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A NOTE ON CONTINUOUS COLLECTIONS OF DISJOINT CONTINUA

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M. E. Hamstrom has shown that if G is a continuous collection of disjoint arcs filling up a compact continuous curve M in the plane such that M/G is an arc, then G^* ($x \in G^*$ if and only if for some $g \in G, x \in g$) is a simple closed curve plus its interior. One purpose of this note is to show that if S is a space satisfying Axioms $0-5$ of R. L. Moore's Foundations of Point Set Theory, and $M \subset S$ such that (1) M has one and only one complementary domain, and (2) there exists a continuous collection of disjoint nondegenerate continua filling up M, then M is a simple closed curve J plus one of the complementary domains of J. Another purpose of this note is to state and prove some consequences of this theorem.

Both *circle-like continuum* and an *annulus-like point set* are defined in a natural way. It is shown that if M is a compact continuum in a separable space satisfying Axioms $0-5$ of R. L. Moore, such that (1) M is filled up by a continuous collection of disjoint continua, (2) all but countably many elements of G are circle-like, and (3) there exists a point 0 not in M such that no element of G separates two elements of G from one another and both from 0 in S, then M/G is an arc. An example exists where the conclusion of the theorem fails if condition (3) is omitted, but it is not included in this note. From the preceding theorem, it follows that if M is a compact continuum in the plane which is filled up by a continuous collection of disjoint circle-like continua, then M is annulus-like.

Throughout this note Bd(M) will denote the boundary of M and Cl(M) will denote the closure of M. Furthermore, S will denote a space satisfying Axioms $0-5$ of [3] unless otherwise stated, and "simple closed curve" will be abbreviated scc.

DEFINITION 1. The collection G is a continuous collection if and only if $\{g_n\}$ is a sequence of elements of G such that for some $g \in G$, $\liminf \{g_n\} \cap g \neq \emptyset$, then $\liminf \{g_n\} = \limsup \{g_n\} = g$ (We shall denote this by $\{g_n\} \rightarrow g$).

LEMMA 1. *Let M denote a point set and let N denote a proper subset of M such that $M-N$ is connected. Let $J \subset N$ such that $M-J$ is the union of two mutually separated point sets H and K (H and K are disjoint and neither contains a limit point of the other). Then one of H and K is a subset of N.*

Proof. Let $D = M - N$. Since D is connected $D \subset H$ or $D \subset K$, say K. Now suppose $H \not\subset N$. Then $H \cap D \neq \emptyset$. This means that $H \cap K \neq \emptyset$, a contradiction.

LEMMA 2. *Suppose M is a closed subset of S such that $M \neq S$ and $S - M$ is connected and every component of $S - \mathrm{Cl}(D)$ is the interior of a scc, where $D = S - M$ and the ideal point is a point of D. Then if X and Y are components of $S - \mathrm{Cl}(D)$ such that $\mathrm{Cl}(X) \cap \mathrm{Cl}(Y) \neq \emptyset$, then $\mathrm{Cl}(X) \cap \mathrm{Cl}(Y)$ is degenerate.*

Proof. Let us suppose the contrary. Let P and O denote points of $\mathrm{Bd}(X) \cap \mathrm{Bd}(Y)$. Both P and O are accessible from both X and Y (Theorem 59, Chapt. IV of [3]). There exist arcs (OAP) and (OBP) such that $[(OAP) - (A \cup P)] \subset X$ and $[(OBP) - (O \cup P)] \subset Y$. Now the scc $J = [(OBP) \cup (OAP)] \subset M$. It follows from Lemma 1 that the interior I of J is a subset of M. Since $B \in Y$, $I \cap Y \neq \emptyset$. Likewise, $I \cap X \neq \emptyset$. But $X \cup I$ is a connected subset of $S - \mathrm{Cl}(D)$ of which X is a proper subset contrary to the fact that X is a component of $S - \mathrm{Cl}(D)$. Hence $\mathrm{Cl}(X) \cap \mathrm{Cl}(Y)$ is degenerate.

LEMMA 3. *If G is a continuous collection of disjoint closed point sets such that each element of G is a limit element of G, then no element of G contains a domain.*

Proof. Let $g \in G$ and $D = G^* - g$. Since each element of G is a limit element of G, some point of G is a limit point of D. Since g is closed, there exists a sequence $\{g_n\}$ of elements of G such that for each n, $g_n \subset D$ and $\{g_n\} \rightarrow g$. If g contains a domain U this would imply that for some k, $g_k \cap U \neq \emptyset$ since G is a continuous collection, contrary to the fact that the elements of G are disjoint.

THEOREM 1. *If M is a compact continuous curve in S such that $M \neq S$ and does not separate S, and M is filled up by a continuous collection G of disjoint continua not all degenerate, then M contains a domain.*

Proof. Let $g \in G$ such that g is nondegenerate, and let P_1 and P_2 denote points of g. Let T denote a component of $M - g$. Since M is compact and G is a continuous collection, $g = \mathrm{Bd}(T)$ with respect to M. Since M is a continuous curve, there exist open connected subsets R_1 and R_2 of M such that $P_1 \in R_1$, $P_2 \in R_2$, and $\mathrm{Cl}(R_1) \cap \mathrm{Cl}(R_2) = \emptyset$. In the following $i = 1, 2$. Let $0_i \in R_i \cap T$. Now there exist arcs $0_iP_i \subset R_i$. Let A_i denote the first point in the order from 0_i to P_i of $(0_iP_i) \cap g$. Now $[(0_iA_i) - A_i] \subset T$. Since T is a connected open subset of M, there is an arc $(0_1 0_2) \subset T$ (Theorem 13, Chapt. II of [3]). Let $U = (0_1 0_2) \cup (0_1 A_1) \cup (0_2 A_2)$. Now U is a continuous curve. Hence, there is an arc $(A_1 A_2) \subset U$. Now $[(A_1 A_2) - (A_1 \cup A_2)] \subset T$, and $(A_1 A_2) \cup g$ separates S (Theorem 22, Chapt. IV of [3]). Let H denote one of the complementary domains of $(A_1 A_2) \cup g$ and let K denote all other complementary domains of $(A_1 A_2) \cup g$. According to Lemma 1, one of the point sets H and K is a subset of M. This means that M contains a domain.

THEOREM 2. *Under the hypothesis of Theorem* 1, *it follows that each component of* $M - \mathrm{Bd}(D)$ *is bounded by a scc, where* $D = S - M$.

Proof. Let E denote a complementary domain of $S - \mathrm{Cl}(D)$. By Theorem 1, $E \neq \varnothing$. Let J denote the outer boundary of D with respect to E (the outer boundary of D with respect to E is the boundary of the component of $S - \mathrm{Cl}(D)$ containing E). Now $\mathrm{Bd}(D)$ is a continuous curve (Theorem 42, Chapt. IV of [3]). Furthermore, J is a scc (Theorem 43, Chapt. IV of [3]). Since E is a component of $S - \mathrm{Cl}(D)$, $\mathrm{Bd}(E) = J$.

THEOREM 3. *Under the hypothesis of Theorem* 1 *with the additional stipulation that no element of* G *is degenerate, it follows that* $\mathrm{Bd}(M)$ *is a scc.*

Proof. Let C denote a complementary domain of $S - \mathrm{Cl}(D)$, where $D = S - M$. By Theorem 2, $\mathrm{Bd}(C)$ is a scc. Let $g \in G$. It follows that if $g \cap (C \cup \mathrm{Bd}(C)) \neq \varnothing$, $g \subset (C \cup \mathrm{Bd}(C))$. Let us prove this by contradiction. Let $P \in g - (C \cup \mathrm{Bd}(C))$. Now $g \cap \mathrm{Bd}(C) \neq \varnothing$ since g is a continuum. There is a subcontinuum g' of g which is irreducible from P to $\mathrm{Bd}(C)$. Let 0 denote a point of C and let R denote an open connected subset of M such that $P \in R$ and $\mathrm{Cl}(R) \cap \mathrm{Bd}(C) = \varnothing$. By Lemma 3, $C \not\subset g$. Furthermore, $\mathrm{Bd}(C) \not\subset g$. Suppose this is the case. Now

$$H = (C \cup \mathrm{Bd}(C)) - [(C \cup \mathrm{Bd}(C)) \cap g] = \varnothing .$$

Furthermore, $(C \cup \mathrm{Bd}(C)) \cap g$ must contain a limit point of H. If $t \in G$ and $H \cap t \neq \varnothing$, then $t \in C$. But there exists a sequence $\{g_n\}$ of elements of G such that $\{g_n\} \to g$. This means that there is a g_k of $\{g_n\}$ such that $g_k \cap \mathrm{Bd}(C) \neq \varnothing$, contrary to the fact that the elements of G are disjoint. Hence, $\mathrm{Bd}(C) - (g \cap \mathrm{Bd}(C)) \neq \varnothing$. Now C lies in some component E of $M - g'$. Furthermore, each component of $[\mathrm{Bd}(C) - (g \cap \mathrm{Bd}(C))] \subset E$ since each point of $\mathrm{Bd}(C) - (g \cap \mathrm{Bd}(C))$ is a limit point of C. Let K denote a component of $\mathrm{Bd}(C) - (g \cap \mathrm{Bd}(C))$.

There exists a sequence of elements $\{g_n\}$ of G such that $\{g_n\} \to g$ and for each n, $g_n \cap R \neq \emptyset$. This in turn means that P is a limit point of E. Since P is a limit point of E, R contains some point X of E. There is an arc $(XP) \subset R$. Let A denote the first point in the order from X to P of $(XP) \cap g'$. Since E is connected open subset of M, there is an arc $(X0) \subset E$. Let B denote the first point in the order from X to 0 of $(X0) \cap \mathrm{Bd}(C)$. Now there exists an arc (AB) such that $(AB) \subset ((XA) \cup (XB))$. Let U denote a point of $g' \cap \mathrm{Bd}(C)$. There is an arc (AU) such that $[(AU) - (A \cup U)] \subset C$. Let Z denote a point of the arc (AB) such that $Z \neq A$ and $Z \neq B$. Then we have two disjoint compact continua H and g' and two arcs (ZA) and (BU) such that $H \cap (ZA) = A$, $H \cap (BU) = B$, $(ZA) \cap g' = A$ and $(BU) \cap g' = U$. Let D_H denote the component of $S - g'$ such that $H \subset D_H$, and let $D_{g'}$ denote the component of $S - H$ such that $g' \subset D_{g'}$. Moreover, let $H' = S - D_{g'}$, $K' = S - D_H$ and let $N = H' \cup K' \cup (ZA) \cup (BU)$. $S - N$ is the union of two disjoint connected domains D_1 and D_2 such that (ZA) and (BU) is a subset of both $\mathrm{Bd}(D_1)$ and $\mathrm{Bd}(D_2)$ (Theorem 38, Chapt. IV of [3]). It follows from Lemma 1 that either $D_1 \subset M$ or $D_2 \subset M$, say D_1. Since $[(ZA) \cup (BU)] \subset \mathrm{Bd}(D_1)$, it follows that $D_1 \cap C \neq \emptyset$ and $D_1 \cap (S - C) \neq \emptyset$. Hence, C is a proper subset of D_1 which means that C is not a component of $S - \mathrm{Cl}(D)$, a contradiction. Hence, $g \subset (C \cup \mathrm{Bd}(C))$. By a similar argument, it may be shown that if $g \in G$, there is some component C of $S - \mathrm{Cl}(D)$ such that $g \cap \mathrm{Cl}(C) \neq \emptyset$.

Now let us suppose that $S - \mathrm{Cl}(D)$ has more than one component. Let W denote a set such that $X \in W$ if and only if for some component C of $S - \mathrm{Cl}(D)$, $X = C \cup \mathrm{Bd}(C)$. Now $W^* = M$. This follows from the immediately preceding paragraph. For some two point sets X and Y of W, $X \cap Y \neq \emptyset$. Otherwise, M would be the union of a countable number of closed and disjoint point sets. Let P denote a point of $\mathrm{Bd}(X) \cap \mathrm{Bd}(Y)$, and let $g \in G$ such that $P \in g$. Since $g \cap X \neq \emptyset$, $g \subset X$. Likewise, $g \subset Y$. By Lemma 2, $X \cap Y$ is degenerate, a contradiction. Hence, $W = X$. But $\mathrm{Bd}(X)$ is a scc. Hence, M is bounded by a scc.

COROLLARY. *A necessary and sufficient condition that a compact continuous curve M which does not separate the plane be a simple closed curve plus its interior is that there exist a continuous collection of nondegenerate disjoint continua filling up M.*

Proof. The sufficiency follows from Theorem 3. The necessity follows from the fact that each scc plus its interior in the plane is homeomorphic to the unit square plus its interior, and there is a continuous collection of disjoint nondegenerate continua filling up the unit square. This collection is invariant under a homeomorphism.

THEOREM 4. *Let M denote a compact continuous curve in S such that $M \neq S$ and M is filled up by a continuous collection G of disjoint nondegenerate continua. Then the boundary of every complementary domain of M is a scc.*

Proof. If M has only one complementary domain, the theorem follows from Theorem 3. Let D denote a complementary domain of M and let H denote the collection of all other complementary domains of M. Let $L = M \cup H^*$ and let C denote a component of $S - \mathrm{Cl}(D)$. It follows as in Theorem 2, $\mathrm{Bd}(C)$ is a simple closed curve. Let $g \in G$ such that $g \cap \mathrm{Cl}(C) \neq \varnothing$. Then $g \subset \mathrm{Cl}(C)$. Suppose the contrary. Let $P \in (g - g \cap \mathrm{Bd}(C))$. Now suppose $\mathrm{Bd}(C) \subset g$. There is some component V of $M - g$ which is not a subset of C. For if all components of $M - g$ lie in C, then G is not a continuous collection. Now g is $\mathrm{Bd}(V)$ with respect to M. A process like that used in Theorem 3 is used to construct two arcs (AXB) and (AYB) such that A and B are points of $\mathrm{Bd}(C)$, $[(AXB) - (A \cup B)] \subset V$, and $[(AYB) - (A \cup B)] \subset C$. Then by Lemma 1, one of the complementary domains of the scc $(AXB) \cup (AYB) \subset L$. This means that C is not a component of $S - \mathrm{Cl}(D)$, a contradiction. Hence, $\mathrm{Bd}(C) \not\subset g$. Let g' denote a subcontinuum of g which is irreducible from P to $g \cap \mathrm{Bd}(C)$. Now every component of $\mathrm{Bd}(C) - (g \cap \mathrm{Bd}(C))$ has a limit point in $g \cap \mathrm{Bd}(C)$. Now $C \cup (\mathrm{Bd}(C) - g \cap \mathrm{Bd}(C))$ lies in some component U of $L - g'$. Let K denote a component of $\mathrm{Bd}(C) - g \cap \mathrm{Bd}(C)$. There exists a sequence of elements $\{g_n\}$ of G such that $\{g_n\} \rightarrow g$ and $g_n \cap K \neq \varnothing$ for each n. This means that P is a limit point of U. Let Z denote a point of $g' \cap \mathrm{Bd}(C)$. As in Theorem 3, we construct arcs as there and again apply Theorem 36 of Chapt. IV of [3] and reach the contradiction that C is not a component of $S - \mathrm{Cl}(D)$. Hence, as in Theorem 3, if $g \in G$ such that $g \cap \mathrm{Cl}(C) \neq \varnothing$, $g \subset \mathrm{Cl}(C)$. Likewise, it easily follows that if $g \in G$, then for some C of $S - \mathrm{Cl}(C)$, $g \cap \mathrm{Cl}(C) \neq \varnothing$. Again we may define the point set W as in Theorem 3 and show that it contains only one point set using the fact that no compact continuous curve has uncountably many complementary domains (Theorem 63, Chapt. IV of [3]), i.e., the point set W again is a countable collection. It then follows that L is a simple closed curve plus one of its complementary domains. Consequently, D is bounded by a scc.

COROLLARY 1. *Let G be a continuous collection of nondegenerate disjoint continua filling up S. Then there is no nondegenerate subcollection G' (contains at least two elements of G) of G such that $(G')^*$ is a continuous curve.*

Proof. Suppose the contrary. Since $(G')^*$ satisfies the hypothesis of Theorem 4, each complementary domain of $(G')^*$ is bounded by a simple closed curve. Let D denote a complementary domain of $(G')^*$, and let $g \in G$ such that $P \in g$. Now $g \subset \mathrm{Bd}(D)$ since G is a continuous collection. This means that $\mathrm{Bd}(D)$ must be the union of a countable collection of nondegenerate disjoint continua.

R. D. Anderson has stated in [1] that there exists a continuous collection G of pseudo arcs filling up a 2-sphere. It follows from Corollary 1 that if G' is a proper nondegenerate subcollection of G, $(G')^*$ is not a continuous curve.

COROLLARY 2. *Suppose M is a compact continuous curve in the plane which is filled up by a continuous collection G of disjoint nondegenerate continua such that M has only two complementary domains such that their boundaries do not intersect. Then M is an annulus.*

Proof. Let D_1 and D_2 denote the complementary domains of M. By Theorem 4, $\mathrm{Bd}(D_1)$ and $\mathrm{Bd}(D_2)$ are scc. Now one of $\mathrm{Bd}(D_1)$ and $\mathrm{Bd}(D_2)$ separates the other from the ideal point. Otherwise, it follows that M is not compact. Suppose $\mathrm{Bd}(D_1)$ separates $\mathrm{Bd}(D_2)$ from the ideal point. Now $M \subset H = I_1 - \mathrm{Cl}(D_2)$, where I_1 is the interior of $\mathrm{Bd}(D_1)$. Furthermore, each point of H belongs to some element of G. If not, it follows that M would have more than two complementary domains. Hence, M is an annulus.

THEOREM 5. *Let G denote a continuous collection of disjoint continua is S such that* (1) *G is a compact continuous curve with respect to its elements,* (2) *no element of G separates S, and* (3) *if H is a subcollection of G which is an arc with respect to its elements, then H^* is a continuous curve and does not separate S. Then G is either an arc or a scc with respect to its elements.*

Proof. Let us suppose the contrary. Then G must contain a simple triod T of elements of G. Let $T = H_1 \cup H_2 \cup H_3$ where the H_i are arcs of elements in G having a common end element. By Theorem 3, H_i^* is a scc plus its interior. Let I_i denote the interior of $\mathrm{Bd}(H_i^*)$. It easily follows that $h \subset \mathrm{Bd}(H_i^*)$. Let A and B denote the end points of h, and let $X \in h$ which is distinct from both A and B. Let $A_i \in (\mathrm{Bd}(H_i^*) - h)$. Now $(\mathrm{Bd}(H_i^*) - h) = \varnothing$. Otherwise, it would follow that the elements of G are not disjoint. Now the arc (AA_2B) must lie outside $\mathrm{Bd}(H_i^*)$. Moreover, either X is without $J = \mathrm{Bd}(H_1^*) \cup \mathrm{Bd}(H_2^*)$ or A_1 is without $\mathrm{Bd}(H_2^*)$. Let us suppose X is without J. Then $A \in I_2$, contrary to the fact that $H_1 \cap H_2 = h$. Hence, A_1 is without $\mathrm{Bd}(H_2^*)$, and $(AXB) - (A \cup B)$ is a subset of the interior

I of J. Furthermore, $I = I_1 \cup I_2 \cup [(AXB) - (A \cup B)]$. Now $(AA_3B) - (A \cup B)$ must lie outside J since $\cap H_i = h$. Now each element of H_3 distinct from h must lie outside J. Otherwise, $\cap H_i \neq h$. But h is a limit element of H_3, and since $X \in h \cap I$, and G is a continuous collection, $I \cap (H_3 - h) \neq \emptyset$. Hence, G does not contain a simple triod. By Theorem 75, Chapt. IV of [3], G is an arc or a simple closed curve with respect to its elements.

THEOREM 6. *Under the hypothesis of Theorem* 5 *plus the additional stipulation that one of the elements is not an end element and is not a continuous curve, it follows that* G *must be an arc with respect to its elements.*

Proof. Let us suppose G is a scc with respect to its elements, and $g \in G$ is not a continuous curve. There is an arc hk of elements of G such that g is not an end element of hk. Let H denote the arc hg and K the arc gk. By Theorem 3, H^* and K^* arc scc plus their interiors. It easily follows that $g \subset \mathrm{Bd}(H^*) \cup \mathrm{Bd}(K^*)$. Since $\mathrm{Bd}(H^*)$ and $\mathrm{Bd}(K^*)$ are scc, it follows that g is a continuous curve, a contradiction. Since G is either a scc or arc with respect to its elements, G must be an arc with respect to its elements.

DEFINITION 2. The statement that the compact continuum C in S is circle-like means that C has only two complementary domains and C is the boundary of each.

In the following theorem, S is also assumed to be separable.

THEOREM 7. *Let* M *denote a compact continuum such that* (1) M *is filled up by a continuous collection* G *of disjoint continua*, (2) *all but countably many elements of* G *are circle-like, and* (3) *there is a point* $0 \notin M$ *such that no element of* G *separates two elements from one another and both from* 0 *in* S. *Then* G *is an arc with respect to its elements.*

Proof. First let us show that if $g_1, g_2 \in G$, then one of g_1 and g_2 separates the other from 0. Suppose the contrary. Then there exist elements g_1 and g_2 of G such that neither separates the other from 0. Let us now suppose that there is some element $g \in G$ which separates both g_1 and g_2 from 0. Let H_1 denote a collection such that $X \in H_1$ if and only if X separates g_1 from both 0 and g_2, or $X = g_1$. Let H_2 denote a collection such that $X \in H_2$ if and only if $X = g_2$ or X separates g_2 from both g_1 and 0. Let (AB) denote an arc such that $(AB) \cap g_1 = A$ and $(AB) \cap g_2 = B$. Let A' denote the first point in the order from B to A of $(AB) \cap \mathrm{Cl}(H_1^*)$, and let B' denote the first point in the order from A to B of $(AB) \cap \mathrm{Cl}(H_2^*)$. Since M is closed, there exist $h_1, h_2 \in G$ such that $A' \in h_1$ and $B' \in h_2$. Let (CD) denote an arc such that $g_1 \cap (CD) = C$ and $g \cap (CD) = D$. Let D' denote the first point in the order from C to D of $(CD) \cap \mathrm{Cl}(H^*)$, and let $h \in G$ such that $D' \in h$. Now $h_1 \in H_1$. Suppose the contrary, and suppose h_1 does not separate g_1 from 0. Then it follows that h_1 is a limit element of H_1. Let α denote an arc such that $0 \in \alpha$ and such that $\alpha \cap g_1 \neq \emptyset$ but $\alpha \cap h_1 = \emptyset$. There is a domain U containing h_1 such that $U \cap (g_1 \cup \alpha) = \emptyset$. Since G is a continuous collection and h_1 is a limit element of H_1, there is some $t \in H_1$ such that $t \subset U$. But since each element of H_1 separates g_1 from $0, t \cap \alpha \neq \emptyset$, a contradiction. Similarly, it may be shown that h_1 separates g_1 from g_2. Thus $h_1 \in H_1$. Likewise, it may be shown that $h \in H$ and $h_2 \in H_2$. Now $h_1 \neq h_2$. Otherwise this would imply that h_1 separates g_1 from g_2 and also separates both from 0, contrary to the hypothesis.

Now h and $h_1 \cup h_2$ are two closed subsets of G. Since G is a compact continuum with respect to its elements, there is a subcontinuum L of G which is irreducible from h to $h_1 \cup h_2$. Suppose t_1 and t_2 are two circle-like elements of $L - (h \cup h_1 \cup h_2)$ such that t_1 separates t_2 from 0. Let L_1 and L_2 denote subcontinua of L which are irreducible respectively from $h_1 \cup h_2$ to t_2 and from h to t_1, and let E denote the complementary domain of t_1 such that $t_2 \subset E$. It easily follows that $0 \in E$. Now $t_2 \subset L_1$ or $t_2 \subset L_2$. Suppose $t_2 \subset L_1$. Then $h_1 \subset E$ or $h_2 \subset E$, say h_1. Then it follows that $g_1 \subset E$ also. This means that t_1 separates g_1 from 0. If $g_2 \subset E$ also, it follows that $t_1 \in H_1$ contrary to the fact that h_1 is the outer most element of H_1. If $g_2 \subset E$, then $g_2 \in H_1$ again contrary to the fact that h_1 is the outer most such element. The assumption that $g_2 \subset E$ leads to the same contradictions. Hence, $t_2 \notin L_1$. Now suppose $t_2 \in L_2$. Then $h \in E$. As before neither $g_1 \subset E$ nor $g_2 \subset E$. This means that h does not separate both g_1 and g_2 from 0. Hence, it follows that $t_2 \in L_1$, a contradiction. Thus, no one of the circle like elements of $L - (h \cup h_1 \cup h_2)$ separates the other from 0. This means that there exist uncountably many disjoint domains in a separable space, a contradiction.

Let us suppose now that there is no element of G separating both g_1 and g_2 from 0. Let L denote a subcontinuum of G which is irreducible from h_1 to h_2, and again suppose there exist two circle-like elements t_1 and t_2 of $L - (h_1 \cup h_2)$ such that t_1 separates t_2 from 0. Again let L_1 and L_2 denote two subcontinua of L which are irreducible from h_1 to t_1 and from h_2 to t_1 respectively. Let E denote the complementary domain of t_1 such that $t_2 \subset E$. Now suppose $t_2 \in L_1$. Then it follows $g_1 \subset E$ which means that t_1 separates g_1 from 0. Now $g_2 \subset E$ since we are assuming that no element of G separates both g_1 and g_2 from 0. Hence, $t_1 \in H_1$, contrary to the fact that h_1 is the outer most element of H_1. A similar contradiction is reached when it is assumed that $t_2 \in L_2$. This again means that there are uncountably many disjoint domains. Hence, it must follow that if $g_1, g_2 \in G$, then one must separate the other from 0. Since G is a compact continuum, G has two noncut "points" A_1 and A_2. Now one must separate the other from 0, say A_1 separates A_2 from 0. Let $A_3 \in G$ such that $A_3 \neq A_1$ and $A_3 \neq A_2$. Now A_3 separates A_2 from 0 or vice versa. Suppose A_3 separates A_1 from 0. Then A_1 does not separate A_3 from 0. But A_1 does separate A_2 from 0. This means that A_1 is a cut point of G, a contradiction. Hence, A_3 does not separate A_1 from 0. Suppose A_2 separates A_3 from 0. Since A_2 does not separate A_1 from 0, we would have A_2 as a cut point of G, a contradiction. Hence A_3 separates A_2 from 0. This means that A_3 is a cut point of G. By definition, G is an arc from A_1 to A_2.

DEFINITION 3. We shall say that the compact continuum M in the plane is annulus-like if and only if there exist two circle-like subcontinua C_1 and C_2 of M such that if I_2 is the interior of C_2 and I_1 is the interior of C_1, then $M = C_1 \cup C_2 \cup (I_1 - \mathrm{Cl}(I_2))$.

THEOREM 8. *Let M denote a compact continuum in the plane which is filled up by a continuous collection of disjoint circle-like continua. Then M is annulus-like.*

Proof. By Theorem 7, G is an arc with respect to its elements. Let C_i denote the end points of G, and let I_i denote the interior of C_i, $i = 1, 2$. Let $L = I_1 - (I_2 \cup C_2)$, and let F denote a component of L. Then $F \cap (G - (C_1 \cup C_2)) \neq \emptyset$. Suppose the contrary. Then $\mathrm{Bd}(F) \subset C_2$ or $\mathrm{Bd}(F) \subset C_1$. For if $\mathrm{Bd}(F) \cap C_1 \neq \emptyset$ and $\mathrm{Bd}(F) \cap C_2 \neq \emptyset$, then since each element of $G - (C_1 \cup C_2)$ separates C_1 from C_2, then some element of G must intersect F. Suppose $\mathrm{Bd}(F) \subset C_1$. Now $I_1 - F \neq \emptyset$ since $I_2 \subset I_1$, and I_1 contains a point of F since $F \subset I_1$. This means $I_1 \cap C_1 \neq \emptyset$, a contradiction. A similar contradiction is reached by assuming $\mathrm{Bd}(F) \subset C_2$ using the exterior of C_2. Now L has only one component since $G - (C_1 \cup C_2)$ is connected and each component of L intersects some element of G. Now suppose $L - M \neq \emptyset$, and let $P \in L - M$. Let H_i denote a collection such that $X \in H_i$ if and only if $X = C_i$ or $X \in G$ which separates C_i from P, where $i = 1, 2$. As in Theorem 6, it follows that there is some outermost element h_i in H_i, $i = 1, 2$. Now $h_1 \neq h_2$ since each element is circle-like. Let h denote a point of the interval h_1h_2 of G distinct from both h_1 and h_2. It then follows that either h separates C_1 from P or h separates C_2 from P. In either case a contradiction would be reached since it would mean that h_i is not the outermost element in H_i, $i = 1, 2$.

Hence $M = C_1 \cup C_2 \cup L$. Thus M is annulus-like by definition.

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