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RESIDUATED MAPPINGS

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In a series of papers, D. J. Foulis developed a theory in the course of which he obtained analogues of Von Neumann's Coordinatization Theorem by making use of *-monotone mappings. A generalization of these mappings, residuated mappings, leads to extensions of his results. Residuated mappings also arise independently in studies of R. Croisot and G. Nöbeling. The purpose of this paper is to develop their properties systematically. Of particular help is the link established with the basic properties of M-homomorphisms between groups with operators yielding analogues of the Fundamental Theorem of Homomorphisms and Fitting's Lemma, and with the study of residuation especially in Noetherian rings.

Preliminaries. Unless further restricted, P, Q, R denote arbitrary posets whose order relations are all written $\leqq$.

DEFINITION 1. An isotone mapping $\varphi: P \rightarrow Q$ is said to be *residuated* (resp. *residual*) if there is an isotone mapping $\psi: Q \rightarrow P$ such that:

(i) $x\varphi\psi \geqq x$ (resp. $x\varphi\psi \leqq x$) for all x in P

(ii) $x\psi\varphi \leqq x$ (resp. $x\psi\varphi \geqq x$) for all x in Q.

An antitone mapping $\varphi: P \rightarrow Q$ is said to be a *Galois connection* if there is an antitone mapping $\psi: Q \rightarrow P$ satisfying

(i) $x\varphi\psi \geqq x$ for all x in P

(ii) $x\psi\varphi \geqq x$ for all x in Q.

Since one may pass from one type of mapping to either of the other two by dualizing either one or both of the posets involved, we shall record only the results for the residuated case, though using them in whatever form required later.

We list some facts that will be used in the sequel and can be found in, say, [1] and [2] possibly after applying the aforementioned duality:

A. A necessary and sufficient condition that an isotone mapping $\varphi: P \rightarrow Q$ be residuated is that $\mathrm{Max}\{z \in P : z\varphi \leqq x\}$ exists for all x in Q; moreover, if this is the case, ψ is given by the rule $x\psi = \mathrm{Max}\{z \in P : z\varphi \leqq x\}$. Thus ψ is uniquely determined by φ and will be denoted φ^+.

B. If $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow R$ are residuated so is $\varphi\psi: P \rightarrow R$; moreover, $(\varphi\psi)^+ = \psi^+\varphi^+$. We denote by $S(P, Q)$ the set of all residuated mappings $\varphi: P \rightarrow Q$ and write $S(P)$ when $P = Q$. Note that $S(P)$ is a semigroup.

C. $\varphi\varphi^+\varphi = \varphi$ for all φ in $S(P, Q)$.

D. For any φ in $S(P)$ $\varphi|_{P\varphi\varphi^+}: P\varphi\varphi^+ \to P\varphi\varphi^+$ is an order isomorphism.

E. For φ in $S(P, Q)$ if $\bigvee_{\lambda\in\Lambda} x_\lambda$ exists in P $\bigvee_{\lambda\in\Lambda}(x_\lambda\varphi)$ exists in Q and equals $(\bigvee_{\lambda\in\Lambda} x_\lambda)\varphi$. If P has a zero 0_P so does Q and $0_Q = 0_P\varphi$. In a complete lattice φ is residuated if and only if it is a complete joinhomomorphism.

We now sketch two examples to provide a perspective for the theory:

F. Let R be a binary relation on a set X, $\varphi_R: \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $A\varphi_R = AR$ for $A \subseteqq X$ is residuated and its residual is given by $A\varphi_R^+ = ((A')R^{-1})'$, where $'$ denotes set-theoretic complementation.

G. Let G_1, G_2 be groups with operators M, let $L(G_i)$ be the lattice of M-subgroups of G_i and $f: G_1 \to G_2$ an M-homomorphism. Then $\varphi: L(G_1) \to L(G_2)$ given by the rule $H\varphi = \{hf : h \in H\}$ for $H \in L(G_1)$ is residuated and its residual is given by the rule $H\varphi^+ = \{g \in G_1 : gf \in H\}$. The exact relations are of interest:

(i) $H\varphi\varphi^+ = (H)(\mathrm{ker} f) = H \vee \{1\}\varphi^+$,

(ii) $H\varphi^+\varphi = H \cap G\varphi$.

All the above is true for the lattice of normal M-subgroups and for normal M-endomorphisms.

2. Range-closed mappings. Henceforth P, Q, R are supposed to contain a least and a largest element 0 and 1 respectively. We know by definition that $x\varphi^+\varphi \leqq x$; also $x\varphi^+\varphi \leqq 1\varphi$. When is $x\varphi^+\varphi$ as large as it can be: $x\varphi^+\varphi = x \wedge 1\varphi$?

PROPOSITION 1. For any $\varphi \in S(P, Q)$ the following conditions are equivalent:

(i) $\varphi: P \to Q(0, 1\varphi)$ is onto

(ii) $x \wedge 1\varphi$ exists for all x in Q and equals $x\varphi^+\varphi$.

(iii) $\varphi^+|_{Q(0,1)}: Q(0, 1\varphi) \to P$ is one-to-one.

Proof. (i) $\Rightarrow$ (ii): Suppose $z \leqq x$ and $z \leqq 1\varphi$, then, by (i), there exists w in P such that $z \leqq w\varphi$. Since $w\varphi \leqq x$, $w \leqq w\varphi\varphi^+ \leqq x\varphi^+$, and $z = w\varphi \leqq x\varphi^+\varphi$; therefore, $x\varphi^+\varphi = x \wedge 1\varphi$.

(ii) $\Rightarrow$ (iii): Suppose $x, y \leqq 1\varphi$ and $x\varphi^+ = y\varphi^+$, then $x = x \wedge 1\varphi = x\varphi^+\varphi = y\varphi^+\varphi = y \wedge 1\varphi = y$.

(iii) $\Rightarrow$ (i): Suppose $x \leqq 1\varphi$, then $x\varphi^+\varphi \leqq 1\varphi\varphi^+\varphi \leqq 1\varphi$. By the dual of C $x\varphi^+\varphi\varphi^+ = x\varphi^+$ for arbitrary x; thus, by (iii), $x\varphi^+\varphi = x$. Therefore φ is onto $Q(0, 1\varphi)$.

DEFINITION 2. If $\varphi \in S(P, Q)$ satisfies any of the conditions of the proposition it is said to be *range-closed*. The set of all such maps is denoted $S_{RC}(P, Q)$. If φ^+ satisfies the conditions of the dual proposition, then φ itself is said to be *dually range-closed;* the set of all such maps is denoted $S_{DRC}(P, Q)$. This terminology is suggested by [4].

PROPOSITION 2. Let P be a lattice, the following mappings on P are residuated:

$$x\omega_e = \begin{cases} x \text{ if } x \leqq e \\ e \text{ if } x \nleqq e \end{cases}, \qquad x\alpha_e = \begin{cases} 0 \text{ if } x \leqq e \\ x \vee e \text{ if } x \nleqq e \end{cases},$$

their residuals are

$$x\omega_e^+ = \begin{cases} 1 \text{ if } x \geqq e \\ x \wedge e \text{ if } x \ngeqq e \end{cases}, \qquad x\alpha_e^+ = \begin{cases} x \text{ if } x \geqq e \\ x \text{ if } x \ngeqq e \end{cases},$$

Proof. By computation.

Note also that ω_e is range-closed, α_e dually range-closed, both are idempotent and $1\omega_e = e = 0\alpha_e^+$.

PROPOSITION 3. For any poset P (with 0 and 1), the following are equivalent:

(i) P is a lattice.

(ii) For all x in P there exists $Q_x, R_x, \varphi_x \in S_{RC}(Q_x, P)$, $\psi_x \in S_{DRC}(P, R_x)$ such that $1\varphi_x = x = 0\psi_x^+$.

Proof. (i) $\Rightarrow$ (ii): ω_x and α_x have the requisite properties.

(ii) $\Rightarrow$ (i). follows from Proposition 1 and its dual.

PROPOSITION 4. If $\varphi \in S_{RC}(P, Q) \cap S_{DRC}(P, Q)$, then

$$\varphi|_{P(0\varphi^+, 1)}: P(0\varphi^+, 1) \rightarrow Q(0, 1\varphi)$$

is an order-isomorphism.

Proof. Follows from D and Proposition 1.

In the case of example G this is the Fundamental Theorem of Homomorphisms.

DEFINITION 3. $a, b \in P$ are said to form a modular pair, in symbols $M(a, b)$, if $(x \vee a) \wedge b$ and $x \vee (a \wedge b)$ exist and are equal whenever $x \leqq b$. The dual statement is denoted $M^*(a, b)$.

PROPOSITION 5. For arbitrary P, Q, R, with $\varphi \in S_{RC}(P, Q)$ and

$$\psi \in S_{RC}(P, Q) \cap S_{DRC}(Q, R)\ ,$$

the following conditions are equivalent:

(i) $\varphi\psi \in S_{RC}(P, R)$.

(ii) $M^*(1\varphi, 0\psi^+)$ and $x \wedge 1\varphi\psi$ exists for all x in R.

Proof. (i) $\Rightarrow$ (ii): Suppose $\varphi\psi \in S_{RC}(P, R)$, then

$$a\psi^+\varphi^+\varphi\psi = a(\varphi\psi)^+\varphi\psi = a \wedge 1\varphi\psi .$$

Also, if $a \geqq 0\psi^+$, $a = b\psi^+$ for some b in R. We thus have

$$b \wedge 1\varphi\psi = b\psi^+\varphi^+\varphi\psi = a\varphi^+\varphi\psi = (a \wedge 1\varphi)\psi$$

where the last term exists by hypothesis. Finally, we get successively:

$$\begin{aligned}(a \wedge 1\varphi) \vee 0\psi^+ &= (a \wedge 1\varphi)\psi\psi^+ = (b \wedge 1\varphi\psi)\psi^+ = b\psi^+ \wedge 1\varphi\psi\psi^+ \\ &= a \wedge 1\varphi\psi\psi^+ = a \wedge (1\varphi \vee 0\psi^+)\end{aligned}$$

where all the meets and joins exist.

(ii) $\Rightarrow$ (i): Suppose $M^*(1\varphi, 0\psi^+)$, then

$$\begin{aligned}a\psi^+\varphi^+\varphi\psi\psi^+ &= (a\psi^+ \wedge 1\varphi) \vee 0\varphi^+ = a\psi^+ \wedge (1\varphi \vee 0\psi^+) \\ &= a\psi^+ \wedge 1\varphi\psi\psi^+ = (a \wedge 1\varphi\psi)\psi^+\end{aligned}$$

since $a\psi^+ \geqq 0\psi^+$. Therefore

$$\begin{aligned}a(\varphi\psi)^+\varphi\psi &= a\psi^+\varphi^+\varphi\psi = a\psi^+\varphi^+\varphi\psi\psi^+\psi = (a \wedge 1\varphi\psi)\psi^+\psi \\ &= (a \wedge 1\varphi\psi) \wedge 1\psi = a \wedge 1\varphi\psi .\end{aligned}$$

Thus $\varphi\psi$ is range-closed.

DEFINITION 4. The mapping $\varphi \in S(P, Q)$ where P is a lattice is said to be *totally range-closed* if $\omega_e\varphi$ is range-closed for all e in P. The set of all such maps is denoted $S_{TRC}(P, Q)$. $\varphi \in S(P, Q)$ is said to be *dually totally range-closed* if $\varphi^+\alpha_e^+$ is dually range-closed for all e in P; the set of all such maps is denoted by $S_{DTRC}(P, Q)$.

PROPOSITION 6. If P is a lattice, the following conditions are equivalent for $\varphi \in S(P, Q)$:

(i) $\varphi \in S_{TRC}(P, Q)$.

(ii) $g \wedge e\varphi$ exists for all $g, e \in P$ and equals $(g\varphi^+ \wedge e)\varphi$.

(iii) $\psi \in S_{RC}(P) \Rightarrow \psi\varphi \in S_{RC}(P, Q)$.

Proof. (i) $\Leftrightarrow$ (ii): $\omega_e\varphi$ is range-closed if and only if $g(\omega_e\varphi)^+\omega_e\varphi = g\varphi^+\omega_e^+\omega_e\varphi = g \wedge 1\omega_e\varphi = g \wedge e\varphi$. Since ω_e is range-closed, we have $g\varphi^+\omega_e^+\omega_e = g\varphi^+ \wedge 1\omega_e = g\varphi^+ \wedge e$. Therefore $\varphi \in S_{TRC}(P)$ if and only if $(g\varphi^+ \wedge e)\varphi = g \wedge e\varphi$ for all e, g in P.

(ii) $\Rightarrow$ (iii): Since ψ is range-closed

$$x(\psi\varphi)^+\psi\varphi = x\varphi^+\psi^+\psi\varphi = x\varphi^+\psi^+\psi\varphi = (x\varphi^+ \wedge 1\psi)\varphi = x \wedge 1\psi\varphi \ .$$

(iii) $\Rightarrow$ (i): obvious.

Next we study the relationship between the properties 'range-closed' and 'totally range-closed'. To this end we set:

DEFINITION 5. Let P be a lattice, $\psi: P \to P$ is called a weak quantifier if it is a closure operator satisfying: $(e \wedge f\psi)\psi = e\psi \wedge f\psi$ for all e, f in P.

LEMMA. *For a lattice P:*

(i) *If $\varphi \in S_{TRO}(P, Q)$, then $\varphi\varphi^+$ is a weak quantifier.*

(ii) *If $\varphi \in S_{RO}(P, Q)$ and $\varphi\varphi^+$ is a weak quantifier then $\varphi \in S_{TRO}(P, Q)$, whenever $g \wedge h\varphi$ exists for all g, h.*

Proof. (i) Suppose φ is range-closed; this equivalent to $(g\varphi^+ \wedge h)\varphi = g \wedge h\varphi$ for all g, h. A fortiori $(g\varphi\varphi^+ \wedge h)\varphi = g\varphi \wedge h\varphi$ for all g, h. Apply φ^+ to both sides:

$$(g\varphi\varphi^+ \wedge h)\varphi\varphi^+ = (g\varphi \wedge h\varphi)\varphi^+ = g\varphi\varphi^+ \wedge h\varphi\varphi^+$$

for all g, h.

(ii) Since φ is range-closed and $\varphi\varphi^+$ is a weak quantifier,

$$(g\varphi^+ \wedge h)\varphi\varphi^+ = (g\varphi^+\varphi\varphi^+ \wedge h)\varphi\varphi^+ = g\varphi^+\varphi\varphi^+ \wedge h\varphi\varphi^+ = (g \wedge h\varphi)\varphi^+ \ .$$

Since $(g\varphi^+ \wedge h)\varphi \leqq 1\varphi$ and $g \wedge h\varphi \leqq 1\varphi$ and φ is range closed we get $g \wedge h\varphi = (g\varphi^+ \wedge h)\varphi$.

PROPOSITION 7. For a lattice P and $\varphi \in S_{RO}(P, Q) \cap S_{DRO}(P, Q)$, the following conditions are equivalent:

(i) $\varphi \in S_{TRO}(P, Q)$.

(ii) $M^*(f, 0\varphi^+)$ for all f in P.

Proof. By the lemma, since $\varphi \in S_{RO}(P, Q)$, $\varphi \in S_{TRO}(P, Q)$ if and only if $\varphi\varphi^+$ is a weak quantifier. This is equivalent to $(e \wedge f\varphi\varphi^+)\varphi\varphi^+ = e\varphi\varphi^+ \wedge f\varphi\varphi^+$ for all e, f, and, in turn, to $(e \wedge (f \vee 0\varphi^+)) \vee 0\varphi^+ = (e \vee 0\varphi^+) \wedge (f \vee 0\varphi^+)$ since $\varphi \in S_{DRO}(P, Q)$. Now, if $M^*(e, 0\varphi^+)$ for all e in P, then $(e \wedge (f \vee 0\varphi^+)) \vee 0\varphi^+ = (e \vee 0\varphi^+) \wedge (f \vee 0\varphi^+)$ since $f \vee 0\varphi^+ \geqq 0\varphi^+$. Conversely, since $\varphi\varphi^+$ is a weak quantifier and $f = f \vee 0\varphi^+ = f\varphi\varphi^+$ whenever $f \geqq 0\varphi\varphi^+$, we have successively

$$(f \wedge e) \vee 0\varphi^+ = (e \wedge f\varphi\varphi^+)\varphi\varphi^+ = e\varphi\varphi^+ \wedge f\varphi\varphi^+ = (e \vee 0\varphi^+) \wedge f \ .$$

For $\varphi \in S(P)$, i is called φ-invariant if $i\varphi \leqq i$.

PROPOSITION 8. Let P be a lattice, $\varphi \in S_{TRC}(P)$, i φ-invariant:

(i) $\varphi|_{P(0,i)}: P(0, i) \to P(0, i)$ is residuated, range-closed and its residual is given by the rule $x(\varphi|_{P(0,i)})^+ = x\varphi^+ \wedge i$.

(ii) If, in addition, φ is dually range closed $\varphi|_{P(0,i)}$ is dually range closed if and only if $M(i, 0\varphi^+)$.

Proof. (i) By computation.

(ii) $\varphi|_{P(0,i)}$ is dually range-closed if and only if

$$x(\varphi|_{P(0,i)})(\varphi|_{P(0,i)})^+ = x \vee 0(\varphi|_{P(0,i)})^+ = x \vee (0\varphi^+ \wedge i)$$

for x in $P(0, i)$. But $x(\varphi|_{P(0,i)})(\varphi|_{P(0,i)})^+ = x\varphi\varphi^+ \wedge i$ for all x in $P(0, i)$. Hence the original statement is equivalent to $M(0\varphi^+, i)$.

3. **Mappings of finite ascent and descent.** Since φ and φ^+ are isotone, it follows immediately that $1 \geqq 1\varphi \geqq 1\varphi^2 \geqq \cdots$ and $0 \leqq 0\varphi^+ \leqq 0(\varphi^+)^2 \leqq \cdots$. If there exists an integer $n \geqq 0$ such that $0(\varphi^+)^n = 0(\varphi^+)^{n+1}$, there is a smallest such integer; it will be called the *ascent* of φ and will be denoted $A(\varphi)$. If no such integer exists we write $A(\varphi) = \infty$. Dually the least integer such that $1\varphi^n = 1\varphi^{n+1}$ is called the *descent* of φ and denoted $D(\varphi)$; we will set $D(\varphi) = \infty$ if no such integer exists. We investigate the relationship between $D(\varphi)$ and $A(\varphi)$. First we exhibit an extreme case.

EXAMPLE. Let Z denote the nonnegative integers and let $\varphi: \mathscr{P}(Z) \to \mathscr{P}(Z)$ be induced by the relation $R = \{(n, 2n) : n \in Z\}$. We see that $D(\varphi) = 0$ while $A(\varphi) = \infty$.

PROPOSITION 9. For any $\varphi \in S_{RC}(P)$, if $A(\varphi)$ is finite and $D(\varphi) = 0$ then $A(\varphi) = 0$.

Proof. Suppose $A(\varphi) > 0$, then for some $0 \neq x_1 \in P$, $x_1\varphi = 0$. Since, by hypothesis $1\varphi = 1$ and φ is range-closed, we can define a sequence $\{x_n\}$ such that $x_{n+1}\varphi = x_n$. Now,

$$x_{n+1}\varphi^{n+1} = (x_{n+1}\varphi)\varphi^n = x_n\varphi^n = \cdots = x_1\varphi = 0 ;$$

therefore $x_{n+1} \leqq x_{n+1}\varphi^{n+1}(\varphi^+)^{n+1} = 0(\varphi^+)^{n+1}$. But $x_{n+1} \nleqq 0(\varphi^+)^n$, for, if $x_{n+1} \leqq 0(\varphi^+)^n$, we would have $x_1 = x_{n+1}\varphi^n \leqq 0(\varphi^+)^n\varphi^n \leqq 0$ which would be a contradiction. We would therefore have $0(\varphi^+)^n \neq 0(\varphi^+)^{n+1}$ for all $n \geqq 0$ which is impossible since $A(\varphi)$ is finite.

PROPOSITION 10. If P is a lattice, $\varphi \in S_{TRC}(P) \cap S_{DTRC}(P)$, and $A(\varphi)$, $D(\varphi)$ are both finite, then $A(\varphi) = D(\varphi)$.

Proof. Suppose $D(\varphi) = k$. $1\varphi^k$ is a φ-invariant, hence by Proposition 8 $\varphi_1 = \varphi \mid_{P(0,1\varphi^k)}: P(0, 1\varphi^k) \to P(0, 1\varphi^k)$ is residuated and range closed. Also $(1\varphi^k)\varphi_1 = 1\varphi^k\varphi = 1\varphi^k$, thus $D(\varphi_1) = 0$. Recall that $0\varphi_1^+ = 0\varphi^+ \wedge 1\varphi^k$, hence

$$0(\varphi^+)^2 = (0\varphi^+ \wedge 1\varphi^k)\varphi^+ \wedge 1\varphi^k = 0(\varphi^+)^2 \wedge 1\varphi^k\varphi^+ \wedge 1\varphi^k = 0(\varphi^+)^2 \wedge 1\varphi^k$$

since $1\varphi^k\varphi^+ = 1\varphi^{k-1}\varphi\varphi^+ \geqq 1\varphi^{k-1} \geqq 1\varphi^k$. Inductively we get $0(\varphi_1^+)^n = 0(\varphi^+)^n \wedge 1\varphi^k$, hence if $0(\varphi^+)^m = 0(\varphi^+)^n$, we have $0(\varphi_1^+)^m = 0(\varphi_1^+)^n$. This means $A(\varphi_1) \leqq A(\varphi)$; therefore $A(\varphi_1)$ is finite. Applying Proposition 9 to φ_1 we get $A(\varphi_1) = 0$. Now, suppose $x \leqq 0(\varphi^+)^{k+1}$ and let $y = x\varphi^k$. We have $y\varphi_1 = x\varphi^k\varphi_1 = x\varphi^{k+1} = 0$. But $x\varphi^k = y \leqq y\varphi_1\varphi_1^+ = 0\varphi_1 = 0$ and $x \leqq 0(\varphi^+)^k$. We have shown $0(\varphi^+)^{k+1} \leqq 0(\varphi^+)^k$. Since we always have $0(\varphi^+)^k \leqq 0(\varphi^+)^{k+1}$ we may conclude that $0(\varphi^+)^k = 0(\varphi^+)^{k+1}$. Therefore $A(\varphi) \leqq D(\varphi)$. The other inequality follows by duality.

We now focus our attention to mappings for which $A(\varphi) = 1$ or $D(\varphi) = 1$.

PROPOSITION 11. For arbitrary P:

(i) If $\varphi \in S_{RC}(P)$ and $A(\varphi) = 1$, then $1\varphi \wedge 0\varphi^+ = 0$

(ii) If $\varphi \in S_{DRC}(P) \cap S_{RC}(P)$ and $A(\varphi) = 1 = D(\varphi)$, then $1\varphi \dot{\vee} 0\varphi^+ = 1$.

Proof. By Proposition 1, $1\varphi \wedge 0\varphi^+$ exists. We have successively $1\varphi \wedge 0\varphi^+ = 0\varphi^+\varphi^+\varphi = 0\varphi^+\varphi \leqq 0$.

(ii) follows from (i) and its dual.

COROLLARY 1. *For any P, let $A(\varphi)$ and $D(\varphi)$ be finite, $n = \operatorname{Max}\{A(\varphi), D(\varphi)\}$, then if $\varphi^n \in S_{RC}(P) \cap S_{DRC}(P)$, $1\varphi^n \dot{\vee} 0(\varphi)^n = 1$.*

Proof. Note that $A(\varphi^n) = 1 = D(\varphi^n)$, hence it suffices to apply the above to φ^n.

Next, we restrict P to be a lattice:

COROLLARY 2. *For any P, let $A(\varphi)$ and $D(\varphi)$ be finite, then if $\varphi \in S_{TRC}(P) \cap S_{DTRC}(P)$, $1\varphi^n \vee 0(\varphi^+)^n = 1$ where $A(\varphi) = D(\varphi) = n$.*

Proof. By Proposition 6 $\varphi^n \in S_{RC}(P) \cap S_{DRC}(P)$. By Proposition 10 $A(\varphi) = D(\varphi)$. The remainder follows from Corollary 1.

It seems appropriate at this point to point out that if P satisfies the ascending chain condition $A(\varphi)$ is automatically finite. The same applies to the descending chain condition and $D(\varphi)$. In the case of Example G Corollary 1 or 2 are known as Fitting's Lemma. Ore's Theorem ([9], pp. 203–4) can be formulated and proven in terms of residuated mappings.

PROPOSITION 12. Let P be a lattice and i a φ-invariant:

(i) If $\varphi \in S_{TRC}(P)$ and $A(\varphi) = n$, then $i\varphi^n \wedge 0(\varphi^+)^n \wedge i = 0$.

(ii) If $\varphi \in S_{DTRC}(P) \cap S_{RC}(P)$, $A(\varphi) = n$ and $M^*(i, 1\varphi)$, then $i = i(\varphi^+)^n \wedge (1\varphi^n \vee i)$.

Proof. (i) We first show that if $\varphi \in S_{TRC}(P)$ and $A(\varphi) = 1$ then $0 = i\varphi \wedge 0\varphi^+ \wedge i$. Since φ is totally range-closed $\varphi\,|_{P(0,i)}$ is residuated and range-closed; moreover,

$$0((\varphi\,|_{P(0,i)})^+)^2 = 0(\varphi^+)^2 \wedge 0\varphi^+ \wedge i = 0\varphi^+ \wedge i = 0(\varphi\,|_{P(0,i)})^+ .$$

Hence, applying Proposition 11, we get $0 = i(\varphi\,|_{P(0,i)}) \wedge 0(\varphi\,|_{P(0,i)})^+ = i\varphi \wedge 0\varphi^+ \wedge i$. Now note that $\varphi^n \in S_{TRC}(P)$ and $0(\varphi^+)^n = 0(\varphi^+)^{2n}$ thus, if we apply the above to φ^n, we get $0 = i\varphi^n \wedge 0(\varphi^+)^n \wedge i$.

(ii) follows by a dual argument.

EXAMPLE. Let R be a commutative ring with unit, $L(R)$ the lattice of ideals of R. Define $\rho_A\colon L(R) \to L(R)$ for $A \subset R$, by the rule $B\rho_A = BA$. Denote $\rho_{\{a\}}$ by ρ_a. One verifies very easily that:

(i) ρ_a is totally range-closed and dually totally range-closed for all a in R.

(ii) $R\rho_a = R\rho_{(a)}$ and $\{0\}\rho^+_{(a)} = \{0\}\rho^+_a$.

For R Noetherian, I an ideal and a an element of R, part (ii) of the above proposition yields $I = I : (a^r) \cap (I + (a^r))$ for some integer r which as is well known implies that every irreducible ideal is primary.

Extending a notion of Kurosh we set:

DEFINITION 6. Let P be a lattice, suppose $e \dot{\vee} f = 1$, $M(e, f)$, $M^*(e, f)$, define the mappings $\varphi_{e,f}$, $\varphi^+_{e,f}\colon P \to P$ by the rules $x\varphi_{e,f} = (x \vee e) \wedge f$, $x\varphi^+_{e,f} = (x \wedge f) \vee e$ for all x in L.

PROPOSITION 13. (i) $\varphi^2_{e,f} = \varphi_{e,f}$, $\varphi^+_{e,f} = (\varphi^+_{e,f})^2$.

(ii) $\varphi_{e,f} \in S_{RC}(P) \cap S_{DRC}(P)$.

(iii) $1\varphi_{e,f} = f$, $0\varphi^+_{e,f} = e$.

(iv) If, in addition, $M(x, e)$ and $M^*(x, f)$ for all x in L, then $\varphi_{e,f} \in S_{TRC}(P) \cap S_{DTRC}(P)$.

Proof. By computation.

PROPOSITION 14. If P is a lattice, the following conditions are equivalent:

(i) $\varphi^2 = \varphi \in S_{RC}(P) \cap S_{DRC}(P)$.

(ii) $0\varphi^+ \dot{\vee} 1\varphi = 1$, $M(0\varphi^+, 1\varphi)$, $M^*(1\varphi, 0\varphi^+)$ and $\varphi_{0\varphi^+, 1\varphi} = \varphi$.

Proof. (ii) $\Rightarrow$ (i) is Proposition 13.

(i) $\Rightarrow$ (ii): Since $\varphi^2 = \varphi \in S_{RO}(P) \cap S_{DRO}(P)$ we have $M(0\varphi^+, 1\varphi)$ and $M^*(1\varphi, 0\varphi^+)$ from Proposition 5 and its dual and $0\varphi^+ \vee 1\varphi = 1$ from Proposition 11. Furthermore we have successively

$$x\varphi = x\varphi\varphi^+\varphi = x\varphi\varphi^+\varphi^+\varphi = (x \vee 0\varphi^+) \wedge 1\varphi = x\varphi_{0\varphi^+,1\varphi}\,.$$

COROLLARY 1. *For a lattice P the following conditions are equivalent:*

(i) $\varphi^2 = \varphi \in S_{TRO}(P) \cap S_{DTRO}(P)$.

(ii) $0\varphi^+ \dot{\vee} 1\varphi = 1$, $M(x, 1\varphi)$, $M^*(x, 0\varphi^+)$ *for all* x *in* P *and* $\varphi_{0\varphi^+,1\varphi} = \varphi$.

COROLLARY 2. *For a lattice P the following conditions are equivalent:*

(i) *P is modular complemented.*

(ii) *For every* x *in* P *there is* $\varphi_x^2 = \varphi_x \in S_{TRO}(P) \cap S_{DTRO}(P)$ *such that* $1\varphi_x = x$.

(iii) *For every* x *in* P *there is* $\varphi_x^2 = \varphi_x \in S_{TRO}(P) \cap S_{DTRO}(P)$ *such that* $0\varphi_x^+ = x$.

(iv) *P is isomorphic to the lattice of left annihilating ideals of* $S_{TRO}(P) \cap S_{DTRO}(P)$.

Proof. All equivalences are immediate except those with (iv) which follow from paragraph 3 of [7].

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