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ON THE SCARCITY OF LATTICE-ORDERED MATRIX RINGS

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It is well-known that the ring Q_n of $n \times n$ matrices over a lattice-ordered ring Q may be lattice-ordered by prescribing that a matrix is positive exactly when each of its entries is positive. We conjecture in case Q is the field of rational numbers that this is essentially the only lattice-order of the matrix ring in which the multiplicative identity 1 is positive and settle the conjecture in case $n=2$. There are however other lattice-orders of Q_2 in which 1 is not positive. A complete description of this family is obtained.

THEOREM. *Up to isomorphism there is exactly one lattice-order of the algebra Q_2 of two-by-two matrices over the field Q of rational numbers in which the identity* 1 *is positive.*

For each rational number $B > 1$, there is a lattice-order of Q_2 in which there are distinct positive idempotents f_1, f_2, f_3, and f_4 satisfying:

(i) $(1 - B)(f_1 + f_2) + B(f_3 + f_4) = 1$, *and*

(ii) Q_2 *is the l-group direct sum of the subrings* Qf_i, $1 \leqq i \leqq 4$.

These lattice-orders are not isomorphic, and each lattice-order in which 1 *is not positive is isomorphic to one of them.*

Proof. Any lattice-order of a finite-dimensional semisimple algebra over the field of rational numbers is archimedean [1]. Hence, for any lattice-order of Q_2, Q_2, as an l-group, is the direct sum of (at most four) totally-ordered subgroups of the real numbers [2]. We will consider and eliminate the various cases that might occur depending on the number of summands, the dimensions of the summands, and the number and sign of the nonzero coordinates of the identity matrix 1 in each such decomposition.

In each case $\approx$ denotes l-group isomorphism.

We will begin by considering all possible lattice-orders in which 1 is positive. The reader should note that in this case the components of 1 in a decomposition of Q_2 into the l-group direct sum of totally-ordered groups are pairwise disjoint mutually orthogonal idempotents.

(1) Suppose that $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$, $E_i \neq 0, 0 < 1$, and $1 = e_1 + e_2 + e_3 + e_4$ with $e_i \in E_i$.

(1a) If all of the coordinates of 1 are different from 0, then Q_2

is spanned by commuting elements. This is absurd.

(1b) Suppose that exactly three coordinates of 1 are different from 0: $e_1, e_2, e_3 > 0 = e_4$. Let $0 < n \in E_4$. Then $0 \leqq e_i n \leqq n$ implies $e_i n = k_i n$ for some $k_i \in Q^+$. Moreover, $e_i^2 n = k_i^2 n = k_i n$, so $k_i = 0$ or $k_i = 1$. If, for all i, $k_i = 1$, then $n = e_1 n + e_2 n + e_3 n = 3n$ which is impossible. If, for some i, $k_i = 0$, then $e_i Q_2 = E_i$ is a one-dimensional right ideal. However, all right ideals of Q_2 have even dimension.

(1c) Suppose that exactly two coordinates of 1 are greater than 0: $e_1, e_2 > 0$, $e_3 = e_4 = 0$. In this case there is a lattice-order and we need only show that it is determined up to isomorphism. Let $0 < n_1 \in E_3$ and $0 < n_2 \in E_4$. As in (1b), for each i and j, either $e_i n_j = 0$ or $e_i n_j = n_j$. Moreover, by the Cayley-Hamilton theorem, there are rational numbers q and r such that

$$n_j^2 = q + rn_j = qe_1 + qe_2 + rn_j$$

Thus $E_1 \otimes E_2 \otimes E_3$ and $E_1 \otimes E_2 \otimes E_4$ are subalgebras of Q_2.

Let $e_i n_1 = k_i n_1$, $k_i \in Q^+$. Then $e_1 e_2 n_1 = k_1 k_2 n_1 = 0$, so k_1 or k_2 is 0. Suppose $k_1 = 0$. Then $(e_1 + e_2)n_1 = n_1$, so

(i) $e_2 n_1 = n_1$ and

(ii) $e_1 n_1 = 0$.

If $n_1 e_1 = 0$ as well, then $n_1 e_2 = n_1$. For some $q, r \in Q^+$, $n_1^2 = q + rn_1$, $n_1^2 e_2 = qe_2 + rn_1$, so $q = 0$ and $n_1^2 \in E_3$. Thus $n_1(E_1 \otimes E_2 \otimes E_3) = E_3$. Since $n_1 Q_2$ is at least two-dimensional, $n_1 n_2 > 0$. Similarly

$$e_1(E_1 \otimes E_2 \otimes E_3) = E_1$$

implies $e_1 n_2 > 0$, so $e_1 n_2 = n_2$.

Then $0 = n_1 e_1 n_2 = n_1 n_2 > 0$. Hence

(iii) $n_1 e_1 = n_1$,

(iv) $n_1 e_2 = 0$, and

(v) $n_1^2 = n_1 e_1 n_1 = 0$.

If $e_1 n_2 = 0$ as well, then $e_1 Q_2 = E_1$, so

(vi) $e_1 n_2 = n_2$,

(vii) $e_2 n_2 = 0$, and, as above,

(viii) $n_2 e_1 = 0$

(ix) $n_2 e_2 = n_2$, and

(x) $n_2^2 = 0$.

To complete a multiplication table for Q_2 it suffices to calculate $n_1 n_2$ and $n_2 n_1$:

$n_1 n_2 = ae_1 + be_2 + cn_1 + dn_2$ for some $a, b, c, d \in Q^+$. Then $n_1^2 n_2 = 0 = an_1 + dn_1 n_2$ implies $a = d = 0$, while $n_1 n_2^2 = 0 = cn_1 n_2$ implies $c = 0$, so $n_1 n_2 = be_2$. If $n_1 n_2 = 0$, then $n_1 Q_2$ is one dimensional, so $b > 0$. Observe that replacing n_1 by $b^{-1}n_1$ does not change the validity of any of the equations (i)-(x), so we may suppose

(xi) $n_1n_2 = e_2$.

Similarly, $n_2n_1 = ce_1$ for some $c > 0$. Using the relations already obtained it is now easy to check that $n_1n_2 + n_2n_1 = e_2 + ce_1$ commutes with e_1, e_2, n_1, and n_2 and hence is in the center of Q_2. Thus $c = 1$, and

(xii) $n_2n_1 = e_1$.

The equations (i)-(xii) uniquely determine a multiplication table for Q_2. This lattice-order is evidently the usual order for Q_2.

(1d) Suppose that exactly one coordinate of 1 is greater than 0: $e_1 = e_2 = e_3 = 0$, $e_4 = 1 > 0$. Let $0 < n_i \in E_i$, $i = 1, 2, 3$. Observe that

$$0 \leqq n_in_j \leqq (n_i + n_j)^2 = a + b(n_i + n_j) \text{ for some } a, b \in Q^+$$

implies that each $E_i \otimes E_4$ and each $E_i \otimes E_j \otimes E_4$ is a subalgebra of Q_2. We will consider and successively eliminate several cases depending upon the location of idempotents in the summands.

($1d_1$) Suppose that E_1, E_2, and E_3 contain no nonzero idempotents. Assume that one of the n_i's, say n_1, is invertible. Then $n_1^2 = q + rn_1$; $q, r \in Q^+, q > 0$. We have $n_1n_2 = a + bn_1 + cn_2$ for some $a, b, c \in Q^+$. Since $E_1 \otimes E_4$ is an algebra containing n_1^{-1}, $c > 0$ and $n_1n_2 > 0$. Then $qn_2 + rn_1n_2 = n_1^2n_2 = bq + (a + br)n_1 + cn_1n_2$, and $(r - c)n_1n_2 = bq + (a + br)n_1 - qn_2$, so $bq \leqq 0, a + br \leqq 0$. Thus $a = b = 0$ and $n_1n_2 = cn_2 > 0$. Now, if $n_2^2 = s + tn_2$, then $cn_2^2 = n_1n_2^2 = sn_1 + tcn_2 = cs + ctn_2$, and $s = 0$. If $t > 0$, then $t^{-1}n_2$ is a nonzero idempotent, so $n_2^2 = 0$. Similarly $n_3^2 = 0$.

If none of the n_i's are invertible, then again $n_2^2 = n_3^2 = 0$. Recalling that n_2n_3 and n_3n_2 belong to $E_2 \otimes E_3 \otimes E_4$ one can quickly compute $n_2n_3 = n_3n_2 = 0$ so that $E_2 \otimes E_3$ is a two-dimensional nilpotent subalgebra of Q_2. This is absurd.

($1d_2$) Suppose that at least two summands other than E_4 contain nonzero idempotents: say $0 < n_1 = n_1^2 \in E_1$ and $0 < n_2 = n_2^2 \in E_2$. We have $n_1n_2 = q + un_1 + vn_2$ for some $q, u, v \in Q^+$; $n_1n_2 = n_1n_2^2 = un_1n_2 + (q + v)n_2$, so $un_1n_2 = un_1$, and similarly $vn_1n_2 = vn_2$. Suppose, for example, that

(*) $n_1n_2 = n_1$.

Calculate $n_1n_3 = a + bn_1 + cn_3$ for some $a, b, c \in Q^+$, $n_1^2n_3 = (a + b)n_1 + cn_1n_3 = a + bn_1 + cn_3$, whence $cn_1n_3 = a + cn_3 - an_1$ and $a = 0$. If $c = 0$, then $n_1Q_2 = E_1$, so $b = 0$ and

(**) $n_1n_3 = n_3$.

As above, $n_2n_3 = yn_2 + zn_3$ and $zn_2n_3 = zn_3$. If $z \neq 0$, then $Q_2n_3 = E_3$, so $n_2n_3 = yn_2$ for some $y \in Q^+$. However, by (*) and (**), this yields $(n_1n_2)n_3 = n_1n_3 = n_3 = n_1(n_2n_3) = yn_1n_2 = yn_1$. Hence (*) is false and $n_1n_2 = 0$. Similarly $n_2n_1 = 0$. Calculate, as above, $n_1n_3 = xn_1 + yn_3$ and $yn_1n_3 = yn_3$. If $y = 0$, then $n_1Q_2 = E_1$, so $n_1n_3 = n_3$. Similarly $n_3n_1 = n_3n_2 = n_2n_3 = n_3$; so n_3 belongs to the center of Q_2, which is impossible.

(1d$_3$) Suppose that $0 < n_1 = n_1^2 \in E_1$, but E_2 and E_3 do not contain nonzero idempotents. As in (1d$_2$) either $n_1n_2 = kn_1$ or $n_1n_2 = n_2$; either $n_1n_3 = mn_1$ or $n_1n_3 = n_3$. We cannot have both n_1n_2 and n_1n_3 in E_1, for then $n_1Q_2 = E_1$. We cannot have both $n_1n_2 = n_2$ and $n_1n_3 = n_3$ for then n_1Q_2 is three-dimensional. Thus we may assume that $n_1n_2 = n_2$ and $n_1n_3 = kn_1$ for some $k \in Q^+$. If $k > 0$ we can replace n_3 by $k^{-1}n_3$, obtaining the possible cases:

(i) $n_1n_2 = n_2$ and $n_1n_3 = 0$, or

(ii) $n_1n_2 = n_2$ and $n_1n_3 = n_1$.

Consider (i). Calculate $n_3^2 = a + bn_3$ for some $a, b \in Q^+$. Then $n_1n_3 = 0$ implies $a = 0$, and the fact that E_3 contains no nonzero idempotents implies $b = 0$; i.e., $n_3^2 = 0$. From this we can show $n_2n_3 = 0$, which yields $Q_2n_3 = E_3$.

Consider (ii). As in the first part of the argument for (1d$_3$), $n_3n_1 = n_3$ or $n_3n_1 = kn_1$ for some $k \in Q^+$. If $n_3n_1 = n_3$, then $n_3^2 = n_3n_1n_3 = n_3n_1 = n_3$, although E_3 contains no idempotents. Thus $n_3n_1 = kn_1$; moreover, $k = 1$, so n_3 commutes with n_1. Similarly $n_2n_1 = kn_1$ or $n_2n_1 = n_2$. In the first case, $Q_2n_1 = E_1$. In the second case, n_1 is in the center of Q_2, which is false.

This completes the proof that there is no lattice-order of Q_2 satisfying the hypotheses of (1d).

(2) Suppose that $Q \approx E_1 \otimes E_2 \otimes E_3$, $1 > 0$, E_1 is two-dimensional, and $E_i \neq 0$. Let $1 = e_1 + e_2 + e_3$, $e_i \in E_i$.

(2a) If all $e_i > 0$, then each E_i is an ideal.

(2b) Suppose that $e_1, e_2 > 0 = e_3$. Let $0 < n \in E_3$. As in (1b), for $i = 1$ or 2, $e_in = n$ or $e_in = 0$. If $e_2n = 0$, then $e_1n = n$ and $E_1n = E_3$. Since $n^2 = a + bn$ implies $e_2n^2 = 0 = ae_2$, we also get $n^2 \in E_3$, so $Q_2n = E_3$. Thus $e_2n = n$, $e_1n = 0$, and again $Q_2n = E_3$.

(2c) Suppose that $e_1 = 0 < e_2, e_3$. Let $0 < n \in E_1$. Since e_in and ne_i belong to E_1^+, we can show that E_1 is an ideal if it is a subalgebra. Either e_2n or e_3n, say e_2n, is different from 0. Then $(e_2n)^2 = a + be_2n$, so $e_3(e_2n)^2 = 0 = ae_3$ implies $(e_2n)^2$ and hence $(E_1)^2$ is contained in E_1.

(2d) Suppose that $e_1 > 0 = e_2 = e_3$. Let 1 and y be a positive basis for E_1. Then $y^2 = a + by \in E_1$ implies E_1 is a totally-ordered ring. If $a = 0$, then either E_1 is a zero-ring or E_1 is an archimedean totally-ordered ring with two linearly independent idempotents. Since both of these cases are impossible [2], y, and hence each nonzero element of E_1, is invertible. From this it is easy to see that E_2 is two-dimensional, a contradiction.

(2e) Suppose that $e_1 = e_2 = 0 < e_3$. Let p_1 and p_2 be positive linearly independent elements of E_1, and let $0 < n \in E_2$.

Calculate $p_i^2 = q_i + r_i p_i$ for some $q_i, r_i \in Q^+$. Since E_1, if a subalgebra, can neither be nilpotent nor contain linearly independent idempotents, neither q_i is 0, so both p_1 and p_2 are invertible in Q_2. Calculate

$$p_1 n = a + (bp_1 + cp_2) + dn$$

for some $a, d \in Q^+$; $b, c \in Q$; $bp_1 + cp_2 \geqq 0$. Then

$$q_1 n + r_1 p_1 n = p_1^2 n = ap_1 + p_1(bp_1 + cp_2) + dp_1 n$$

and

$$(r_1 - d)p_1 n = ap_1 + p_1(bp_1 + cp_2) - q_1 n\ .$$

Before proceeding, observe that $p_1 p_2 \leqq (p_1 + p_2)^2 = x + y(p_1 + p_2)$ implies that $E_1 \otimes E_3$ is a subalgebra. Since $q_1 > 0$, $p_1(a + bp_1 + cp_2) = 0$, $a + bp_1 + cp_2 = 0$, and hence $a = b = c = 0$. Thus $p_1 n$, and similarly $p_2 n$, belong to E_2. Since p_1 and p_2 are invertible, this implies that E_2 is two-dimensional which is a contradiction.

(3) Suppose that $Q_2 \approx E_1 \otimes E_2$, $E_i \neq 0$, and $1 = e_1 + e_2 > 0$, $e_i \in E_i$.

(3a) If both coordinates of 1 are greater than 0, then each E_i is an ideal.

(3b) In case E_1 is three-dimensional and $1 \in E_1$, see the argument of (2d).

(3c) Suppose that E_1 is three-dimensional, E_2 is one-dimensional, and $1 \in E_2$. Let $0 < f \in E_1$ and $f^2 = a + bf$ for some $a, b \in Q^+$. Since E_1 cannot be a right ideal, $a > 0$ and f is invertible. Let h be an element of E_1 which is bigger than but not a rational multiple of f. Then $h^2 = x + yh$. Define

$$L = \{r \in Q^+ : rf \leqq h\}$$

and

$$U = \{s \in Q^+ : sf \geqq h\} .$$

Define $t = \sup L = \inf U$.

If $y = 0$, then $fh \in E_2$. In such a case, for each $r \in L$ and $s \in U$, $rf^2 \leqq fh \leqq sf^2$, so $ta = fh$ and t is rational. Since this is impossible, $y \neq 0$. For each $r \in L$ and $s \in U$, $r^2f^2 \leqq h^2 \leqq s^2f^2$ whence $t^2a = x$ and t^2 is rational. Then $t^2bf = yh$ and h is a rational multiple of f.

(3d) Suppose that E_1 and E_2 are two-dimensional and $1 \in E_1$. Let $0 < f \in E_2$, $f^2 = a + bf$, a and b in Q^+. Observe that a must be nonzero in order to prevent E_2 from being an ideal.

If e is a positive element of E_1 which is linearly independent of 1, consider $e^2 = x + ye$, x and y in Q^+. If $x = 0$, then $y > 0$ and $y^{-1}e$ is a nonzero idempotent of E_1 different from 1. Since this is impossible, E_1 is a field. The remainder of the argument for this case resembles that of (3c).

(4) Suppose that $Q_2 = E_1$. Since the field of rational complex numbers is a subalgebra of Q_2 which has no total order, this is impossible.

We now consider the possible lattice orders of Q_2 in which 1 is not positive. Their description is obtained in (7b).

(5) Suppose that $Q_2 \approx E_1 \otimes E_2$, $1 = e_1 + e_2$, $e_i \in E_i$, and $e_1 < 0 < e_2$. One of the summands, say E_1, has dimension bigger than 1. Calculate $e_1^2 = a + be_1 = (a + b)e_1 + ae_2$. If $a = 0$, then $e_1e_2 = e_1 - e_1^2 = e_2e_1 \in E_1$ and E_1 is an ideal. Thus $a > 0$.

Let $0 < f$ be any positive element of E_1 which is linearly independent of e_1. Let $L = \{p \in Q^+ : -pe_1 \leqq f\}$, let $U = \{q \in Q^+ : -qe_1 \geqq f\}$, and let r_f be the common least upper bound of L and greatest lower bound of U in the set of real numbers. Calculate $f^2 = x + yf$ for some $x, y \in Q$. For any p in L and q in U, $p^2a \leqq x \leqq q^2a$, so $r_f^2 = xa^{-1}$. However, r_f and r_{f-e_1} cannot both have rational squares.

(6) Suppose that $Q_2 \approx E_1 \otimes E_2 \otimes E_3$, $E_i \neq 0$; $e_i \in E_i$, and $1 = e_1 + e_2 + e_3$ is not positive. Let E_1 be the two-dimensional summand.

(6a) Suppose $e_1 < 0 < e_2, e_3$. Then $e_2^2 = a + be_2 = ae_1 + (a + b)e_2 + ae_3 \geqq 0$ implies $a = 0$. Thus $e_2^2 = k_2e_2$ and $e_3^2 = k_3e_3$ for some $k_i \in Q^+$. Since E_1 cannot be a nilpotent subalgebra, $e_1^2 = x + ye_1 > 0$. If $x = 0$, then $e_1 1 = e_1 = ye_1 + e_1e_2 + e_1e_3$, and $e_1e_2 + e_1e_3 \in E_1$, so e_1e_2 and $e_1e_3 \in E_1$. However $1e_2 = e_2 = e_1e_2 + e_2^2 + e_3e_2$, $e_3e_2 > e_3$, and $e_2^2 \in E_2$ gives rise to a contradiction. Thus $x > 0$, e_1 is invertible, and each element of E_1 is invertible. This means that $T = E_1(e_2 + e_3)$ is a two-dimensional totally ordered subspace of Q_2 and hence equals E_1, although $e_1(e_2 + e_3) = e_1 - e_1^2 = (1 - x - y)e_1 - x(e_2 + e_3)$ belongs to T.

(6b) Suppose that $e_1, e_2 > 0 > e_3$. Then $e_1^2 = ke_1$, so E_1 is a subalgebra of Q_2. Since E_1 cannot be nilpotent, $k > 0$. Moreover, if $0 < f \in E_1$ is linearly independent of e_1, then $f^2 = tf$ for some $t \in Q$, $t \neq 0$. Unfortunately, this yields linearly independent idempotents $t^{-1}f$ and $k^{-1}e_1$ of a subring of the real field.

(6c) Suppose that $e_1, e_2 < 0 < e_3$. Argue as in (6b).

(6d) Suppose that $e_1 > 0 > e_2, e_3$. Argue as in (6a) to obtain e_1e_2 in E_1. Then $e_3e_2 = (e_2 - e_2^2) - e_1e_2$ and $e_2 - e_2^2 \in E_2$ implies $e_2 \geqq e_2^2$ which is absurd.

(6e) Suppose that $e_1 = 0$, $e_2 < 0 < e_3$. Let $0 < f \in E_1$. Then $f^2 = kf$ for some $k \in Q$. Since E_1 cannot be a nilpotent algebra, $k > 0$. In this way we can produce linearly independent idempotents of the archimedean ordered ring E_1.

(6f). Suppose that $e_1 < 0 < e_2$, $e_3 = 0$. Let $0 < n \in E_3$. In the usual manner it can be shown that $E_1 \otimes E_2$ is a subalgebra of Q_2.

Now $n^2 = a + bn = ae_1 + ae_2 + bn$ implies $a = 0$. Assume that $n^2 = 0$. If, in addition, $e_1n = 0$, then $e_2n = n$ and $Q_2n = E_3$. Thus $e_1n = g + xe_2 + yn \neq 0$ for some $g \in E_1$; $x, y \in Q$. Then $e_1n^2 = 0 = gn + xe_2n$. Since $g \leqq 0$ and $x \leqq 0$, $gn = xe_2n = 0$. Thus $x = 0$, $g = 0$, and $Q_2n = E_3$.

Hence $n^2 = bn$ for some $b > 0$. Without loss of generality we may assume that n is idempotent. Again, $e_1n = g + xe_2 + yn$, $e_1n^2 = e_1n = gn + xe_2n + yn$, and $((1 + x)e_1 - g)n = (x + y)n$. Since $e_1n \notin E_3$, it follows that $x + y = 0$, $x = y = 0$, and $g = e_1$, so $e_1n = e_1$ and $e_2n = n - e_1$. Since Q_2n cannot be three-dimensional, if f is an element of E_1 which is linearly independent of e_1, then $fn = te_1$ for some $t \in Q$, $t \neq 0$. Whence $(e_1 - t^{-1}f)n = 0$, which implies $e_1n = 0$, a contradiction.

(6g) Suppose that $e_1 > 0 > e_2$, $e_3 = 0$. Proceed as in (6f) down to the point where it is concluded that $x + y = 0$. From the two equations for e_1n we calculate $(g - xe_1)n = g + xe_2 - xn = e_1n$, so $g = (1 + x)e_1$. We have $e_1n = (1 + x)e_1 + xe_2 - xn$ and

$$e_2n = -(1 + x)e_1 - xe_2 + (1 + x)n$$

which yields $0 \leqq 1 + x \leqq 0$. Thus $e_1n = -e_2 + n$ and $e_2n = e_2$.

Since Q_2n cannot be three-dimensional, if f is an element of E_1 which is linearly independent of e_1, then $fn = ae_2 + bn$ for some $a, b \in Q$. We have $(f + ae_1)n = (a + b)n$, whence $e_1n \in E_3$, a contradiction.

(7) Suppose that $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$, $E_i \neq 0$, $e_i \in E_i$, and $1 = e_1 + e_2 + e_3 + e_4$ is not comparable to 0.

(7a) Suppose that $e_1 < 0 < e_2, e_3, e_4$. Then $e_2^2 = a + be_2 = ae_1 + (a + b)e_2 + ae_3 + ae_4$ implies $a = 0$ and $e_2^2 \in E_2$. Similarly, $e_3^2 \in E_3$, $(e_2 + e_3)^2 \in E_2 \otimes E_3$, etc. Thus $E_2 \otimes E_3 \otimes E_4$ is a subalgebra of Q_2. Now calculate

$$0 \leqq e_1^2 = (1 - (e_2 + e_3 + e_4))^2 = 1 - 2(e_2 + e_3 + e_4) + (e_2 + e_3 + e_4)^2$$
$$= e_1 + f$$

for some $f \in E_2 \otimes E_3 \otimes E_4$, although $e_1 < 0$.

(7b) Suppose that $e_1, e_2 < 0 < e_3, e_4$. There are lattice-orders of Q_2 in which this situation is realized.

For each i there exists $k_i \in Q$ such that $e_i^2 = k_ie_i$. In addition, $(e_j - e_i)^2 = t(e_j - e_i)$ for some $t \in Q$ as long as $j = 3$ or 4 and $i = 1$ or 2, in which case $E_i \otimes E_j$ is a subalgebra of Q_2. Calculate $e_1e_3 = ae_1 + be_3$ for some $a, b \in Q$, $e_1^2e_3 = k_1e_1e_3 = ak_1e_1 + be_1e_3$, and $e_1e_3^2 = k_3e_1e_3 = ae_1e_3 + bk_3e_3$, which yield $be_1e_3 = bk_1e_3$ and $ae_1e_3 = ak_3e_1$. Either $e_1e_3 = 0$, or $e_1e_3 = k_1e_3$, or $e_1e_3 = k_3e_1$. Similar results hold for e_je_i and e_ie_j as long as $i = 1$ or 2 and $j = 3$ or 4.

Assume that one such product is 0; e.g., $e_1e_3 = 0$. Then $e_1 = e_11 = e_1^2 + e_1e_2 + e_1e_3 + e_1e_4$, and $e_1e_2 = (1 - k_1)e_1 - e_1e_4$. If $e_1e_4 = 0$ or $e_1e_4 = k_4e_1$, then $e_1Q_2 = E_1$ is one-dimensional. If $e_1e_4 = k_1e_4$, then $e_1e_2 = (1 - k_1)e_1 - k_1e_4$ implies $1 \leqq k_1 \leqq 0$ which is absurd. Thus no such product is 0.

Suppose that

(i) $e_1e_3 = k_1e_3$, $k_1 < 0$.

(The case $e_1e_3 = k_3e_1$ will be discussed separately.) Then $e_1 = e_11$ yields $e_1e_2 = (1 - k_1)e_1 - k_1e_3 - e_1e_4$. If $e_1e_4 = k_1e_4$, then $1 - k_1 \leqq 0$ which contradicts $k_1 < 0$, so

(ii) $e_1e_4 = k_4e_1$, $k_4 > 0$, and

$$e_1e_2 = (1 - k_1 - k_4)e_1 - k_1e_3 .$$

Calculating $e_3 = 1e_3$ we get $e_4e_3 = (1 - k_1 - k_3)e_3 - e_2e_3$. If $e_2e_3 = k_2e_3$, then Q_2e_3 is one-dimensional, so

(iii) $e_2e_3 = k_3e_2$, $k_3 > 0$ and

$$e_4e_3 = (1 - k_1 - k_3)e_3 - k_3e_2 .$$

Now $e_1e_2e_3 = k_3e_1e_2 = k_3(1 - k_1 - k_4)e_1 - k_3k_1e_3 = (1 - k_1 - k_4)e_1e_3 - k_1e_3^2 = k_1(1 - k_1 - k_4)e_3 - k_1k_3e_3$, whence

(iv) $1 = k_1 + k_4$ and

(v) $e_1e_2 = - k_1e_3$.

Calculate $e_1e_2^2 = k_2e_1e_2 = - k_1k_2e_3 = - k_1e_3e_2$, so

(vi) $e_3e_2 = k_2e_3$,

and $e_4e_2 = e_2 - e_1e_2 - e_2^2 - e_3e_2 = (1 - k_2)e_2 + (k_1 - k_2)e_3$, whence

(vii) $k_1 = k_2$.

Now $e_4e_1 = e_4 - e_4e_2 - e_4e_3 - e_4^2 = (1 - k_4)e_4 + (k_3 - k_4)e_2 - (1 - k_1 - k_3)e_3$, whence $1 = k_1 + k_3$

(viii) $k_3 = k_4$, and

(ix) $e_4e_1 = k_2e_4$.

Since $e_3e_4 = e_3 - e_3e_2 - e_3^2 - e_3e_1 = - e_3e_1$ and simultaneously $e_3e_4 = e_4 - e_1e_4 - e_2e_4 - e_4^2 = (1 - k_4)e_4 - k_4e_1 - e_2e_4$, we must have

(x) $e_2e_4 = k_2e_4$.

Let $\alpha = k_1 = k_2$, $\beta = k_3 = k_4$, $f_i = k_i^{-1}e_i$. Then $\alpha < 0 < \beta$, $\alpha + \beta = 1$, and the f_i's are nonzero linearly independent idempotents different from the identity and satisfying

$$\alpha(f_1 + f_2) + \beta(f_3 + f_4) = 1 .$$

Moreover, equations (i)—(x) together with the fact that $e_1 + e_2 + e_3 + e_4 = 1$ yield the following multiplication table.

	f_1	f_2	f_3	f_4
f_1	f_1	$-\beta\alpha^{-1}f_3$	f_3	f_1
f_2	$-\beta\alpha^{-1}f_4$	f_2	f_2	f_4
f_3	f_1	f_3	f_3	$-\alpha\beta^{-1}f_1$
f_4	f_4	f_2	$-\alpha\beta^{-1}f_2$	f_4

Thus such a lattice-order would be determined up to isomorphism by the choice of β. The matrices

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} -\beta\alpha^{-1} & 1 \\ -\beta\alpha^{-2} & \alpha^{-1} \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & -\alpha\beta^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 0 \end{pmatrix}$$

fulfill all of the requirements.

Clearly distinct β's yield nonisomorphic lattice-orders.

Finally, suppose that $e_1e_3 = k_3e_1$ (rather than k_1e_3). Now $e_1e_2 = e_1 - e_1e_3 - e_1e_4 - e_1^2 = (1 - k_3 - k_1)e_1 - e_1e_4$. If $e_1e_4 = k_4e_1$, then e_1Q_2 is one-dimensional. Thus $e_1e_4 = k_1e_4$. This indicates that the lattice-order must be isomorphic to one of those already considered.

(7c) Suppose that $e_1, e_2, e_3 < 0 < e_4$. Proceed as in (7a). Then $e_i^2 = k_ie_i$ for $i = 1, 2, 3$ and $E_1 \otimes E_2 \otimes E_3$ is a subalgebra of Q_2. Calculate $e_4^2 = e_4 + f$ for some $f \in E_1 \otimes E_2 \otimes E_3$. Let $e_ie_4 = f_i + d_ie_4$ where $f_i \in E_1 \otimes E_2 \otimes E_3$ and $d_i \in Q^-$. Moreover, $(e_1 + e_2 + e_3)e_4 = e_4 - e_4^2 = (f_1 + f_2 + f_3) + (d_1 + d_2 + d_3)e_4$. Since $e_4 - e_4^2 = -f$, $d_1 = d_2 = d_3 = 0$. This implies that $E_1 \otimes E_2 \otimes E_3$ is a three-dimensional right ideal.

(7d) Suppose that $e_4 = 0$, the other e_i's are not 0, and e_1 and e_2 have the same sign opposite that of e_3. Let $0 < n \in E_4$. Then $e_1^2 = k_1e_1$, $e_2^2 = k_2e_2$, $n^2 = kn$ and k_1 and k_2 have the same sign. Moreover $E_1 \otimes E_4$, $E_2 \otimes E_4$, and $E_1 \otimes E_2 \otimes E_3$ are subalgebras of Q_2.

Let $e_2n = ae_2 + bn$. Then $e_2^2n = k_2e_2n = ak_2e_2 + be_2n$ and $e_2n^2 = ke_2n = ae_2n + bkn$, so $be_2n = bk_2n$ and $ae_2n = ake_2$. Thus $e_2n = 0$ or $e_2n = ke_2$, or $e_2n = k_2n$. Similarly for e_1n, ne_2, and ne_1.

(i) Suppose that $e_2n = 0$. If $e_1n = 0$ or $e_1n = k_1n$, then $Q_2n = E_4$; so $e_1n = ke_1$, $k \neq 0$, and $e_3n = n - ke_1$. For some $x, y, z \in Q$, $e_1e_2 = xe_1 + ye_2 + ze_3$. Then $e_1e_2n = 0 = k(x - z)e_1 + zn$, $z = x = 0$, and $e_1e_2 \in E_2$. By a similar calculation $ne_2 \in E_2$, whence $Q_2e_2 = E_2$.

(ii) Suppose that $e_2n = ke_2$. Then $e_1n = ke_1$ would make Q_2n three-dimensional, so $e_1n = k_1n$. Both k and k_1, by (i), are different from 0. Now

$$e_1e_2 = xe_1 + ye_2 + ze_3,$$
$$e_1e_2n = ke_1e_2 = k(y - z)e_2 + (z - zk_1 + xk_1)n, \quad x = z = 0,$$

and $e_1e_2 = ye_2$. If $ne_2 = ke_2$, then $Q_2e_2 = E_2$, so $ne_2 = k_2n$. Finally, $ne_1 = ke_1$, which yields $ne_1e_2 = ke_1e_2 = yne_2 = yk_2n = kye_2$, and $e_1e_2 = 0$. By symmetry, $e_2e_1 = 0$, whence $e_2Q_2 = E_2$.

(iii) Suppose that $e_2n = k_2n$. Then $e_1n = k_1n$ would make $Q_2n = E_4$; so $e_1n = ke_1$, and we are back to case (ii).

(7e) Suppose that $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$, $E_i \neq 0$, $1 = e_1 + e_2$, $e_1 < 0 < e_2$, and $e_i \in E_i$. Let $0 < n_3 \in E_3$ and $0 < n_4 \in E_4$.

Then $n_i^2 = k_in_i$, and we may assume $k_i = 0$ or $k_i = 1$. Suppose, for example, that $n_3^2 = 0$. Since $E_3 \otimes E_4$ is a subalgebra of Q_2, $n_4n_3 = an_3 + bn_4$ for some $a, b \in Q$; $0 = n_4n_3^2 = bn_4n_3$ yields $n_4n_3 \in E_3$. Since $E_1 \otimes E_2 \otimes E_3$ is a subalgebra of Q_2, $e_1n_3 = xe_1 + ye_2 + zn_3$, and $e_1n_3^2 = 0 = xe_1n_3 + ye_2n_3$. Since $xe_1 \leqq 0$ and $ye_2 \leqq 0$, $xe_1n_3 = ye_2n_3 = 0$. In particular, $x = 0$. If $e_2n_3 = 0$, then $0 \geqq e_1n_3 = n_3 > 0$, so $y = 0$ also. Thus $Q_2n_3 = E_3$.

We may thus assume that n_3 and n_4 are idempotents. This time

$n_3n_4 = an_3 + bn_4$ yields $bn_3n_4 = bn_4$ and $an_3n_4 = an_3$. Either $a = 0$ or $b = 0$. Suppose $a = 0$. Calculate $e_1n_4 = xe_1 + ye_2 + zn_4$, $e_1n_4^2 = e_1n_4 = xe_1n_4 + ye_2n_4 + zn_4$, so $(1 - x + y)e_1n_4 = (y + z)n_4$. Since $n_3n_4 \in E_4$, $e_1n_4 \notin E_4$, so $y = z = 0$ and $x = 1$; i.e., $e_1n_4 = e_1$.

If $n_3n_4 \neq 0$, then $n_3n_4 = n_4$. Calculate $e_1n_3 = ae_1 + be_2 + cn_3$, from which $e_1n_3n_4 = e_1n_4 = e_1 = ae_1 + b(n_4 - e_1) + cn_4$. This yields $b = c = 0$ and $e_1n_3 = e_1$. Similarly $e_1e_2 = ae_1 + be_2$, from which $e_1e_2n_4 = e_1(1 - e_1) = ae_1n_4 + be_2n_4 = ae_1 + b(n_4 - e_1)$. Since $e_1^2 \in E_1 \otimes E_2$, $b = 0$. Thus e_1e_2 and $e_1^2 = e_1 - e_1e_2 \in E_1$, whence e_1Q_2 is one-dimensional.

We must have $n_3n_4 = 0$, and, similarly, $n_4n_3 = 0$. Now $e_1n_4n_3 = e_1n_3 = 0$, although, as in the calculation for e_1n_4, $e_1n_3 = e_1$.

The referee is responsible for an important change in the statement of the theorem. Having detected an error in the original version of (7b), he suggested as a counter example the matrices f_i now listed there.

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