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ON THE EQUATION $\varphi(x) = \int_x^{x+1} K(\xi)f[\varphi(\xi)]d\xi$

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Suppose $K(x)$ measurable and $0 < K(x) \leqq 1$ for $x \in (-\infty, \infty)$. Suppose $f(u)$ convex for $u \in [0,1]$, $f(0) = 0$, $f(u) > 0$ for $u \in (0,1)$, and $f(u) = 1 - f'(1)(1-u) + O(1-u)^{1+\delta}$ as $u \to 1$ for some $\delta > 0$. (Example: $f(u) = u^p$, $p \geqq 1$.)

Theorem: The equation $(*)\varphi(x) = \int_x^{x+1} K(\xi)f[\varphi(\xi)]d\xi$ has a solution $\varphi(x)$ satisfying $0 < \varphi(x) \leqq 1$ for $x \in (-\infty, \infty)$ if and only if $\int^\infty e^{\alpha x}[1 - K(x)]dx < \infty$ where α is the largest real root of $\alpha = f'(1)(1-e^{-\alpha})$. Furthermore, if φ is any such solution of (*), then the limits $\varphi(\pm\infty)$ exist and satisfy

$$\frac{\varphi(+\infty) - \varphi(-\infty)}{2} = \int_{-\infty}^{\infty} [\varphi(x) - K(x)f[\varphi(x)]]dx .$$

In 1960 M. L. Slater and H. S. Wilf [2] studied the linear integral equation $\varphi(x) = \int_x^{x+1} K(\xi)\varphi(\xi)d\xi$, $-\infty < x < \infty$, with $\varphi(+\infty) = 1$, and obtained the following results. Under the assumptions 1° $K(x)$ measurable, 2° $0 < K(x) \leqq 1$, 3° $K(x)$ increasing for sufficiently large x, and 4° $\lim_{x\to\infty} K(x) = 1$, a solution φ of the equation exists satisfying $\varphi(+\infty) = 1$ if and only if $\int^\infty [1 - K(x)]dx < \infty$. (We use the notation "$\int^\infty$" to mean "the integral from any finite limit to infinity.") If in addition 5°

$$\lim_{x\to-\infty} \int_x^{x+1} | K(\xi + 1) - K(\xi) | d\xi = 0,$$

then $\varphi(-\infty)$ exists.

The purpose of this paper is to extend the above results in two directions; namely to generalize the equation and to remove some of the restrictions on $K(x)$.

Accordingly, we consider throughout the paper the equation

$$\varphi(x) = \int_x^{x+1} K(\xi)f[\varphi(\xi)]\, d\xi$$

with the requirement that the solution φ satisfy $0 < \varphi(x) \leqq 1$ for all x. The functions $f(u) = u^p$, $p \geqq 1$, were the prototypes for the analysis and the results which we summarize below are valid for at least these functions. However, for each theorem of the paper a wider class of functions $\{f\}$ is specified in order to clarify the logical structure of the result. The weakening of the restrictions on $K(x)$

is easily stated. Assumptions 3°, 4°, and 5° are dropped completely and without replacement.

In § II we consider the question of existence of the limits $\varphi(\pm\infty)$. Theorem 1 and its corollary establish that under conditions 1° and 2° both of the limits exist. (The order argument used in § II was already used to some extent in [2].)

Section III contains the proofs of two lemmas required for the main existence theorem-Theorem 2 in § IV. This theorem provides a necessary and sufficient condition for the existence of a solution of the required type. The condition reduces in the linear case to that obtained in [2]. The underlying assumptions on K are again only 1° and 2°.

Section V contains an extension of an integral relation proved in [2] (Theorem 3), and § VI gives a brief discussion of the actual range of validity of the results (Theorem 4).

II Existence of $\varphi(\pm\infty)$.

THEOREM 1. *Suppose $K(x)$ measurable and $0 < K(x) \leqq 1$ a.e. for $-\infty < x < \infty$, and suppose $\varphi(x)$ satisfies $0 < \varphi(x) \leqq 1$ and the linear equation*

$$\varphi(x) = \int_x^{x+1} K(\xi)\varphi(\xi)d\xi \tag{1}$$

for all x. Then both $\varphi(+\infty)$ and $\varphi(-\infty)$ exist and satisfy

$$\frac{\varphi(+\infty) - \varphi(-\infty)}{2} = \int_{-\infty}^{\infty} \varphi(\xi)[1 - K(\xi)]d\xi\ . \tag{2}$$

Proof. Define

$$G(x) = \int_0^1 K(x+1-y)\varphi(x+1-y)y\,dy\ .$$

$$G(x) = \int_x^{x+1} K(\xi)\varphi(\xi)(x+1-\xi)d\xi$$

is absolutely continuous over any finite interval, and, by using equation (1), one can verify that $G'(x) = \varphi(x)[1 - K(x)]$ a.e. Thus $G(x)$ is increasing so that $G(\pm\infty)$ exist, are finite, and

$$\infty > G(+\infty) - G(-\infty) = \int_{-\infty}^{\infty} \varphi(x)[1 - K(x)]dx\ . \tag{3}$$

We first prove $\varphi(+\infty)$ exists. Set $M = \limsup_{x\to\infty} \varphi(x)$, $m = \liminf_{x\to\infty} \varphi(x)$, and suppose $M > m$. Set

$$k = \limsup_{x\to\infty} \int_x^{x+1} |\varphi'(\xi)|\, d\xi \,.$$

Almost everywhere,

$$\varphi'(x) = \varphi(x)[1 - K(x)] - \varphi(x+1)[1 - K(x+1)] + \varphi(x+1) - \varphi(x)\,,$$

so that since

$$\infty > \int_{-\infty}^{\infty} \varphi[1 - K]dx\,, \qquad k \leqq M - m\,.$$

Now, it follows from equation (1) that φ cannot have a proper maximum at the left hand endpoint of an interval of length one; that is, it is impossible that for any x, $\varphi(x) > \varphi(y)$ for all y satisfying $x < y \leqq x + 1$. We shall use this fact (which we shall refer to as the "proper maximum property") to show that given any positive $\varepsilon < (M - m)/2$ and X arbitrarily large, there exist triples x, y, z satisfying $X < x < y < z$, and $z - x \leqq 1$, for which $\varphi(x) = \varphi(z) = M - \varepsilon$, and $\varphi(y) = m + \varepsilon$.

Choose $x_0 > X$ so that $\varphi(x_0) = M - \varepsilon$ and let y be the first point greater than x_0 at which $\varphi(y) = m + \varepsilon$. Now let x be the largest point less than y at which $\varphi(x) = M - \varepsilon$. $y - x < 1$; otherwise the proper maximum property would be violated. Finally let z be the first point greater than y at which $\varphi(z) = M - \varepsilon$. $z - x \leqq 1$ for the same reason.

Given $\varepsilon > 0$, choose $X = X(\varepsilon)$ so that for all

$$x \geqq X, \quad k + \varepsilon > \int_x^{x+1} |\varphi'(\xi)|d\xi\,.$$

Now choose x, y, z as described in the preceding paragraph using $X = X(\varepsilon)$. Then

$$\begin{aligned} k + \varepsilon &> \int_x^z |\varphi'(\xi)|\, d\xi \\ &\geqq \left|\int_x^y \varphi'(\xi)d\xi\right| + \left|\int_y^z \varphi'(\xi)d\xi\right| \\ &= 2(M - m - 2\varepsilon). \quad \text{Hence} \\ k &\geqq 2(M - m), \text{ contradicting } k \leqq M - m\,. \end{aligned}$$

Thus $M = m = \varphi(+\infty)$, and incidentally, $k = 0$.

The proof that $\varphi(-\infty)$ exists is similar to the preceding proof. Define M, m, and k as above but with respect to $-\infty$. Then as in the previous case, $k \leqq M - m$. To find the appropriate triples to complete the proof, we proceed slightly differently. Given X choose $y < X - 1$ such that $\varphi(y) = m + \varepsilon$. Then take x to be the first point less than

y at which $\varphi(x) = M - \varepsilon$ and z to be the first point greater than y at which $\varphi(z) = M - \varepsilon$. (The existence of such a z is guaranteed by the proper maximum property.) The remainder of the proof is identical to the corresponding part of the preceding proof.

$G(\pm\infty)$ can be evaluated in terms of $\varphi(\pm\infty)$, yielding the integral formula obtained in [2]. For, using equation (1) and an interchange of the order of integration, we obtain

$$\int_x^{x+1} G(\xi)d\xi = \int_0^1 \varphi(x+1-y)\,y\,dy\,.$$

Hence

$$G(\pm\infty) = \frac{\varphi(\pm\infty)}{2}$$

and so

$$\int_{-\infty}^{\infty} \varphi[1-K]d\xi = \frac{\varphi(+\infty) - \varphi(-\infty)}{2}\,.$$

COROLLARY. *Suppose $f(u)$ is continuous and satisfies $0 < f(u) \leqq u$ for $u \in (0, 1]$, suppose $K(x)$ is measurable and satisfies $0 < K(x) \leqq 1$ for $-\infty < x < \infty$, and suppose $\varphi(x)$ satisfies $0 < \varphi(x) \leqq 1$ and the equation*

$$\varphi(x) = \int_x^{x+1} K(\xi) f[\varphi(\xi)]d\xi \tag{1f}$$

over the same range of x. Then both $\varphi(+\infty)$ and $\varphi(-\infty)$ exist.

Proof. Apply Theorem 1 to $Kf[\varphi]/\varphi$ in place of K.

III. The main lemmas.

LEMMA 1. *Suppose $X \in (-\infty, \infty)$, $a \geqq 1$, and $\mu_0(x)$ measurable, $0 \leqq \mu_0(x) < \infty$, for $x \geqq X$. Then the linear integral inequality*

$$\mu(x) \geqq \mu_0(x) + a\int_x^{x+1} \mu(\xi)d\xi \tag{$*$}$$

has a solution $\mu(x)$ with $0 \leqq \mu(x) < \infty$ for $x \geqq X$ if and only if

$$\int^{\infty} e^{\alpha x}\mu_0(x)dx < \infty\,, \tag{5}$$

where $\alpha = \alpha(a)$ is the largest real root of $\alpha = a(1 - e^{-\alpha})$. (Note that $\alpha > 0$ if $a > 1$ and $\alpha = 0$ if $a = 1$.) Furthermore, if a finite non-negative solution of (*) exists, then there is also such a solution of

(*) *with the inequality replaced by equality which has the additional property that* $\lim_{x\to\infty}[\mu(x)-\mu_0(x)]=0$.

Proof. Let $\mu(x)$ be a finite nonnegative solution of (*). Let $F(x)$ be any increasing continuously differentiable function defined for $x \geqq X-1$. Then for $x \geqq X$

$$\frac{d}{dx}\int_0^1 \mu(x+1-y)[F(x)-F(x-y)]dy$$

$$= F'(x)\int_0^1 \mu(x+1-y)dy + \mu(x)[F(x-1)-F(x)]$$

$$\leqq \mu(x)\left[\frac{F'(x)}{a}+F(x-1)-F(x)\right]-\frac{\mu_0(x)F'(x)}{a}.$$

If $a>1$, set $F(x)=(e^{\alpha x}-1)/\alpha$, where α is defined above, and if $a=1$ set $F(x)=x$, the limiting value as α approaches zero. The expression in square brackets vanishes, and we have

$$(6)\qquad \frac{d}{dx}\int_0^1 \mu(x+1-y)[F(x)-F(x-y)]dy \leqq -\frac{\mu_0(x)F'(x)}{a}$$

Thus, since $\mu(x)\geqq 0$, we find

$$\int_x^\infty \mu_0(\xi)F'(\xi)d\xi \leqq a\int_0^1 \mu(x+1-y)[F(x)-F(x-y)]dy\ ,$$

thereby establishing necessity.

To prove sufficiency we first define

$$\gamma(u) = ae^{-\alpha u} \qquad 0 \leqq u \leqq 1\ ,$$
$$= 0 \qquad u > 1\ ,$$

and show that the solution $\nu(u)$ of the equation

$$(7)\qquad \nu(u)=\gamma(u)+\int_0^u \nu(v)\gamma(u-v)dv$$

is unique, nonnegative, and bounded. Equation (7) is an example of a renewal equation, and uniqueness and nonnegativity follow from the general theory of such equations. (See for example Doetsch [**1**], Volume III, page 145, Theorem I.) Boundedness, which is essential here, can be shown by noting that if ν is unbounded then there is a $\bar{u}>1$ such that if $u<\bar{u}$ then $\nu(u)<\nu(\bar{u})$. But

$$\nu(\bar{u})=\int_{\bar{u}-1}^{\bar{u}} \nu(v)\gamma(\bar{u}-v)dv\ ,$$

and since $\int_0^1 \gamma(v)dv = 1$ (a consequence of $\alpha=\alpha(a)$),

$$\int_{\bar{u}-1}^{\bar{u}} [\nu(\bar{u}) - \nu(v)]\gamma(\bar{u} - v)dv = 0 ,$$

contradicting the positivity of $\gamma(u)$.

We now proceed with the proof of sufficiency and show that

$$\mu(x) = \mu_0(x) + \int_0^\infty \nu(u)\mu_0(x + u)e^{\alpha u}du \tag{8}$$

is a solution of (*). Actually we show that $\mu(x)$ satisfies (*) with equality. To do this we must verify that

$$\int_0^\infty \nu(u)e^{\alpha u}\mu_0(x + u)du = a\int_x^{x+1} \mu(\xi)d\xi . \tag{9}$$

The right hand side of (9) can be rewritten as

$$\int_0^1 ae^{-\alpha u}e^{\alpha u}\mu(x + u)du = \int_0^\infty \gamma(u)e^{\alpha u}\mu(x + u)du ,$$

and substituting (8) this becomes

$$\int_0^\infty \gamma(u)e^{\alpha u}\mu_0(x + u)du + \int_0^\infty\int_0^\infty \nu(v)\gamma(u)e^{\alpha(u+v)}\mu_0(x + u + v)du\,dv .$$

If in the double integral we set $u + v = w$ and $v = z$ and integrate first with respect to z we obtain

$$\int_0^\infty dw\, e^{\alpha w}\mu_0(x + w)\int_0^w \nu(z)\gamma(w - z)dz .$$

Thus, after renaming variables, the right side of (9) becomes

$$\int_0^\infty du\, e^{\alpha u}\mu_0(x + u)\left\{\gamma(u) + \int_0^u \nu(v)\gamma(u - v)dv\right\} ,$$

and the required equality is a consequence of (7).

To prove the last statement of the lemma we show now that

$$\lim_{x\to\infty} \int_0^\infty \nu(u)\mu_0(x + u)e^{\alpha u}du = 0 .$$

This follows from the boundedness of $\nu(u)$ and the fact that

$$\int^\infty e^{\alpha x}\mu_0(x)dx < \infty .$$

LEMMA 2. *Suppose* $a > 1$ *and* $\alpha = \alpha(a)$ *is the largest real root of* $\alpha = a(1 - e^{-\alpha})$. *Then for all* $\beta < \alpha$ $\int^\infty e^{\beta x}\mu(x)dx < \infty$, *where* $\mu(x)$ *is any nonnegative finite-valued solution of* (*) *with the parameter* a.

proof. From (6)

$$\frac{d}{dx}\left[e^{\alpha x}\int_0^1 \mu(x+1-y)(1-e^{-\alpha y})dy\right] \leqq 0 .$$

Hence for some nonnegative A, $\int_0^1 \mu(x+1-y)(1-e^{-\alpha y})dy \leqq Ae^{-\alpha x}$, and

$$\begin{aligned}\frac{A}{\alpha-\beta}\, e^{-(\alpha-\beta)x} &\geqq \int_x^\infty e^{\beta\xi}\, d\xi \int_0^1 \mu(\xi+1-y)(1-e^{-\alpha y})dy \\ &= \int_0^1 e^{-\beta(1-y)}(1-e^{-\alpha y})dy \int_x^\infty e^{\beta(\xi+1-y)}\mu(\xi+1-y)d\xi \\ &\geqq C\int_{x+1}^\infty e^{\beta\xi}\mu(\xi)d\xi, \text{ where } C = \int_0^1 e^{-\beta(1-y)}(1-e^{-\alpha y})dy .\end{aligned}$$

IV. Existence of solutions.

THEOREM 2. *Suppose $K(x)$ measurable and $0 < K(x) \leqq 1$ a.e. in $-\infty < x < +\infty$. Suppose $f(u)$ convex for $0 \leqq u \leqq 1$, $f(0) = 0$, $f(1) = 1$, $f(u) > 0$ for $0 < u < 1$, $f'(1) < \infty$, and $f(u) = 1 - f'(1)(1-u) + O(1-u)^{1+\delta}$ as $u \to 1$ for some $\delta > 0$. Then the equation*

$$\varphi(x) = \int_x^{x+1} K(\xi) f[\varphi(\xi)]d\xi \tag{10}$$

has a solution $\varphi(x)$, $-\infty < x < \infty$, satisfying $0 < \varphi(x) \leqq 1$, if and only if

$$\int^\infty e^{\alpha\xi}(1-K(\xi))d\xi < \infty ,$$

where $\alpha = \alpha(f'(1))$ is the largest real root of $\alpha = f'(1)(1-e^{-\alpha})$. If $f'(1) > 1$, then $1 - \varphi(x) = O(e^{-\beta x})$ as $x \to \infty$ for all $\beta < \alpha = \alpha(f'(1))$.

Sufficiency. Define

$$\varphi_0(x) \equiv 1,\ \varphi_{n+1}(x) = \int_x^{x+1} K(\xi) f[\varphi_n(\xi)]d\xi .$$

Then, since $f(x)$ is increasing, $0 < \varphi_{n+1}(x) \leqq \varphi_n(x)$ for all x and $n \geqq 0$. Thus $\lim_{n\to\infty}\varphi_n(x) = \varphi(x)$ exists and $\varphi(x)$ satisfies equation (10) by the dominated convergence theorem. We must show that $\varphi(x)$ is positive. For $n \geqq 1$

$$\begin{aligned}\varphi_n(x) - \varphi_{n+1}(x) &= \int_x^{x+1} K(\xi)[f(\varphi_{n-1}) - f(\varphi_n)]d\xi \\ &\leqq f'(1)\int_x^{x+1}[\varphi_{n-1}(\xi) - \varphi_n(\xi)]d\xi .\end{aligned}$$

Thus

$$1-\varphi_{n+1}(x) \leqq 1-\varphi_1(x)+f'(1)\int_x^{x+1}[1-\varphi_n(\xi)]d\xi$$

$$(11) \qquad = \int_x^{x+1}[1-K(\xi)]d\xi + f'(1)\int_x^{x+1}[1-\varphi_n(\xi)]d\xi \, .$$

Since

$$\int^\infty e^{\alpha x}\int_x^{x+1}[1-K(\xi)]d\xi\, dx < \infty \, ,$$

since $f'(1) \geqq 1$, and since

$$\lim_{x\to\infty}\int_x^{x+1}(1-K)d\xi = 0 \, ,$$

there is by Lemma 1 a nonnegative function $\mu(x)$ satisfying

$$(12) \qquad \mu(x) = \int_x^{x+1}[1-K(\xi)]d\xi + f'(1)\int_x^{x+1}\mu(\xi)d\xi \quad \text{and} \quad \lim_{x\to\infty}\mu(x) = 0 \, .$$

Now

$$1-\varphi_1(x) = \int_x^{x+1}[1-K(\xi)]d\xi \leqq \mu(x)$$

and by induction using (11) and (12) we see that $1-\varphi_n(x) \leqq \mu(x)$ and consequently $1-\varphi(x) \leqq \mu(x)$. Thus $\lim_{x\to\infty}\varphi(x) = 1$, and if $\varphi(x) = 0$ for some x, there must be a largest x at which φ vanishes. But this clearly contradicts the fact that φ is a solution of (10).

Necessity. Suppose that $\varphi(x)$ is a solution of the required type. By the corollary to Theorem 1, $\varphi(+\infty)$ exists. Now, in fact, $\varphi(+\infty) = \text{lub}\,\varphi(x)$, for if not there would exist an $\bar{x}$ such that for all $x > \bar{x}$, $\varphi(\bar{x}) > \varphi(x)$, which would contradict the fact that $\varphi(x)$ satisfies (10). In particular this means that $\varphi(+\infty) > 0$. If $f(u) \equiv u$, then $\varphi(x)/\varphi(+\infty)$ is a solution whose limit at infinity is one. If $f(u) \not\equiv u$, then $f(u) < u$ for $0 < u < 1$, and from (10) we see that since $\varphi(+\infty) \neq 0$, it must be equal to one. Thus we may always assume $\varphi(+\infty) = 1$.

Writing $f(u) = 1 - f'(1)(1-u) + R(u)$ we have

$$1-\varphi(x) = \int_x^{x+1}[1-K(\xi)][1-f'(1)(1-\varphi(\xi))]d\xi$$
$$- \int_x^{x+1}K(\xi)R[\varphi(\xi)]d\xi + f'(1)\int_x^{x+1}(1-\varphi(\xi))d\xi \, .$$

If $f(u) \equiv u$, then $R(u) \equiv 0$ and $f'(1) = 1$ so that the use of Lemma 1 with $\mu(x) = 1-\varphi(x)$ allows one to conclude that

$$\int^\infty dx\int_x^{x+1}[1-K(\xi)]\varphi(\xi)d\xi < \infty \, .$$

Then, since $\varphi(+\infty)=1$, we obtain the desired result that

$$\int^{\infty}[1-K(\xi)]d\xi<\infty\ .$$

If $f(u)\not\equiv u$, then $f'(1)>1$. We first show that if $\delta>0$, then

$$\int^{\infty}e^{\alpha\xi}[1-\varphi(\xi)]^{1+\delta}d\xi<\infty\ .$$

Define

$$g(x)=\int_0^1\{1-K(x+1-y)f[\varphi(x+1-y)]\}y\,dy\ .$$

Now $g(x)$ is absolutely continuous over any finite interval and since for almost all x, $g'(x)=-[\varphi(x)-K(x)f[\varphi(x)]]\leqq 0$, $g(x)$ is decreasing. Furthermore from (10)

$$\int_x^{x+1}g(\xi)d\xi=\int_0^1[1-\varphi(x+1-y)]y\,dy\ .$$

Thus for any $\varepsilon\in(0,\ f'(1)-1)$ and for sufficiently large x, since $\varphi(+\infty)=1$, we have $1-f[\varphi(x)]\geqq(f'(1)-\varepsilon)(1-\varphi(x))$, so that

$$\begin{aligned}\int_x^{x+1}g(\xi)d\xi&\leqq\frac{1}{f'(1)-\varepsilon}\int_0^1\{1-f[\varphi(x+1-y)]\}y\,dy\\&\leqq\frac{g(x)}{f'(1)-\varepsilon}\ .\end{aligned}$$

Hence by Lemma 2,

$$\int^{\infty}e^{\beta x}g(x)\,dx<\infty\ \text{ for all }\beta<\alpha=\alpha(f'(1))\ .$$

Since $g(x)$ is decreasing,

$$g(x+1)e^{\beta x}\leqq\int_x^{x+1}e^{\beta\xi}g(\xi)\,d\xi<A=A(\beta)\ ,$$

and so $g(x)=O(e^{-\beta x})$ for all $\beta<\alpha$. On the other hand

$$\begin{aligned}1-\varphi(x)&=\int_x^{x+1}\{1-K(\xi)f[\varphi(\xi)]\}d\xi\\&=\int_0^1\{1-K(x+1-y)f[\varphi(x+1-y)]\}dy\\&\leqq 2g(x)+2g(x+1/2)=O(e^{-\beta x})\ ,\end{aligned}$$

so that if we now choose β so that $\beta(1+\delta)>\alpha$, we have the required result.

Since $R(\varphi)$ by hypothesis is $O\{(1-\varphi)^{1+\delta}\}$, the equation

$$\mu(x) = \int_x^{x+1} K(\xi)R[\varphi(\xi)]d\xi + f'(1)\int_x^{x+1} \mu(\xi)d\xi \,, \tag{13}$$

has by Lemma 1 a nonnegative solution $\mu(x)$ for which $\lim_{x\to\infty} \mu(x) = 0$. $(R(\varphi) \to 0.)$ Now,

$$\varphi(x) = \int_x^{x+1} K(\xi)R(\varphi)d\xi + \int_x^{x+1} K(\xi)[1 - f'(1)(1 - \varphi(\xi))]d\xi \,.$$

Define $\psi_0(x) = \varphi(x)$, and for $n \geqq 0$,

$$\psi_{n+1}(x) = \int_x^{x+1} K(\xi)[1 - f'(1)(1 - \psi_n(\xi))]d\xi \,. \tag{14}$$

Since $R(\varphi) \geqq 0$ (by the convexity of f), $\varphi(x) = \psi_0(x) \geqq \psi_1(x)$, and we see by induction using (14) that each $\psi_n(x) \geqq \psi_{n+1}(x)$. Thus $\varphi(x) - \psi_n(x)$ is increasing with respect to n. Again,

$$\varphi(x) - \psi_{n+1}(x) = \int_x^{x+1} K(\xi)R(\varphi)d\xi + f'(1)\int_x^{x+1} K(\xi)[\varphi(\xi) - \psi_n(\xi)]d\xi \,. \tag{15}$$

Now, $\varphi(x) - \psi_0(x) = 0 \leqq \mu(x)$, and by a second induction using (13) and (15) we see that $\varphi(x) - \psi_n(x) \leqq \mu(x)$. Thus $\psi_n \downarrow_n \psi(x)$ (say) satisfying $\varphi(x) \geqq \psi(x) \geqq \varphi(x) - \mu(x)$, and

$$\psi(x) = \int_x^{x+1} K(\xi)[1 - f'(1)(1 - \psi(\xi))]d\xi \,. \tag{16}$$

We rewrite (16) as

$$\begin{aligned} 1 - \psi(x) = &\int_x^{x+1} [1 - K(\xi)][1 - f'(1)(1 - \psi(\xi))]d\xi \\ &+ f'(1)\int_x^{x+1} [1 - \psi(\xi)]d\xi \,, \end{aligned}$$

and note that since $\lim_{x\to\infty} \mu(x) = 0$ there is an $X = X(\varepsilon)$ such that for $x \geqq X$, $0 \leqq 1 - \psi(x) \leqq \varepsilon$. Thus

$$1 - \psi(x) \geqq (1 - f'(1)\varepsilon)\int_x^{x+1} [1 - K(\xi)]d\xi + f'(1)\int_x^{x+1} [1 - \psi(\xi)]d\xi \,,$$

and so by Lemma 1,

$$\int^\infty e^{\alpha\xi}[1 - K(\xi)]d\xi < \infty \,.$$

V. An integral relation. Suppose $f(u)$ is as in Theorem 2 and in addition $f(u) \not\equiv u$. Then $\varphi(+\infty) = 1$ and from equation (10) we see that $\varphi(-\infty) = 0$ or 1. Apply Theorem 1 with K replaced by $Kf(\varphi)/\varphi$. Then equation (2) becomes

$$\frac{1-\varphi(-\infty)}{2} = \int_{-\infty}^{\infty} \{\varphi(\xi) - K(\xi)f[\varphi(\xi)]\}d\xi \ .$$

If $\varphi(-\infty) = 1$, then $\varphi(x) = K(x)f[\varphi(x)]$ for almost all x, and since $\varphi > 0$, this means that $\varphi \equiv 1$ and $K \equiv 1$ a.e. This yields the following relation.

THEOREM 3. *Let f and K be as in Theorem* 2 *and in addition assume $f(u) \not\equiv u$ and $K(x) \not\equiv 1$* a.e. *Then a solution φ of equation* (10) *satisfies*

$$\int_{-\infty}^{\infty} \{\varphi(\xi) - K(\xi)f[\varphi(\xi)]\}d\xi = 1/2 \ . \tag{17}$$

VI. Concluding remarks. The hypotheses in Theorem 2 were chosen to make, in some sense, a "clean" theorem, and as is usually the case more is actually proved than is stated. Thus in proving sufficiency, no use is made of the assumption $R(u) = O(1-u)^{1+\delta}$. Furthermore very weak use is made of the convexity of f and, in fact, the behavior of $f(u)$ in the neighborhood of $u = 1$ is all that is significant in the following sense.

THEOREM 4. *Let $\mathfrak{F}$ be the class of increasing, nonnegative, continuous functions defined on the unit interval such that if $f \in \mathfrak{F}$, then $f(1) = 1$. Suppose that for a certain $f_1 \in \mathfrak{F}$ equation* (10) *has a nonnegative solution φ_1 satisfying $\varphi_1 \leqq 1$ and $\varphi_1(+\infty) = 1$. Then if some other $f_2 \in \mathfrak{F}$ coincides with f_1 in some neighborhood of* 1, *equation* (10) *with $f = f_2$ has a nonnegative solution φ_2 satisfying $\varphi_2 \leqq 1$ and $\varphi_2(+\infty) = 1$.*

Proof. Suppose $f_1(u) = f_2(u)$ for $u_0 \leqq u \leqq 1$. There is an X such that for $x \geqq X$, $\varphi_1(x) \geqq u_0$. Set $\psi_0(x) = 0$ for $x < X$ and $\psi_0(x) = \varphi_1(x)$ for $x \geqq X$. Then for $-\infty < x < +\infty$

$$\psi_0(x) \leqq \int_x^{x+1} K(\xi)f_2[\psi_0(\xi)]d\xi \ . \tag{18}$$

Now for $n \geqq 0$ define

$$\psi_{n+1}(x) = \int_x^{x+1} K(\xi)f_2[\psi_n(\xi)]d\xi \ .$$

Since f_2 is increasing, $\psi_{n+1}(x) \geqq \psi_n(x)$ for all n and x and in addition $\psi_n(x) \leqq 1$. Thus $\psi_n(x) \uparrow_n \varphi_2(x)$, a solution with $f = f_2$.

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