], p. 98), to each $\varphi \in H_0^m(G)$ there corresponds an element $A\varphi \in H_0^m(G)$ such that

$$(9) \qquad B[A\varphi, \psi] = (\varphi, \psi)_m , \qquad \text{for all } \psi \in H_0^m(G) .$$

Moreover $A\colon H_0^m(G) \to H_0^m(G)$ is bounded, one-to-one, and hence onto. By the open mapping theorem, A^{-1} is also bounded.

Next, if T is the operator defined in the theorem, we will show that

$$(10) \qquad ((T+\lambda I)\varphi, \psi) = (A^{-1}\varphi, \psi)_m , \quad \varphi \in \mathscr{D}(T),\ \psi \in H_0^m(G) .$$

This relation is evident for $\varphi, \psi \in C_0^\infty(G)$. If $\varphi \in H_0^m(G)$, $\psi \in C_0^\infty(G)$, and if $\varphi_n(\in C_0^\infty(G)) \to \varphi$ in the norm of $H_0^m(G)$, then $\varphi_n \to \varphi$ in the sense of distributions on G, so that $((a+\lambda)\varphi_n, \psi) \to ((a+\lambda)\varphi, \psi)$, and therefore

$$((a+\lambda)\varphi, \psi) = (A^{-1}\varphi, \psi)_m , \quad \varphi \in H_0^m(G),\ \psi \in C_0^\infty(G) .$$

This implies (10) immediately.

By (8), (9), and (10) we have for $\varphi \in \mathscr{D}(T)$

$$(11) \qquad \begin{aligned} \|(T+\lambda I)\varphi\|_0 \cdot \|\varphi\|_m &\geqq |((T+\lambda I)\varphi, \varphi)_0| \\ &= |(A^{-1}\varphi, \varphi)_m| = |B[\varphi, \varphi]| \geqq k_2 \|\varphi\|_m^2 . \end{aligned}$$

Hence $(T+\lambda I)^{-1}$ exists and is bounded on Range $(T+\lambda I)$. Another simple argument shows that Range $(T+\lambda I) = \mathscr{L}_2(G)$. We therefore conclude that T is closed and $\lambda \in \rho(T)$, the resolvent set of T.

By (11) we have

$$\|(T+\lambda I)^{-1}\varphi\|_m \leqq k_2^{-1} \|\varphi\|_0 , \quad \varphi \in \mathscr{L}_2(G) .$$

Thus $(T+\lambda I)^{-1}$ maps a bounded set in $\mathscr{L}_2(G)$ into a bounded set in $H_0^m(G)$, which, according to Theorem 3, is precompact in $\mathscr{L}_2(G)$. Therefore $(T+\lambda I)^{-1}$ is a completely continuous operator in $\mathscr{L}_2(G)$.

The remaining assertions of the theorem follow from the Riesz-Schauder theory of completely continuous operators.

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