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ASYMPTOTIC EXPANSIONS OF FOURIER TRANSFORMS AND DISCRETE POLYHARMONIC GREEN'S FUNCTIONS

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The results in M. J. Lighthill's book, Fourier Analysis and Generalized Functions, dealing with the asymptotic developments of Fourier transforms and Fourier series coefficients, are extended to the n-dimensional case. Together with several theorems due to L. Schwartz's work on distribution theory, integral representations and asymptotic developments of the n-dimensional discrete (generalized) polyharmonic Green's functions, are then obtained. A few examples of these Green's functions are illustrated and compared with known results.

With a considerable simplification of L. Schwartz's *Theory of Distributions* [6], Lighthill [4] has developed through the theory of generalized functions of a single variable, an asymptotic technique which leads quickly to estimating asymptotically Fourier transforms (F.T.). This technique was also applied without change, to the asymptotic determination of Fourier coefficients in trigonometrical series.

In the papers by Duffin and others [1-3], classical techniques were employed to estimate asymptotically two and three dimensional Fourier transforms. These techniques were then applied to determine the asymptotic behavior of discrete harmonic and biharmonic Green's functions. However, only the leading asymptotic terms of the n-dimensional discrete polyharmonic ($p > 3$) Green's functions, were obtained.

This paper is primarily concerned with the extension of Lighthill's one-dimensional asymptotic theory into n-dimensions. Using this extension, together with several results due to L. Schwartz, a method for obtaining all the terms of the asymptotic expansion of the n-dimensional discrete polyharmonic Green's functions, is derived. Known results [1-3] and more generalized ones concerning these Green's functions, are noted here.

Since the concern here is with functions of n independent variables, the following notations and conventions, unless otherwise specified, will be employed:

$$x \equiv (x_1, x_2, \cdots, x_n)\ ,$$
$$a \equiv (a_1, a_2, \cdots, a_n)\ ,$$
$$x/2k \equiv (x_1/2k_1, x_2/2k_2, \cdots, x_n/2k_n)\ ;$$

$$\begin{aligned} r &\equiv |x| \equiv (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}\,, \\ k &\equiv |a| \equiv (k_1^2 + k_2^2 + \cdots + k_n^2)^{1/2}\,; \\ a\cdot x &\equiv a_1x_1 + a_2x_2 + \cdots + a_nx_n\,; \\ \Delta^p &\equiv \frac{\partial^{p_1+p_2+\cdots+p_n}}{\partial x_1^{p_1}\partial x_2^{p_2}\cdots\partial x_n^{p_n}}\,; \\ g(a) &\equiv \text{F.T.}[f(x)]\,; \\ C_m &\equiv C_{m_1m_2\cdots m_n}\,, \end{aligned}$$

$$\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty} C_{m_1m_2\cdots m_n} \equiv \sum_{m_i} C_m\,.$$

Finally, f will designate a generalized function whereas f^* will represent an ordinary function.

2. The asymptotic estimation of Fourier transforms in n-dimensions. An n-dimensional asymptotic method involving F.T.'s is developed here. The method involves writing a given function $f(x)$ as $f(x) = F(x) + f_R(x)$, where $F(x)$ is a simpler function whose F.T. $G(a)$ is known, and $f_R(x)$ is a remainder such that F.T. $[\Delta^p f_R(x)] \to 0$ as $k \to \infty$. Then, $g(a) \equiv \text{F.T.}[f(x)]$ satisfies

$$g(a) = G(a) + g_R(a) = G(a) + o(k^{-p})$$

as $k \to \infty$. To develop such a method, a simple technique of identifying functions whose F.T.'s tend to zero as $k \to \infty$, is needed. The Rieman-Lebesgue lemma as we know, is the classical result which does this for ordinary integrable functions.

By means of the following two definitions, the Rieman-Lebesgue lemma may be extended to generalized functions.

DEFINITION 2.1. For a generalized function $f(x)$, any statement like $f(x) \to 0$, $f(x) = O[h(x)]$, or $f(x) = o[h(x)]$ as $x \to c$ (or $x \to \infty$) means that $f(x)$ is equal in some n-dimensional parallelepiped $x = c$ (or outside some n-dimensional parallelepiped containing $|x_i| > p_i$, $i = 1, 2, \cdots, n$) to an ordinary function $f^*(x)$ satisfying the stated condition.

DEFINITION 2.2. If $f(x) = f^*(x)$ in the n-dimensional parallelepiped P: $c_j < x_j < d_j$, $j = 1, 2, \cdots, n$, and $f^*(x)$ is absolutely integrable there, then we say that $f(x)$ is absolutely integrable in P.

By use of the above two definitions, the extension of the Rieman-Lebesgue lemma to generalized functions, follows immediately.

To obtain a criteria for estimating asymptotically F.T.'s of generalized functions, the following definition will be used.

DEFINITION 2.3. The generalized function $f(x)$ is said to be "well behaved at infinity", if for some number R, the function $f(x) - F(x)$ is absolutely integrable over the region $|x_1| > R, |x_2| > R, \cdots, |x_n| > R$, where $F(x)$ is absolutely integrable in every finite region and $G(a) \equiv$ F.T.$[F(x)] \to 0$ as $k \to \infty$.

THEOREM 2.1. *If a generalized function $f(x)$ is "well behaved at infinity" and absolutely integrable over every finite region of E^n, then its* F.T.$g(a) \to 0$ *as* $k \to \infty$.

Proof. Consider $F(x) - f(x)$, where $F(x)$ is the function defined in Definition 2.3. Since both $f(x)$ and $F(x)$ are absolutely integrable in every finite region of E^n, so is $f(x) - F(x)$. Furthermore, by Definition 2.3, $f(x) - F(x)$ is absolutely integrable over the region $|x_j| > R, j = 1, 2, \cdots, n$. Hence, $f(x) - F(x)$ is absolutely integrable in the entire E^n space. Thus, in view of Rieman-Lebesgue lemma for generalized functions, F.T.$[f(x) - F(x)] \equiv [g(a) - G(a)] \to 0$ as $k \to \infty$. But $G(a) \to 0$ as $k \to \infty$ by Definition 2.3. Therefore, $g(a) \equiv$ F.T.$[f(x)] \to 0$ as $k \to \infty$.

DEFINITION 2.4. A generalized function is said to have a finite number M of singularities at the points $Q_1, Q_2, \cdots, Q_M$, if in any region $G \subset E^n$ not containing any of these points, $f(x)$ is equal to an ordinary function with partial derivatives of all orders at every point of the region.

THEOREM 2.2. *Assume the following*:

(i) *$f(x)$ has M singularities at the points $Q_1, Q_2, \cdots, Q_M$ and $\Delta^p f(x)$ where $p \equiv p_1 + p_2 + \cdots + p_n$, is well behaved at infinity.*

(ii) *For $m = 1, 2, \cdots, M$, $\Delta^p[f(x) - F_m(x)]$ are absolutely integrable in a region containing Q_m but no other singularity. Also, $\Delta^p F_m(x)$ are absolutely integrable in every finite region not containing Q_m and are well behaved at infinity.*

(iii) *Let N be a positive even integer and let $p_1, p_2, \cdots, p_n$ above hold not only for a single n-tuple but for all such n-tuples with $p_1 + p_2 + \cdots + p_n = N$. Then,*

$$
(2.1) \qquad \begin{aligned} g(a) &\equiv \text{F.T.}[f(x)] = \sum_{m=1}^{M} G_m(a) + o(k^{-N}) \qquad \textit{as } k \to \infty\,, \\ G_m(a) &\equiv \text{F.T.}[F_m(x)]\,. \end{aligned}
$$

Proof. Defining $f_R(x) = f(x) - \sum_1^M F_m(x)$ whose F.T. is $g_R(a)$, then by [5] or [6],

$$\text{(2.2)} \qquad \text{F.T.}[\Lambda^p f_R(x)] = (2\pi i a_1)^{p_1}(2\pi i a_2)^{p_2} \cdots (2\pi i a_n)^{p_n} g_R(a) .$$

Next, $\Lambda^p f_R(x)$ is absolutely integrable in a finite region containing Q_m but no other singularity. This is so because $\Lambda^p[f(x) - F_m(x)]$ is absolutely integrable in the same region as are

$$\Lambda^p F_1(x), \Lambda^p F_2(x), \cdots, \Lambda^p F_{m-1}(x), \Lambda^p F_{m+1}(x), \cdots, \Lambda^p F_M(x) ,$$

by hypothesis. Since this is true for $m = 1, 2, \cdots, M$, it follows that $\Lambda^p f_R(x)$ is absolutely integrable in every finite region of E^n. Furthermore, since each $\Lambda^p F_m(x)$ and $\Lambda^p f(x)$ are well behaved at infinity by (ii), so is $\Lambda^p f_R(x)$. Thus, by Theorem 2.1,

$$\text{F.T.}[\Lambda^p f_R(x)] = (2\pi i a_1)^{p_1}(2\pi i a_2)^{p_2} \cdots (2\pi i a_n)^{p_n} g_R(a) \longrightarrow 0$$

as $k \longrightarrow \infty$, i.e.,

$$\text{(2.3)} \qquad \lim_{k\to\infty} (2\pi i a_1)^{p_1}(2\pi i a_2)^{p_2} \cdots (2\pi i a_n)^{p_n}\left[g(a) - \sum_{m=1}^{M} G_m(a)\right] = 0 .$$

Using now hypothesis (iii), then

$$\text{(2.4)} \qquad \begin{aligned} &\lim_{k\to\infty} \sum_{p_1+p_2+\cdots+p_n=N} \frac{N!}{p_1!\, p_2! \cdots p_n!} a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}\left\{ g(a) - \sum_{m=1}^{M} G_m(a)\right\} \\ &\quad = \lim_{k\to\infty}\left[k^N\left\{ g(a) - \sum_{m=1}^{M} G_m(a)\right\}\right] = 0 . \end{aligned}$$

3. Asymptotic expansion of n-dimensional Fourier coefficients. In dealing with Fourier coefficients of generalized periodic functions, integration according to [4, 5], must be carried out over the entire E^n space, rather than over the period parallelepiped as done ordinarily with ordinary functions. To overcome this, one can extend Lighthill's "unitary" function [4] into n-dimensions and then utilize it to show the equivalency of the two schemes. The extension goes as follows.

LEMMA 3.1. *If $V_1(x_1), V_2(x_2), \cdots, V_n(x_n)$ are one dimensional unitary functions in Lighthill's sense, then*

$$V(x) \equiv V_1(x_1) V_2(x_2) \cdots V_n(x_n)$$

is a good (testing) function [5, p. 3] *satisfying*

(i) $V(x) = 0$ *for* $|x_j| \geqq 1, j = 1, 2, \cdots, n,$

$$\text{(3.1)} \quad \text{(ii)} \quad \sum_{m_i} V(x + m) = \sum_{m_i} V_1(x_1 + m_1) V_2(x_2 + m_2) \cdots V_n(x_n + m_n) = 1$$

for all x,

(iii) $W(a) \equiv \text{F.T.}[V(x)]$ *is such that* $W(0) = 1$ *but* $W(m) = 0$ *if otherwise.*

Proof. The proof follows immediately from the definition of $V(x)$ and the proof of Lighthill [4, p. 61].

The idea of integrating a generalized periodic function $f(x)$ over its period $2k$ can now be replaced by the idea of integrating $f(x)V(x/2k)$ over the entire E^n space. This is so because each value of $f(x)$ which also equals $f(x+2k)$ is multiplied by just $\sum_{m_i} V(m + x/2k) = 1$.

Since the primary concern here is with asymptotic estimates of Fourier coefficients, the following three useful theorems which are well known in classical theory, will be stated without proofs (their proofs for generalized functions are found in [5, § 3]).

THEOREM 3.1. *The multiple trigonometrical series*

$$\sum_{m_i} C_m \exp[\pi i(m \cdot x/k)] \tag{3.2}$$

converges to a generlized function $f(x)$, *if and only if*, $C_m = O(|m|^N)$ *for some* N *as* $|m| \to \infty$, *in which case*

$$g(a) \equiv \text{F.T.}[f(x)] = \sum_{m_i} C_m \delta(a - m/2k) . \tag{3.3}$$

Here δ *is the n-dimensional dirac delta function.*

THEOREM 3.2. *If* $f(x) = \sum_{m_i} C_m \exp[\pi i(m \cdot x/k)]$, *then*

$$\begin{aligned} C_m = (1/2^n k_1 k_2 \cdots k_n) \int_{-\infty}^{\infty} \cdots \int f(x) V(x/2k) \\ \times \exp[-\pi i(m \cdot x/k)] dx_1 dx_2 \cdots dx_n . \end{aligned} \tag{3.4}$$

THEOREM 3.3. *If* $f(x)$ *is any periodic generalized function with periods* $2k_1, 2k_2, \cdots, 2k_n$ *in* $x_1, x_2, \cdots, x_n$, *respectively and if* C_m *is as stated in Eq.* (3.4), *then*

$$f(x) = \sum_{m_i} C_m \exp[\pi i(m \cdot x/k)] \tag{3.5}$$

and

$$\begin{aligned} g(a) \equiv \text{F.T.}[f(x)] &= \sum_{m_i} C_m \delta(a - m/2k) \\ &= \sum_{m_i} g(a) V(2k \cdot a - m) \end{aligned} \tag{3.6}$$

COROLLARY. *Under the hypotheses of Theorem* 3.3, $C_m = O(|m|^N)$ *for some* N *as* $|m| \to \infty$. *This follows from Theorems* 3.1 *and* 3.3.

We wish to apply now the asymptotic method of § 2 to the asymptotic estimation of Fourier coefficients. To do so, the following theorem is needed.

THEOREM 3.4. *If $f(x)$ is a generalized periodic function with periods $2k_1, 2k_2, \cdots, 2k_n$, in $x_1, x_2, \cdots, x_n$ respectively, then*

$$C(a) \equiv \text{F.T.}[(1/2^n k_1 k_2 \cdots k_n) f(x) V(x/2k)] \tag{3.7}$$

is a continuous function whose value for $a_1 = m_1/2k_1, \cdots, a_n = m_n/2k_n$, is the Fourier coefficient C_m of $f(x)$, i.e., $C(m/2k) = C_m$.

Proof. It is well known [6] that one may take the F.T.'s of an infinite series of generalized functions, term by term, i.e., $C(a)$ may be obtained by taking the F.T. of

$$(1/2^n k_1 k_2 \cdots k_n) \sum_{m_i} C_m \exp[\pi i(m \cdot x/k)] V(x/2k) ,$$

term by term. Thus, $C(a) = \sum_{m_i} C_m W(2k \cdot a - m)$, which is an absolutely and uniformly convergent series of continuous functions in any finite region. This follows from the corollary above and from the fact that $W(2k \cdot a - m)$ is a good (testing) function in view of Lemma 3.1. Hence, $C(a)$ is a continuous function. The second part of the theorem follows from property (iii) in Lemma 3.1.

If a periodic generalized function $f(x)$ has any singularities, it has an infinite number. However, if $f(x/2k)$ has a finite number of singularities, then Theorem 3.4 shows that the methods of § 2 may be applied to determine the asymptotic behavior of $C(a)$ and therefore of C_m.

DEFINITION 3.1. The periodic generalized function $f(x)$ with periods $2k_1, 2k_2, \cdots, 2k_n$ in $x_1, x_2, \cdots, x_n$, respectively, is said to have a finite number M of singular points $Q_1, \cdots, Q_M$, in the n-dimensional parallelepiped $-k_j < x_j \leqq k_j$, $j = 1, 2, \cdots. n$, if, for some $\varepsilon_j > 0$, $f(x)$ is equal to an ordinary function differentiable any number of times in the region $S - T$. Here S is the region $\{-k_j < x_j \leqq k_j(1 + \varepsilon_j)$, $j = 1, 2, \cdots, n,\}$ and $T = \{Q_1, Q_2, \cdots, Q_M\}$.

Using Definition 3.1, the following important theorem follows.

THEOREM 3.5. *Let $f(x)$ be a generalized periodic function with singularities at the points $Q_1, Q_2, \cdots, Q_M$, in the period parallelepiped: $-k_j < x_j \leqq k_j$, $j = 1, 2, \cdots, n$. Let also*

(i) *for each $m = 1, 2, \cdots, M$, $\Lambda^p[f(x) - F_m(x)]$ is absolutely integrable in a region containing Q_m but no other singularities,*

(ii) *for each* $m = 1, 2, \cdots, M$, $\Lambda^p F_m(x)$ *is absolutely integrable in every finite region not containing* Q_m *and is well behaved at infinity.*

(iii) *Let* N *be an even positive integer and* $p \equiv (p_1, p_2, \cdots, p_n)$ *above holds not only for a single n-tuple* p *but for all such n-tuples with* $p_1 + p_2 + \cdots + p_n = N$. *If* $G_t(a) = \text{F.T.}[F_t(x)]$, *then*

$$C_m = (1/2^n k_1 k_2 \cdots k_n) \sum_{t=1}^{M} G_t(m/2k) + o(|m|^{-N}) \tag{3.8}$$

$$\text{as } |m| \to \infty .$$

Proof. Take the "unitary" function $V(x)$ in Theorem 3.4 to be defined as

$$\begin{aligned} V(x) &= 1 \text{ for } -(1-\varepsilon_j)/2 \leqq x_j \leqq (1+\varepsilon_j/2)/2 \\ &= 0 \text{ for } x_j \leqq -(1-\varepsilon_j/2)/2 \text{ or } x_j \geqq (1+\varepsilon_j)/2 , \\ &\qquad j = 1, 2, \cdots, n . \end{aligned} \tag{3.9}$$

Here ε_j are those of Definition 3.1 assumed chosen so small that every singularity in the period parallelepiped is such that x_m^j ($=j$th coordinate of Q_m) $> -k_j(1-\varepsilon_j)$. One such "unitary" function is $V(x) \equiv V_1(x_1) V_2(x_2) \cdots V_n(x_n)$, where each $V_j(x_j)$ is taken in Lighthill's sense. Then, $(1/2^n k_1 k_2 \cdots k_n) f(x) V(x/2k)$ is a generalized function with only the singularities $Q_1, \cdots, Q_M$ and equals $(1/2^n k_1 k_2 \cdots k_n) f(x)$ in the n-dimensional parallelepiped:

$$-k_j(1-\varepsilon_j) < x_j < k_j(1+\varepsilon_j/2) , \qquad j = 1, 2, \cdots, n ,$$

including all of them, and all its partial derivatives are "well behaved at infinity". This is so since they all vanish outside the region $|x_j| > k_j(1+\varepsilon_j)$, $j = 1, 2, \cdots, n$. So, the F.T. by Theorem 2.2 satisfies

$$\lim_{k\to\infty} a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \Big[C(a) - (1/2^n k_1 k_2 \cdots k_n) \sum_{t=1}^{M} G_t(a) \Big] = 0 ,$$

or in view of Eq. (3.7),

$$\lim_{|m|\to\infty} a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \Big[C_m - (1/2^n k_1 k_2 \cdots k_n) \sum_{t=1}^{M} G_t(m/2k) \Big] = 0 .$$

The theorem therefore follows from hypothesis (iii).

4. The asymptotic development of discrete polyharmonic Green's functions. In applying the theory developed in § 3 to the asymptotic development of discrete polyharmonic Green's functions, the following is noted. According to Theorem 3.1, a discrete function $\mu(x)$ may be identified with the coefficients C_m of the Fourier series of a generalized periodic function, if and only if, $\mu(x) = O(r^N)$ for some N as $r \to \infty$. Conversely, by Theorem 3.3 and its corollary, the Fourier series representation of any generalized periodic function $f(x)$, defines a discrete function which is $O(r^N)$ for some N as $r \to \infty$.

The polyharmonic difference operator D^p is defined by

$$
\begin{aligned}
&DU(m_1, m_2, \cdots, m_n) \\
&\qquad = U(m_1 + 1, m_2, \cdots, m_n) + U(m_1 - 1, m_2, \cdots, m_n) + \cdots \\
&\qquad\quad + U(m_1, m_2 + \cdots, m_n + 1) + U(m_1, m_2, \cdots, m_n - 1) \\
&\qquad\quad - 2nU(m_1, m_2, \cdots, m_n) ; \\
&D^{p+1}U(m) = D^p[DU(m)] , \qquad p = 1, 2, \cdots .
\end{aligned} \tag{4.1}
$$

The following relation may easily be verified

$$
D^p\{\exp[-2\pi i(m\cdot x)]\} = \exp[-2\pi i(m\cdot x)]\Big(-4 \sum_{j=1}^{n} \sin^2 \pi x_j\Big)^p . \tag{4.2}
$$

THEOREM 4.1. *Let $g(x)$ be a periodic generalized function with period* 1, *i.e.*,

$$
g(x) = \sum_{m_i} \mu_m \exp[2\pi i(m\cdot x)] . \tag{4.3}
$$

If the discrete function μ_m is a fundamental solution for D^p (i.e., $D^p\mu_m = \delta_m = 1$ if *$m = 0$; 0 if $m \neq 0$), then*

$$
\Big(-4 \sum_{1}^{n} \sin^2 \pi x_j\Big)^p g(x) = 1 . \tag{4.4}
$$

Proof. By hypothesis, Theorem 3.2, and relation (4.2),

$$
\begin{aligned}
\delta_m &= D^p\mu_m = D^p\{\text{F.T.}[g(x)V(x)]\} \\
&= D^p\Big\{\int_{-\infty}^{\infty} \cdots \int g(x)V(x) \exp[-2\pi i(m\cdot x)]dx_1 \cdots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int g(x)V(x)\Big(-4 \sum_{1}^{n} \sin^2 \pi x_j\Big)^p \\
&\qquad\qquad \times \exp[-2\pi i(m\cdot x)]dx_1dx_2 \cdots dx_n .
\end{aligned} \tag{4.5}
$$

Next, $(-4 \sum_1^n \sin^2 \pi x_j)^p g(x)$ is a periodic generalized function, since $g(x)$ is. Hence, the result follows by Theorem 3.3. Indeed,

$$
\Big(-4 \sum_{1}^{n} \sin^2 \pi x_j\Big)^p g(x) = \sum_{mi} \delta_m \exp[2\pi i(m\cdot x)] = 1 .
$$

By means of a "Laurent-type" expansion, one may define $(\sum_1^n \sin^2 \pi x_j)^{-p}$ as follows:

DEFINITION 4.1. In the region

$$B: |x_j| < 1,\ j = 1, 2, \cdots, n,\ \left(\sum_1^n \sin^2 \pi x_j\right)^{-p}$$

has the following Laurent-type expansion

$$\begin{aligned}\left(\sum_1^n \sin^2 \pi x_j\right)^{-p} = (\pi^2 r^2)^{-p}\Big\{1 &- (\pi/r)^2\Big[pA_2 \sum_1^n x_j^4\Big] \\ &+ (\pi/r)^4\Big[\frac{p(p+1)}{2} A_2^2\Big(\sum_1^n x_j^4\Big)^2 - pA_3 r^2 \sum_1^n x_j^6\Big] \\ &- (\pi/r)^6\Big[\frac{(7p^3 - 3p^2 + 2p)}{6} A_2^3\Big(\sum_1^n x_j^4\Big)^3 \\ &+ p(p+1) A_2 A_3 r^2 \Big(\sum_1^n x_j^4\Big)\Big(\sum_1^n x_j^6\Big) \\ &+ pA_4 r^4 \sum_1^n x_j^8\Big] + \cdots\Big\},\end{aligned} \tag{4.6}$$

where

$$A_m = (-1)^{m+1} \sum_{j=1}^m [(2j-1)!\,(2m-2j+1)!]^{-1}, \tag{4.7}$$

$$m = 1, 2, \text{ etc.}$$

Proof of consistency. The first term of the expansion, $(\pi^2 r^2)^{-p}$, is a generalized function defined by L. Schwartz [6] as the solution of

$$(\pi r)^{2p} f(x) = 1 \tag{4.8}$$

such that F.T. $[f(x)]$ is given by Eqs. (4.13) and (4.14). The series in the curly parenthesis of Eq. (4.6) converges uniformly everywhere in the region B. Hence, expansion (4.6) satisfactorily defines the generalized function $(\sum_1^n \sin^2 \pi x_j)^{-p}$ in the region B.

We can therefore say now that the generalized function $(-4 \sum_1^n \sin^2 \pi x_j)^{-p}$ is a solution of $(-4 \sum_1^n \sin^2 \pi x_j)^p g(x) = 1$ in the interior to the region B.

The discussion above together with Theorems 3.2 and 4.1, lead to the following important representation:

THEOREM 4.2. *A fundamental solution μ_m for the polyharmonic difference operator D^p is given by*

$$\mu_m = \text{F.T.}\left[\left(-4 \sum_1^n \sin^2 \pi x_j\right)^{-p} V(x)\right]. \tag{4.9}$$

Note. This particular solution will designate the so-called "normalized" fundamental solution or the discrete Green's function $g_p(a)$ corresponding to the n-dimensional polyharmonic difference operator D^p.

In relation to expansion (4.6), let us note first that the only singularities of $(\sum_1^n \sin^2 \pi x_j)^{-p}$ or of its partial derivatives in the region $|x_j| < 1, j = 1, 2, \cdots, n$, are at the origin. Next, only a finite number of them are singular at the origin. This is so because the limit as $r \to \infty$ of the successive terms of the series in the curly bracket of (4.6), are increasingly higher orders at zero. Thus, in view of the discussion above and Theorem 3.5, the following result is valid.

THEOREM 4.3. *Let*

$$h(a) = \text{F.T.}\Big[(-4)^{-p}\Big(\sum_1^l h_j(x)\Big)\Big], \tag{4.10}$$

where $h_j(x)$ are all the terms of the series (4.6) *which are singular at the origin. If $g_p(a) \equiv \mu_m$ is the discrete Green's function for D^p defined in Eq.* (4.9), *then*

$$g_p(a) = h(a) + o(1) \qquad \text{as } k \to \infty . \tag{4.11}$$

Proof. This theorem is just a direct application of Theorem 3.5 for $N = 0, M = 1$ and $F_1(x) = (-4)^{-p} \sum_1^l h_j(x)$.

Next, since the successive nonsingular terms of the series (4.6) (after a finite number l say) have zeros of higher order at the origin, Theorem 3.5 may be used again to obtain the asymptotic expansion for $g_p(a)$ to any desired order by taking more terms in the series (4.6).

THEOREM 4.4. *If $H(x) = \sum_1^{m>l} h_j(x)$ extended to include not only the singular terms of the expansion* (4.6) *but also terms with zero at the origin of order s or less, the for an appropriate s,*

$$g_p(a) = \text{F.T.}[H(x)] + o(k^{-N}) \qquad \text{as } k \to \infty . \tag{4.12}$$

Proof. For an appropriate s, the conditions of Theorem 3.5 can be satisfied for any positive even integer N.

Theorem 4.4 reduces the problem of finding the asymptotic estimates for $g_p(a)$ to the problem of finding the F.T.'s of the functions appearing in Eq. (4.6). These functions are of the type of a polynomial divided by r to an even integral power.

L. Schwartz [6, Vol. II pp. 113–114] has computed the following expressions:

$$\text{F.T.}[r^{-m}] = \pi^{m-n/2}\Gamma\Big(\frac{n-m}{2}\Big)k^{m-n}\Big/\Gamma(m/2) , \tag{4.13}$$

when n is odd, or when n is even but $m < n$; $p = 1, 2$, etc.

$$\begin{aligned} &\text{F.T.}[\text{P.V.}(r^{-n-2h})] = (-1)^h 2\pi^{2h+n/2}[h!\,\Gamma(h+n/2)]^{-1}k^{2h} \\ &\times \left[\log\frac{1}{\pi k} + \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{h} - \gamma\right) + \frac{\Gamma'(h+n/2)}{2\Gamma(h+n/2)}\right], \end{aligned} \tag{4.14}$$

where n is even but $m - n = 2h \geqq 0$, $h = 0, 1, 2, \cdots$, and where $1 + (1/2) + (1/3) + \cdots + (1/h)$ is replaced by 0 if $h = 0$. γ is Euler's constant. The F.T.'s taken in Eq. (4.14) are in the principal value (P.V.) sense because of the logarithmic singularity involved at the origin. One may take explicit expressions for F.T.$[P(x)r^{-2m}]$, where $P(x)$ is a polynomial, as follows: if $P(x) = \sum_{j=0}^{t} C_m x_1^{q_{1j}} \cdots x_n^{q_{nj}}$, then

$$\begin{aligned} &\text{F.T.}[P(x)r^{-2m}] = \sum_{j=0}^{t} C_j(-2\pi i)^{-q}\Delta^q\{\text{F.T.}[r^{-2m}]\}, \\ &q_{1j} + q_{2j} + \cdots + q_{nj} = q. \end{aligned} \tag{4.15}$$

Having established relation (4.15), the asymptotic expansion of $g_p(a)$ may now be evaluated explicitly to within $o(k^{-N})$ as $k \to \infty$ for any even positive integer N, by computing the F.T.'s of the series (4.6), by means of Eqs. (4.13)-(4.15), term by term. For simplicity and in order to compare the results here with those which are known [1-3], only the first two terms of the asymptotic expansion will be calculated below.

From Eqs. (4.12) and (4.6) with $A = -1/3$ and from Eqs. (4.13), (4.15), the following holds.

THEOREM 4.5. *If the dimension n of the space is odd, then*

$$\begin{aligned} g_p(a_1, a_2, \cdots, a_n) \sim\ & B_{p,n}k^{2p-n} - \frac{p}{12}B_{p+1,n}k^{m-4}m(m-2) \\ &\times\left[\frac{(m-4)(m-6)}{k^4}\sum_1^n a_j^4 + (6m - 24 + 3n)\right], \end{aligned} \tag{4.16}$$

where

$$B_{p,n} = \frac{(-1)^p\Gamma(-p+n/2)}{2^{2p}\pi^{n/2}(p-1)!}$$

and $m = 2p + 2 - n$.

Note. Relation (4.16) is also valid when n is even but $2p < n$ for the leading asymptotic term and $2p + 2 < n$ for the second asymptotic term, etc.

EXAMPLES.

$$g_1(a_1, a_2, a_3) \sim -\frac{1}{4\pi k} - \frac{1}{32\pi k^3}\left[-3 + \frac{5}{k^4}(a_1^4 + a_2^4 + a_3^4)\right],$$

$$g_2(a_1, a_2, a_3) \sim -\frac{k}{8\pi} + \frac{1}{64\pi k}\left[1 + \frac{a_1^4 + a_2^4 + a_3^4}{k^4}\right],$$

$$g_1(a_1, a_2, a_3, a_4, a_5) \sim -\frac{1}{8\pi^2 k^3} - \frac{1}{64\pi^2 k^5}\left[-15 + 35\frac{(a_1^4 + a_2^4 + \cdots + a_5^4)}{k^4}\right],$$

$$g_2(a_1, a_2, a_3, a_4, a_5) \sim \frac{1}{16\pi^2 k} + \frac{1}{128\pi^2 k^3}\left[-1 + \frac{5(a_1^4 + a_2^4 + \cdots + a_5^4)}{k^4}\right],$$

$$g_1(a_1, a_2, a_3, a_4) \sim -\frac{1}{4\pi^2 k^2} - \frac{1}{4\pi^4 k^4}\left[1 + \frac{2(a_1^4 + \cdots + a_4^4)}{k^4}\right].$$

The first example above agrees with the result obtained by Duffin [1] except for sign. The difference in sign is due to the difference in the definition of fundamental solutions being defined here by the equation $D^p \mu_m = \delta_m$ and being defined in Duffin's paper by $D^p \mu_m = (-1)^n \delta_m$. The remaining examples were not obtained previously.

Now, from Eqs. (4.12) and (4.6) with $A_2 = -1/3$ and from Eqs. (4.14) and (4.15), the following asymptotic expansion holds.

THEOREM 4.6. *If the dimension n of the space is even but $2(p+j) - n = 2h \geqq 0$, $h = 0, 1, 2, \cdots$ ($j = 0$ corresponds to the first asymptotic term and $j = 1$ corresponds to the second term), then*

$$\begin{aligned}
&g_p(a_1, a_2, \cdots, a_n)\\
&= (-4)^{-p}\mathrm{F.T.}\left[\mathrm{P.V.}\left\{(\pi^2 r^2)^{-p}\left\{1 + p\pi^2 \sum_1^n x_j^4 \Big/ 3r^2\right\}\right\}\right]\\
&\sim \frac{(-1)^{n/2} k^{2p-n}}{2^{2p-1}\pi^{n/2}(p-1)!\,(p-n/2)!}\\
&\quad \times \left[\log\frac{1}{\pi k} + \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{p-n/2} - \gamma\right) + \frac{\Gamma'(p)}{2\Gamma(p)}\right]\\
&\quad + \frac{(-1)^{n/2}}{3\cdot 2^{2p+3}\pi^{n/2}(p-1)!\,(p+1-n/2)!}\\
&\quad \times \left[k^{m-4}\left\{6n(m-1) + 6(3m^2 - 12m + 8)\right.\right.\\
&\qquad\qquad \left. + 4(m^3 - 9m^2 + 22m - 12)\frac{\sum_1^n a_j^4}{k^4}\right\}\\
&\qquad + k^{m-4}\log k\left\{3nm(m-2) + 6m(m-2)(m-4)\right.\\
&\qquad\qquad \left. + m(m-2)(m-4)(m-6)\frac{\sum_1^n a_j^4}{k^4}\right\}
\end{aligned} \tag{4.17}$$

$$+ k^{m-4}m(m-2)\left\{(m-4)(m-6)\frac{\sum_1^n a_j^4}{k^4} + (6m-24+3n)\right\}$$

$$\times \left\{\log \pi - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{p+1-n/2} - \gamma\right)\right.$$

$$\left.\left. - \frac{\Gamma'(p+1)}{2\Gamma(p+1)}\right\}\right],$$

where γ is Euler's constant,

$$m = 2p + 2 - n\,, \qquad 1 + \frac{1}{2} + \cdots + \frac{1}{p - \frac{n}{2}}$$

in the first term is replaced by zero if $m = 0$.

EXAMPLES. By use of Theorems 4.4 and 4.5, we obtain:

$$g_1(a_1, a_2) \sim \frac{1}{2\pi}[\log(\pi k) + \gamma] - \frac{1}{24\pi k^2}\left[-3 + \frac{4(a_1^4 + a_2^4)}{k^4}\right]$$

$$g_2(a_1, a_2) \sim \frac{k^2}{8\pi}[\log(\pi k) + \gamma - 1]$$

$$+ \frac{1}{192\pi}\left[-3 - 12(\log \pi k + \gamma) + \frac{4(a_1^4 + a_2^4)}{k^4}\right]$$

$$g_1(a_1, a_2, a_3, a_4) \sim \frac{-1}{4\pi^2 k^2} - \frac{1}{4\pi^2 k^4}\left[1 + \frac{2(a_1^4 + a_2^4 + a_3^4 + a_4^4)}{k^4}\right]$$

$$g_2(a_1, a_2, a_3, a_4) \sim -\frac{1}{8\pi^2}[\log \pi k + \gamma - 1/2] + \frac{a_1^4 + a_2^4 + a_3^4 + a_4^4}{24\pi^2 k^6}\,.$$

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