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RESTRICTED BIPARTITE PARTITIONS

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Let $\pi_k(n, m)$ denote the number of partitions

$$n = n_1 + n_2 + \cdots + n_k$$
$$m = m_1 + m_2 + \cdots + m_k$$

subject to the conditions

$$\min(n_j, m_j) \geqq \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, \cdots, k-1) .$$

Put

$$\xi^{(k)}(x, y) = \sum_{n,m=0}^{\infty} \pi_k(n, m) x^n y^m .$$

We show that

$$\xi^{(k)}(x, y) = \prod_{j=1}^{k} \frac{1 - x^{2j-1}y^{2j-1}}{(1 - x^j y^j)(1 - x^j y^{j-1})(1 - x^{j-1} y^j)} ,$$

$$\sum_{n,m=0}^{\infty} \pi(n, m; \lambda) x^n y^m = 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y) ,$$

$$\sum_{n\ m=0}^{\infty} \phi(n, m) x^n y^m = \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2) ,$$

where $\pi(n, m; \lambda)$ denotes the number of "weighted" partitions of (n, m) and $\phi(n, m)$ is the number of partitions into odd parts (n_j, m_j all odd).

Consider partitions of the bipartite (n, m) of the type

$$n = n_1 + n_2 + n_3 + \cdots$$
$$m = m_1 + m_2 + m_3 + \cdots , \tag{1.1}$$

where the n_j, m_j are nonnegative integers subject to the conditions

$$\min(n_j, m_j) \geqq \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots) . \tag{1.2}$$

For brevity we may write (1.2) in the form

$$(n_j, m_j) \geqq (n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots)$$

and say that the "parts" of the partition (1.1) decrease.

Let $\pi(n, m)$ denote the number of partitions (1.1) that satisfy (1.2) and let $\rho(n, m)$ denote the numbers of partitions (1.1) that satisfy

$$(n_j, m_j) > (n_{j+1}, m_{j+1}) \qquad (j=1, 2, 3, \cdots) . \tag{1.3}$$

By the inequality (1.3) is understood

$$\min(n_j, m_j) > \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots) .$$

The generating functions for $\pi(n, m)$ and $\rho(n, m)$ are given by [2]

$$(1.4) \qquad \prod_{j=1}^{\infty} (1 - x^{2j}y^{2j})^{-1} (1 - x^j y^{j-1})^{-1} (1 - x^{j-1}y^j)^{-1} ,$$

$$(1.5) \qquad \frac{1-xy}{(1-x)(1-y)} \sum_{n=0}^{\infty} (xy)^{n(n+1)/2} \prod_{j=1}^{n} \frac{1-x^{2j+1}y^{2j+1}}{(1-x^j y^j)(1-x^{j+1}y^j)(1-x^j y^{j+1})} ,$$

respectively.

For the case of unipartite (natural) numbers generating functions are known for partitions with parts restricted in various ways [3]. The notion of a part of the partition (1.1) implied by the conditions (1.2) suggests that these results can be extended to bipartite numbers. For example, we may think of $\rho(n, m)$ as the number of partitions of (n, m) with unequal parts. We shall find generating functions for bipartite partitions with at most k parts, weighted parts, and odd parts.

2. Partitions with at most k parts. We consider partitions of the type

$$(2.1) \qquad \begin{aligned} n &= n_1 + n_2 + \cdots + n_k \\ m &= m_1 + m_2 + \cdots + m_k , \end{aligned}$$

where the n_j, m_j are nonnegative integers subject to the conditions

$$(2.2) \qquad (n_j, m_j) \geqq (n_{j+1}, m_{j+1}) \qquad (j = 1, 2, \cdots, k-1) .$$

Let $\pi_k(n, m)$ denote the number of partitions (2.1) subject to the conditions (2.2) and let $\pi_k(n, m \mid a, b)$ denote the numbers of these partitions that also satisfy

$$(2.3) \qquad (a, b) \geqq (n_1, m_1) .$$

Note that $\pi(n, m)$ defined in § 1 satisfies

$$(2.4) \qquad \pi(n, m) = \lim_{k=\infty} \pi_k(n, m) .$$

We define the rational function $\xi_{ab}^{(k)}$ of x and y by the recurrence

$$(2.5) \qquad \xi_{ab}^{(0)} = 1, \qquad \xi_{ab}^{(k)} = \sum_{r,s=0}^{\min(a,b)} x^r y^s \xi_{rs}^{(k-1)} \qquad (k \geqq 1) .$$

If we put

$$(2.6) \qquad \xi^{(k)} = \xi_{\infty\infty}^{(k)} ,$$

then in the limit (2.5) becomes

(2.7) $$\xi^{(k)} = \sum_{r\ s=0}^{\infty} x^r y^s \xi_{rs}^{(k-1)} \qquad (k \geqq 1)\ .$$

It is clear from (2.5) that $\xi_{ab}^{(k)}$ is the generating function for $\pi_k(n, m \mid a, b)$. Thus it follows from (2.6) that $\xi^{(k)}$ is the generating function for $\pi_k(n, m)$. Explicitly, we have

(2.8) $$\xi_{ab}^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m \mid a, b) x^n y^m\ ,$$

(2.9) $$\xi^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m) x^n y^m\ .$$

We define the generating functions

(2.10) $$F_k(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \xi_{rs}^{(k-1)}\ ,$$

(2.11) $$F_k^{(u)} = \sum_{n=0}^{\infty} u^n \xi_{nn}^{(k-1)}\ ,$$

so that

(2.12) $$F_k(x, y) = \xi^{(k)}\ .$$

Using (2.10), (2.11) and

(2.13) $$\xi_{rr}^{(k)} = \xi_{ab}^{(k)} \qquad (r = \min(a, b))\ ,$$

we get

$$\begin{aligned} F_k(u, v) &= \sum_{r \geqq s} u^r v^s \xi_{ss}^{(k-1)} + \sum_{s \geqq r} u^r v^s \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} u^r v^r \xi_{rr}^{(k-1)} \\ &= \left(\frac{1}{1-u} + \frac{1}{1-v} - 1\right) F_k(uv)\ . \end{aligned}$$

It follows that

(2.14) $$F_k(u, v) = \frac{1 - uv}{(1-u)(1-v)} F_k(uv)\ .$$

On the other hand, using (2.5), (2.11), and (2.13), we get

$$\begin{aligned} F_k(u) &= \sum_{n=0}^{\infty} u^n \sum_{r,s=0}^{n} x^r y^s \xi_{rs}^{(k-2)} \\ &= \frac{1}{1-u}\left(\sum_{r \geqq s} u^r x^r y^s\, \xi_{ss}^{(k-1)} + \sum_{s \geqq r} u^s y^s x^r \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} (xyu)^r \xi_{rr}^{(k-1)}\right) \\ &= \frac{1}{1-u}\left(\frac{1}{1-ux} + \frac{1}{1-uy} - 1\right) F_{k-1}(xyu)\ , \end{aligned}$$

which implies

$$(2.15)\qquad F_k(u) = \frac{1 - xyu^2}{(1-u)(1-xu)(1-yu)} F_{k-1}(xyu) \qquad (k \geqq 1)\,.$$

It follows from (2.5), (2.11), and (2.15) that

$$(2.16)\qquad F_k(u) = \frac{1}{1-u} \prod_{j=0}^{k-2} \frac{1 - x^{2j+1}y^{2j+1}u^2}{(1-x^{j+1}y^{j+1}u)(1-x^jy^{j+1}u)(1-x^{j+1}y^ju)}\,.$$

Thus, using (2.12) and (2.14), we have evidently proved

THEOREM 1. *If $\xi^{(k)}$ is defined by (2.9) then*

$$(2.17)\qquad \xi^{(k)} = \prod_{j=1}^{k} \frac{1 - x^{2j-1}y^{2j-1}}{(1-x^jy^j)(1-x^jy^{j-1})(1-x^{j-1}y^j)}\,.$$

We may now write (1.5) in the form

$$(2.18)\qquad \sum_{n=1}^{\infty} (xy)^{n(n-1)/2}(1 - x^ny^n)\xi^{(n)}(x, y)\,,$$

which is analogous to the well-known identity

$$(2.19)\qquad \prod_{n=1}^{\infty}(1 + x^n) = \sum_{n=1}^{\infty} x^{n(n-1)/2} \prod_{j=1}^{n-1}(1 - x^j)^{-1}\,.$$

3. A q-identity. If we put

$$(3.1)\qquad \xi = \xi^{(\infty)}\,, \quad \xi_{ab} = \xi_{ab}^{(\infty)}\,,$$

then it follows from (2.4) and (2.9) that ξ is the generating function for $\pi(n, m)$. Moreover, it is clear from (2.14) and (2.16) that

$$(3.2)\qquad F(u, v) = \sum_{r,s=0}^{\infty} u^rv^s\xi_{rs} = \frac{1 - uv}{(1-u)(1-v)} F(uv)\,,$$

$$(3.3)\qquad \begin{aligned} F(u) &= \sum_{n=0}^{\infty} u^n\xi_{nn} \\ &= e(u, xy)\, e(xu, xy)\, e(yu, xy) \prod_{j=0}^{\infty}(1 - x^{2j+1}y^{2j+1}u^2)\,, \end{aligned}$$

where

$$(3.4)\qquad \begin{aligned} e(t) = e(t, q) &= \prod_{0}^{\infty}(1 - q^nt)^{-1} = \prod_{0}^{\infty}\frac{t^n}{(q)_n}\,, \\ (q)_n &= (1-q)(1-q^2)\cdots(1-q^n)\,. \end{aligned}$$

We define the polynomial

$$(3.5)\qquad H_n(x) = H_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r\,,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r (q)_{n-r}} .$$

It has been shown [1] that

$$\sum_0^\infty \frac{H_k(x)H_k(y)}{(q)_k} t^k = \frac{e(t)\, e(xt)\, e(yt)\, e(xyt)}{e(xyt^2)} . \tag{3.6}$$

Using (3.3), (3.4), and (3.6), we then have

$$\sum_0^\infty u^n \xi_{nn} = \sum_0^\infty \frac{H_k(x)H_k(y)}{(xy)_k} u^k \sum_0^\infty (-1)^r \frac{x^r y^r u^r}{(xy)_r} .$$

Comparing coefficients of u^n, we get

$$\xi_{nn} = \frac{1}{(xy)_n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^{n-k} H_k(x) H_k(y) . \tag{3.7}$$

Note that $xy = q$ in the right member of (3.7).

It is clear from (3.7) that

$$P_n(x, y) = (xy)_n \xi_{nn} \tag{3.8}$$

is a polynomial in x, y with integral coefficients which satisfies

$$P_n(x, y) = P_n(y, x) ,$$
$$P_n(x, 0) = \frac{1 - x^{n+1}}{1 - x} ,$$
$$x^n P_n\left(x, \frac{1}{x}\right) = (x^2 + x + 1)^n .$$

Also it follows from (2.15) that $P_n(x, y)$ satisfies the recurrence

$$P_n - (1 + x + y)P_{n-1} + [n-1]\,(x + y + xy + x^{n-1}y^{n-1})P_{n-2} - xy[n-1]\,[n-2]P_{n-3} = 0 , \tag{3.9}$$

where $[j] = 1 - x^j y^j$.

4. Weighted partitions. We define $\pi(n, m; \lambda)$, the number of weighted partitions of the bipartite (n, m), by the relation

$$\pi(n, m; \lambda) = \sum_{k=0}^\infty \lambda^k \sum 1 , \tag{4.1}$$

where the inner sum is extended over all partitions of the form (2.1) subject to the conditions (2.2) and the additional condition $\max(n_k, m_k) > 0$; that is, over all partitions with exactly k parts. It follows from the definition of $\pi_k(n, m)$ that we may write (4.1) in the form

$$\pi(n, m; \lambda) = \sum_{k=0}^{\infty} \lambda^k(\pi_k(n, m) - \pi_{k-1}(n, m)) . \tag{4.2}$$

It should be remarked that the sum in (4.2) is finite, the upper bound for k being $\max(n, m)$.

Multiplying both members of (4.2) by $x^n y^m$ and summing over n, m it follows from (2.9) and (2.17) that we have established

THEOREM 2. *We have*

$$\sum_{n,m=0}^{\infty} \pi(n, m; \lambda)x^n y^m = 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y) . \tag{4.3}$$

Note that (4.3) is a direct analogue of the well-known identity

$$\prod_{n=1}^{\infty} (1 - \lambda x^n)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n \sum_{j=1}^{n} (1 - x^j)^{-1} . \tag{4.4}$$

We remark that (4.3) may be proved in a different manner. If we put

$$\xi_{ab}(\lambda) = 1 + \lambda \sum_{r,s=0}^{\min(a,b)} x^r y^s \xi_{rs} , \tag{4.5}$$

where the prime denotes that we sum over all r, s in the indicated range except $r = s = 0$, then it follows from (4.1) that

$$\xi(\lambda) = \xi_{\infty\infty}(\lambda) \tag{4.6}$$

is the generating function for $\pi(n, m; \lambda)$. We may then evaluate $\xi(\lambda)$ by the methods of §2.

5. Partitions into odd parts.

We shall say that the j-th part of the partition (1.1) is odd if each of n_j, m_j is odd.

Let $\psi(n, m)$ denote the number of partitions of the form (1.1) with parts odd and subject to the conditions (1.2). Let $\psi(n, m \mid a, b)$ denote the number of these partitions that satisfy the additional condition

$$(2a + 1, 2b + 1) \geqq (n_1, m_1) . \tag{5.1}$$

We define the rational function $\beta_{2a+1,2b+1}$ of x, y by the relation

$$\beta_{2a+1,2b+1} = 1 + \sum_{r,s=0}^{\min(a\ b)} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1} , \tag{5.2}$$

so that

$$\beta_{2r+1,2r+1} = \beta_{2a+1,2b+1} \qquad (r = \min(a, b)) . \tag{5.3}$$

If we put

$$\beta = \beta_{\infty\infty} , \tag{5.4}$$

then in the limit (5.2) becomes

$$\beta = 1 + \sum_{r,s=0}^{\infty} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1} . \tag{5.5}$$

It follows from (5.2) that

$$\beta_{2a+1,2b+1} = \sum_{n,m=0}^{\infty} \psi(n, m \mid a, b) x^n y^m . \tag{5.6}$$

Thus, using (5.5), we get

$$\beta = \sum_{n,m=0}^{\infty} \psi(n, m) x^n y^m . \tag{5.7}$$

We define the generating functions

$$H(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \beta_{2r+1,2s+1} , \tag{5.8}$$

$$H(u) = \sum_{n=0}^{\infty} u^n \beta_{2n+1,2n+1} , \tag{5.9}$$

so that

$$\beta = 1 + xy\, H(x^2, y^2) . \tag{5.10}$$

Using (5.3), (5.8) and (5.9), we have

$$H(u, v) = \frac{1 - uv}{(1-u)(1-v)} H(uv) . \tag{5.11}$$

The proof of (5.11) is exactly like that of (2.14).

On the other hand, it follows from (5.2), (5.3), and (5.9) that

$$\begin{aligned} H(u) &= \sum_{n=0}^{\infty} u^n \Big(1 + \sum_{r,s=0}^{n} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1}\Big) \\ &= \frac{1}{1-u} + \frac{xy}{1-u} \sum_{r,s=0}^{\infty} x^{2r} y^{2s} u^{\max(r,s)} \beta_{2r+1,2s+1} \\ &= \frac{1}{1-u} + \frac{xy}{1-u}\Big(\frac{1}{1-x^2u} + \frac{1}{1-y^2u} - 1\Big) H(x^2y^2u) , \end{aligned}$$

which implies

$$H(u) = \frac{1}{1-u}\Big(1 + \frac{1 - x^2y^2u^2}{(1-x^2u)(1-y^2u)} H(x^2y^2u)\Big). \tag{5.12}$$

Repeated applications of (5.12) yield

$$(5.13)\quad H(u) = \frac{1}{1-u}\sum_{n=0}^{\infty} x^n y^n \prod_{j=1}^{n} \frac{1 - x^{4j+2}y^{4j+2}u^2}{(1 - x^{2j+2}y^{2j+2}u)(1 - x^{2j}y^{2j+2}u)(1 - x^{2j+2}y^{2j}u)} .$$

Thus, using (5.10), (5.11), and (2.17), we may state

THEOREM 3. *If* $\psi(n, m)$ *denotes the number of partitions of* (n, m) *with odd parts, then*

$$(5.14)\qquad \sum_{n,m=0}^{\infty} \psi(n, m)x^n y^m = \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2) ,$$

where $\xi^{(n)}(x, y)$ *is defined by* (2.17).

The fact that (2.18) and (5.14) are analogous to well-known identities for unipartite numbers leads one to conjecture that $\rho(n, m) = \psi(n, m)$. There are, however, counterexamples to this conjecture. For example, it is easily verified that

$$\rho(5, 4) = 6 \neq 4 = \psi(5, 4) .$$

It would be of interest to know whether generally

$$\rho(n, m) \geqq \psi(n, m) .$$

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