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A TOPOLOGICAL CHARACTERIZATION OF GLEASON PARTS

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Let A be a function algebra on its maximal ideal space $M(A)$, and let P be a Gleason part of $M(A)$. It is easily seen that P is then a σ-compact completely regular space. We prove the converse: if K is completely regular and σ-compact, then there exists a function algebra whose maximal ideal space contains a part homeomorphic to K. Every bounded continuous function on that part is the restriction of a function in the given algebra. Consequently no subset of the part can have an analytic structure.

Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X)$, the algebra of continuous complex valued functions on X. Assume A separates the points of X, contains the constant functions, and is uniformly closed. A is then called a function algebra on X. With the weak star topology, the maximal ideal space $M(A)$ of A is a compact Hausdorff space. We consider X as embedded in $M(A)$ and A as a function algebra on $M(A)$.

In [4] Gleason noted that an equivalence relation could be defined on $M(A)$ by setting $x \sim y$ when the functional norm $\| x - y \|_{A^*} < 2$. The equivalence classes for this relation are called the "parts" of $M(A)$. In certain cases parts have been used to impose an analytic structure on $M(A)$ (see for example [7]).

Let P be a part of some $M(A)$. Then clearly P is a completely regular space and fixing $p \in P$ we have

$$P = \bigcup_{n=1}^{\infty} \{q \in M(A) : \| p - q \| \leqq 2 - 1/n\} ,$$

where each term in the union is weak star closed, and hence compact, so that P is σ-compact.

Some results in this paper have been announced in [3].

2. We begin with a theorem which will be our basic tool in constructing parts.

THEOREM 1. *Let A be a function algebra, S a hull-kernel closed subset of $M(A)$ and P a part of $M(A)$. Then there is a function algebra B such that $M(B)$ contains a part Q homeomorphic to $P \cap S$. Moreover, $B \mid Q$ is isometrically isomorphic to $A \mid P \cap S$.*

Let α be a positive irrational number, and denote by A_α the function algebra on the torus T^2 generated by the functions $z \rightarrow z_1^m z_2^n$ where $m + n\alpha \geqq 0$. Let m^0 be the point in $M(A_\alpha)$ represented by Haar measure on T^2 (which is multiplicative on A_α). Then $m^0 \notin T^2$ and $\{m^0\}$ is a part of $M(A_\alpha)$ ([5] p. 316). If J is a proper closed subset of T^2, then $A_\alpha \mid J$ is dense in $C(J)$ ([8] pp. 69–70), so that when $x \in M(A_\alpha)\backslash J$ there is a function $f \in A_\alpha$ such that $|f(x)| > \max_{z \in J} |f(z)|$, as otherwise evaluation at x would induce a complex homomorphism of $C(J)$.

Proof of Theorem 1. Let $A_\alpha \otimes A$ be the function algebra on $M(A_\alpha) \times M(A)$ generated by the functions of the form $(x, y) \rightarrow f(x)g(y)$ where $f \in A_\alpha$ and $g \in A$. $M(A_\alpha \otimes A)$ is homeomorphic to $M(A_\alpha) \times M(A)$ in a natural fashion.

Set $J = \{z \in T^2 : \text{Real } z_1 \leqq 0\}$ and

$$X = (J \times M(A)) \cup (M(A_\alpha) \times S) .$$

X is a compact subset of $M(A_\alpha) \times M(A)$. Our algebra B is the uniform closure on X of $\{h \mid X : h \in A_\alpha \otimes A\}$. $M(B)$ is then the $A_\alpha \otimes A$-hull of X,

$$\{q \in M(A_\alpha \otimes A) : |g(q)| \leqq \max_{p \in X} |g(p)| \text{ for all } g \in A_\alpha \otimes A\} .$$

If $(x^0, y^0) \in M(A_\alpha \otimes A)\backslash X$, then $x^0 \notin J$ and $y^0 \notin S$. As S is hull-kernel closed, there is a function g in A with $g(y^0) = 1$ and $g(S) = 0$. As $x^0 \notin J$, there is a function f in A_α with $f(x^0) = 1$ and $\max_{z \in J} |f(z)| < 1$. Replace f by a suitable power f^n so that $\max_{z \in J} |f^n(z)| < (1/\|g\|)$. Then $h(x, y) = f^n(x)g(y)$ is in $A_\alpha \otimes A$ and $h(x^0, y^0) = 1$ while $\max_{p \in X} |h(p)| < 1$. Hence $M(B) = X$.

Take $Q = \{m^0\} \times (P \cap S)$. Then Q is subset of X. For $s \in P \cap S$, let $p_s = (m^0, s) \in Q$. Let $(x^0, y^0) \in X\backslash Q$. If $x^0 \neq m^0$, then using functions of the variable $x \in M(A_\alpha)$ alone we see that $(x^0, y^0) \not\sim p_s$ for any $s \in P \cap S$. Similarly if $y^0 \notin P$, then $(x^0, y^0) \not\sim p_s$ for all such s. Finally if $y^0 \notin S$, then by the choice of X, $x^0 \neq m^0$. Hence Q is a union of parts.

If $s \in P \cap S$ and $g \in B$, then

$$g(p_s) = \int_{T^2} g(x, s) d\lambda$$

where λ is normalized Haar measure on T^2, because λ represents m^0 for A_α. Take s and t in $P \cap S$, $g \in B$ with $\|g\| \leqq 1$. Then

$$\begin{aligned} |g(p_s) - g(p_t)| &\leqq \int_{T^2} |g(x, s) - g(x, t)| d\lambda \\ &\leqq \int_{T^2\backslash J} 2d\lambda + \int_J |g(x, s) - g(x, t)| d\lambda . \end{aligned}$$

Now if $x \in J$, then $\{x\} \times M(A) \subset X$ so that $y \to g(x, y)$ is in A with norm $\leqq 1$. Therefore there is a constant $c < 2$ such that for each $x \in J$ $|g(x, s) - g(x, t)| < c$, because $s \sim t$. Hence

$$|g(p_s) - g(p_t)| \leqq \int_{T^2 \setminus J} 2d\lambda + \int_J cd\lambda < 2$$

and $p_s \sim p_t$. Thus Q is a part.

It is obvious that Q is homeomorphic to $P \cap S$ and that $B|Q = A|P \cap S$, because the coordinate x is constant on Q.

As a corollary to Theorem 1 we now prove a special case of our main result because in this case the proof is much simpler.

D denotes the closed unit disc in the complex plane and D^0 its interior. A_0 is the algebra of all functions continuous on D and analytic on D^0. If K is a locally compact Hausdorff space, then $K^* = K \cup \{\infty\}$ is its one point compactification.

COROLLARY. *Let K be a locally compact σ-compact Hausdorff space. Then there exists a function algebra B such that $M(B)$ contains a part Q homeomorphic to K. Moreover $B|Q$ is isometrically isomorphic to $C(K^*)|K$.*

Proof. Let $A = \{f \in C(K^* \times D) : f|\{x\} \times D \in A_0$ for each $x \in K^*$ and $f|K^* \times \{0\}$ is constant$\}$. Then $M(A) = K^* \times D/\approx$ where $\approx$ identifies $K^* \times \{0\}$ to a point, and $P = \{(x, z) \in M(A) : |z| < 1\}$ is a part in $M(A)$, as P is a union of discs with the centers identified.

Since K is σ-compact, $\{\infty\}$ is a G_δ set in K^*. Hence there is a continuous function $h: K^* \to [1/2, 1]$ such that $h^{-1}(1) = \{\infty\}$. Let $S \subset M(A)$ be the graph of h, $S = \{(x, h(x)) : x \in K^*\}$. Then the function $g(x, z) = (h(x) - z/3h(x) - z)$ is in A and vanishes exactly on S, so that S is hull-kernal closed. And clearly $S \cap P$ is homeomorphic to K. Finally if $f \in C(K^*)$, then $f'(x, z) = zf(x)/h(x)$ is in A and $f'(x, h(x)) = f(x)$ when $x \in K$. Thus $A|S \cap P \cong C(K^*)|K$. The conclusion of the corollary now follows directly from Theorem 1.

3. Before proving our main theorem we construct the algebra to be used in place of the disc algebra A_0. Let I be an index set, and let Y_I be the product of discs, $Y_I = \prod(D : i \in I)$. Denote by A_I the subalgebra of $C(Y_I)$ generated by the coordinate functions $z_i, i \in I$ where $z_i(p) = p_i$. Then $M(A_I) = Y_I$, for if $\varphi \in M(A_I)$, then $|\varphi(z_i)| \leqq i$ so that φ is evaluation at $\lambda \in Y_I$ where $\lambda_i = \varphi(z_i)$. Let θ be the "origin" in Y_I, $z_i(0) = 0$ for all $i \in I$, and let P_0 be the part of $M(A_I)$ containing θ. We now need a well known fact which is proved using elementary conformal mappings of the disc D. If $(g_n)_{n=1}^\infty$ is a sequence in A with $||g_n|| \leqq 1$ and $g_n(x) \to 1$, then $x \sim y$ implies $g_n(y) \to 1$.

LEMMA. *Let $p \in M(A_I)$. Then $p \in P_0$ if and only if there exists $a < 1$ such that $|z_i(p)| \leqq a$ for all $i \in I$.*

Proof. If no such number exists, then there is a sequence $(g_n)_{n=1}^{\infty}$ of coordinate functions z_i such that $g_n(p) \to 1$ while $g_n(\theta) = 0$. Hence $p \not\sim \theta$, by the above remark. If such an a exists, then let $\rho: D \to Y_I$ by $(\rho(t))_i = t/a \cdot p_i$. Then $\rho(0) = \theta$, $\rho(a) = p$ and for $f \in A_I, f \circ \rho \in A_0$ with $\|f \circ \rho\| \leqq \|f\|$. Then as $0 \sim a$ for A_0 we have $0 \sim p$.

THEOREM 2. *Let K be a σ-compact completely regular space. Then there is an algebra B and a part $Q \subset M(B)$ such that Q is homeomorphic to K and $B | Q \cong C^b(K)$, the algebra of bounded continuous functions on K.*

Proof. Let βK be the Stone-Čech compactification of K. Take[1] $I = \beta K \backslash K$ and set $A = \{f \in C(\beta K \times Y_I) : f | \{x\} \times Y_I \in A_I$ for all $x \in \beta K$ and $f | \beta K \times \{\theta\}$ is constant$\}$. Then $M(A) = \beta K \times Y_I / \approx$ where $\approx$ identifies $\beta K \times \{\theta\}$ to a point, and $P = \{(x, z) \in M(A) : z \in P_0\}$ is a part of $M(A)$.

Write $K = \bigcup_{n=1}^{\infty} K_n$, where $K_n \subset K_{n+1}$ and each K_n is compact. Then for each $t \in \beta K \backslash K$ there exists a continuous function $h_t: \beta K \to [1/2, 1]$ with $h_t(t) = 1$ and $h_t(x) \leqq 1 - 2^{-n}$ when $x \in K_n$. Let $\rho: \beta K \to M(A)$ be defined by $\rho(x) = (x, H(x))$ where $(H(x))_t = h_t(x)$ for each $t \in \beta K \backslash K$. Then ρ is a homeomorphism of βK onto $S = \rho(\beta K)$ and $\rho(K) = S \cap P$ by the above lemma. S is hull kernel closed in $M(A)$ because $S = \bigcap \{g_t^{-1}(0) : t \in BK \backslash K\}$ where $g_t(x, z) = (h_t(x) - z_t / 3h_t(x) - z_t)$. And $A | S \cap P \cong C^b(K)$, because if $f \in C^b(K)$ and $\tilde{f}$ is its unique extension to βK, then for any $t \in \beta K \backslash K, f'(x, z) = z_t \tilde{f}(x)/h_t(x)$ is in A and $f' = \tilde{f} \circ \rho^{-1}$ on S. The conclusion now follows from Theorem 1.

We remark that with these arguments one can get some restriction algebras $B | Q$ different from $C^b(K)$. For example, if K is compact and A_1 is an algebra with $M(A_1) = K$, then there is an algebra B with part Q homeomorphic to K and $B | Q = A_1$.

4. **Acknowledgments.** Professor Eva Kallin Pohlmann[2] simplified the example given in [3], enabling us to improve our arguments significantly. We thank Professor Lewis Robertson for a helpful conversation. Finally, we are deeply indebted to Professor Irving Glicksberg for introducing us to function algebras and for guiding our research throughout.

[1] If K is compact, let I be a singleton, and proceed as in the corollary.

[2] She observed that $X = \{(z,w) : |z| \leqq 1, w = \pm 1/2\} \cup \{(z,w) : |z| = 1, \operatorname{Im} z \geq 0, |w| \leqq 1\}$ is a polynomially convex subset of $\mathbf{C}^2$ containing a part consisting of two discs.

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