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OPERATORS COMMUTING WITH BOOLEAN ALGEBRAS OF PROJECTIONS OF FINITE MULTIPLICITY

L. TZAFRIRI

The purpose of this paper is to study the algebra Ξ of all the operators which commute with a complete countably decomposable Boolean algebra of projections of uniform finite multiplicity. It will be shown that the sum, product and strong limit (under suitable conditions) of spectral operators in Ξ are also spectral.

Foguel [11], using a matricial representation of operators in Ξ, has obtained the first results in this direction; his methods will be often used in this paper.

We shall start by giving a necessary and sufficient condition for an operator in Ξ to be spectral, which is similar to a theorem of Foguel in Hilbert spaces (see [10, Th. 3.2]). Relying on this theorem we can prove that the sum and the product of a finite number of commuting spectral operators in Ξ are spectral even in the case when the underlying space is not weakly complete. This result generalizes a theorem of McCarthy [12] which asserts the boundedness of the Boolean algebra of projections generated by two bounded commuting Boolean algebras of projections such that one of them is complete, countably decomposable and contains no projection of infinite multiplicity.

Later we shall study the convergence of the operators of Ξ. In connection with these problems, it should be pointed out that one of the most important questions in the theory of spectral operators is to decide when the strong limit of a sequence of spectral operators is a spectral operator. Bade has established in [1] that the strong limit of a sequence of spectral operators of scalar type on a reflexive space is a spectral operator of scalar type if their resolutions of the identity are uniformly bounded and their spectra lie in a fixed R-set (for instance the real line). We prove here, and this seems to be the main result of this paper, that the strong limit of a sequence of spectral operators commuting with a Boolean algebra of projections of uniform finite multiplicity (the operators need not commute) is a spectral operator provided that their resolutions of the identity are uniformly bounded (with no restriction on the distribution of their spectra or on the underlying space).

1. Preliminaries. Our notation is essentially that of Bade [3] and Foguel [11]. For convenience we shall summarize here some facts from these two papers.

A Boolean algebra of projections $\mathscr{E}$ will be called complete if for every family $(E_\alpha) \subseteqq \mathscr{E}$ the projections $\bigvee E_\alpha$ and $\bigwedge E_\alpha$ exist in $\mathscr{E}$ and, moreover

$$(1.1) \qquad \begin{aligned} (\bigvee E_\alpha)X &= \operatorname{clm}\{E_\alpha X\} ; \\ (\bigwedge E_\alpha)X &= \bigcap E_\alpha X \end{aligned}$$

A projection $E \in \mathscr{E}$ will be called countably decomposable if every family of disjoint projections in $\mathscr{E}$ bounded by E is at most countable. If the identity $I \in \mathscr{E}$ satisfies this condition $\mathscr{E}$ will be called countably decomposable.

Throughout the paper X will denote a fixed Banach space and $\mathscr{E}$ a complete countably decomposable Boolean algebra of projections having uniform multiplicity N; $N < \infty$. (see [3, Def. 2.1]). In view of [6, Th. XVII-3-9], $\mathscr{E}$ can be considered as the range of a spectral measure $E(\cdot)$ defined on the Borel sets of a compact Hausdorff space Ω. Thus by Bade [3] there exist N vectors $x_1, x_2, \cdots, x_N$ and N bounded linear functionals $x_1^*, x_2^*, \cdots, x_N^*$ such that:

$$(1.2) \qquad X = \bigvee_{k=1}^{N} \mathfrak{M}(x_k)$$

where $\mathfrak{M}(x) = \operatorname{clm}\{E(\alpha)x \mid \alpha \text{ a Borel set}\}$; the measures $\mu_k = x_k^* E(\cdot)x_k$ are positive and equivalent and $x_i^* E(\alpha)x_k = 0$ for every Borel set α and $i \neq k$. Bade has also shown that there exists a linear continous one-to-one map T of X onto a dense subspace of $\sum_{k=1}^{N} L(\Omega, \mu_k)$. Following [11], instead of writing

$$Tx = \begin{pmatrix} f_1(\omega) \\ \vdots \\ f_N(\omega) \end{pmatrix}$$

we shall use the notation

$$x \sim \begin{pmatrix} f_1(\omega) \\ \vdots \\ f_N(\omega) \end{pmatrix}$$

In what follows we shall denote by M_0 a constant such that $\| E \| \leqq M_0$ for every $E \in \mathscr{E}$.

Let Ξ be the algebra of bounded operators which commute with all the projections of $\mathscr{E}$. Foguel has proved in [11] that for every $A \in \Xi$ there corresponds a matrix of measurable functions

$$a_{ij}(\omega) \in L(\Omega, \mu_i) \; ; \qquad 1 \leqq i, j \leqq N$$

such that

$$Ax \sim (a_{ij}(\omega))\begin{pmatrix} f_i(\omega) \\ \vdots \\ f_N(\omega) \end{pmatrix}.$$

This matrix may be decomposed in the form:

$$(a_{ij}(\omega)) = \sum_{k=1}^{N} z_k(\omega)\varepsilon_k(\omega) + \mathcal{N}(\omega) \tag{1.3}$$

where $z_k(\omega); 1 \leqq k \leqq N$ are measurable bounded functions ($|z_k(\omega)| \leqq \|A\|$ a.e.) and $\varepsilon_k(\omega); 1 \leqq k \leqq N$ and $\mathcal{N}(\omega)$ are matrices of measurable functions satisfying:

$$\begin{aligned} \text{if } i \neq j \quad \varepsilon_i(\omega)\varepsilon_j(\omega) &= 0 ; \\ \varepsilon_k^2(\omega) &= \varepsilon_k(\omega) ; \\ \varepsilon_k(\omega)\mathcal{N}(\omega) &= \mathcal{N}(\omega)\varepsilon_k(\omega) ; \\ (\mathcal{N}(\omega))^N &= 0 . \end{aligned} \qquad 1 \leqq k \leqq N \tag{1.4}$$

Moreover, there exist N Borel sets $\beta_1, \beta_2, \cdots, \beta_N$ whose union is Ω such that on β_k the numbers $z_1(\omega), \cdots, z_k(\omega)$ are different while

$$z_{k+1}(\omega) = \cdots = z_N(\omega) = 0 .$$

2. Sums and products of spectral operators. Let $A \in \Xi$ be an operator having the matricial representation (1.3). Foguel [10, Th. 3.2] has proved that if the underlying space is a Hilbert space then A is spectral whenever the components of the matrices $\varepsilon_k(\omega); 1 \leqq k \leqq N$ are a.e. uniformly bounded. Giving an adequate example, [11, p. 690], he has shown that this theorem cannot be generalized to Banach spaces. Changing the statement of this above mentioned theorem of Foguel we can prove the next result.

THEOREM 1. *An operator $A \in \Xi$ having the matricial representation* (1.3) *is spectral if and only if there exist projection $E_k \in \Xi$; $1 \leqq k \leqq N$ corresponding to the matrices $\varepsilon_k(\omega); 1 \leqq k \leqq N$.*

Proof.[1] Assume the existence of the projections $E_k \in \Xi; 1 \leqq k \leqq N$ whose matricial representations are $\varepsilon_k(\omega); 1 \leqq k \leqq N$. By [11, Lemma 2.2] $|z_k(\omega)| \leqq \|A\|$ a.e. $1 \leqq k \leqq N$ and therefore by Th. 2.1 of the same paper

$$A = \sum_{k=1}^{N} \left(\int z_k(\omega) E(d\omega) \right) E_k + \mathcal{N} \tag{2.1}$$

[1] The idea of this proof has been communicated to the author by Foguel.

where $\mathscr{N}$ is a nilpotent of order N commuting with E_k; $1 \leqq k \leqq N$ and E_k are commuting disjoint projections whose sum is the identity (see (1.4)). According to [6, Th. XVI-5-3], A is a spectral operator of finite type and its resolution of the identity is

$$\sum_{k=1}^{N} E(z_k^{-1}(\cdot))E_k \, . \tag{2.2}$$

Conversely, if $A \in \Xi$ is spectral and $\mathscr{S}$ is its scalar part then by [11, Th. 2.3] there exists a sequence of Borel sets $\alpha_p \subseteqq \Omega$; $p = 1, 2, \cdots$, increasing to Ω and such that

$$AE(\alpha_p) = \sum_{k=1}^{N} \Big(\int_{\alpha_p} z_k(\omega) E(d\omega) \Big) E_{k,p} + \mathscr{N}_p \, ; \qquad p = 1, 2, \cdots \tag{2.3}$$

where $E_{k,p}$ are disjoint projections whose sum is $E(\alpha_p)$ and $\mathscr{N}_p$ is a nilpotent of order N. By [4, Th. 8] we get

$$\mathscr{S}E(\alpha_p) = \sum_{k=1}^{N} \Big(\int_{\alpha_p} z_k(\omega) E(d\omega) \Big) E_{k,p} \, ; \qquad p = 1, 2, \cdots \, .$$

Thus $\chi_{\alpha_p}(\omega) \sum_{k=1}^{N} z_k(\omega)\varepsilon_k(\omega)$ corresponds to $\mathscr{S}E(\alpha_p)$. By [11, Th. 2.2] we see that $\sum_{k=1}^{N} z_k(\omega)\varepsilon_k(\omega)$ is a matricial representation for $\mathscr{S}$.

Now, let us assume that $\mathscr{G} = \{G(\delta); \delta$ a Borel set on the plane$\}$ is the resolution of the identity of A (or $\mathscr{S}$) and $\|G\| \leqq M_1$ for every $G \in \mathscr{G}$. By McCarthy [12], the Boolean algebra of projections $\mathscr{F}$ generated by $\mathscr{E}$ and $\mathscr{G}$ is bounded and $\|F\| \leqq 8\sqrt{N\pi}\ M_0 M_1 = M$; $F \in \mathscr{F}$. Thus by [6, Lemmas 1 and 2], the uniformly closed algebra of operators, $\mathfrak{A}(\mathscr{F})$ generated by $\mathscr{F}$ is a full algebra equivalent to the algebra of continuous functions on its own space of maximal ideals $C(\mathfrak{M})$. Furthermore,

$$\|U\| \leqq 4M \sup_{m \in \mathfrak{m}} |U(m)| \, ; \qquad U \in \mathfrak{A}(\mathscr{F}) \, .$$

But from the explicit form of the projections $E_{k,p}$ given in [11, p. 687] (where instead of the powers of A we can put those of $\mathscr{S}$) we have $E_{k,p} \in \mathfrak{A}(\mathscr{F})$; $1 \leqq k \leqq N$; $p = 1, 2, \cdots$. Since $E_{k,p}$ are projections $E_{k,p}(m)$ will be characteristic functions, i.e. $\|E_{k,p}\| \leqq 4M$; $1 \leqq k \leqq N$; $p = 1, 2, \cdots$. Now, let us remark that

$$E_{1,p} + E_{2,p} + \cdots + E_{N,p} = E(\alpha_p) \, ; \qquad p = 1, 2, \cdots$$

and by multiplication with $E_{k,m}$; $1 \leqq k \leqq N$; $m \geqq p$ we get

$$E_{k,p} = E_{k,m} E(\alpha_p)$$

(in fact all these relations are satisfied by the respective matricial representations and thus, by [11, Th. 2.1], hold for operators too). Then for every $x \in X$

$$\| E_{k,m}x - E_{k,p}x \| = \| E_{k,m}(x - E(\alpha_p)x) \| \leqq 4M \| x - E(\alpha_p)x \|$$

and we can define $E_k x = \lim_{m\to\infty} E_{k,m}x$; $x \in X$. It is easy to see that E_k; $1 \leqq k \leqq N$ will be disjoint projections belonging to Ξ and corresponding to the matrices $\varepsilon_k(\omega)$; $1 \leqq k \leqq N$.

LEMMA 2. *Let $P_k \in \Xi$; $k = 1, 2, \cdots, n$ be commuting projections. Then the Boolean algebra of projections generated by them and $\mathscr{E}$ is complete.*

Proof. One can easily see that it suffices to prove the assertion for $n = 1$. Let us remark that the Boolean algebra of projections generated by $\mathscr{E}$ and a projection $P \in \Xi$ has the form:

$$\mathscr{F} = \{EP + \tilde{E}P' \mid E, \tilde{E} \in \mathscr{E}\}$$

where $P' = I - P$. In order to show that $\mathscr{F}$ is complete let us observe that

$$\bigvee_\alpha \left(E_\alpha P + \tilde{E}_\alpha P'\right) = \left(\bigvee_\alpha E_\alpha\right)P + \left(\bigvee_\alpha \tilde{E}_\alpha\right)P' \in \mathscr{F}$$

$$\bigwedge_\alpha \left(E_\alpha P + \tilde{E}_\alpha P'\right) = \left(\bigwedge_\alpha E_\alpha\right)P + \left(\bigwedge_\alpha \tilde{E}_\alpha\right)P' \in \mathscr{F}$$

for $\mathscr{E}$ is complete. Thus $\mathscr{F}$ is complete as an abstract Boolean algebra (see [3, p. 509]) and we will conclude by proving that in $\mathscr{F}$ every family of projections satisfies (1.1). Indeed,

$$\begin{aligned}
&\left[\left(\bigvee_\alpha E_\alpha\right)P + \left(\bigvee_\alpha \tilde{E}_\alpha\right)P'\right]X \\
&= \operatorname{clm}\left[\left(\bigvee_\alpha E_\alpha\right)PX \cup \left(\bigvee_\alpha \tilde{E}_\alpha\right)P'X\right] \\
&= \operatorname{clm}\left\{\left[PX \cap \left(\bigvee_\alpha E_\alpha\right)X\right] \cup \left[P'X \cap \left(\bigvee_\alpha \tilde{E}_\alpha\right)X\right]\right\} \\
&= \operatorname{clm}\left\{\left[PX \cap \operatorname{clm}\left(\bigcup_\alpha E_\alpha X\right)\right] \cup \left[P'X \cap \operatorname{clm}\left(\bigcup_\alpha \tilde{E}_\alpha X\right)\right]\right\} \\
&= \operatorname{clm}\left\{\left(\bigcup_\alpha PE_\alpha X\right) \cup \left(\bigcup_\alpha P'\tilde{E}_\alpha X\right)\right\} \\
&= \operatorname{clm}\left[\bigcup_\alpha (PE_\alpha X \cup P'\tilde{E}_\alpha X)\right] \\
&= \operatorname{clm}\left[\bigcup_\alpha (PE_\alpha + P'\tilde{E}_\alpha)X\right]
\end{aligned}$$

and in a similar way for the greatest lower bound.

McCarthy has proved in [12] that the Boolean algebra of projections generated by two bounded commuting Boolean algebras of projections is bounded provided one of the original algebras is complete, countably decomposable, and contains no projection of infinite multiplicity. Dunford [4, Th. 19] and Foguel [7, Th. 7] have shown that if the Boolean algebra of projections generated by the resolutions of the identity of two bounded commuting spectral operators on a weakly complete Banach space is bounded, then the sum and the product of these operators are spectral. In the following theorem we prove the spectrality of the sum and product of two commuting spectral operators in Ξ with no restriction on the underlying space or on their resolutions of the identity.

THEOREM 3. *Let $A', A'' \in \Xi$ be two commuting spectral operators. Then $A' + A''$ and $A'A''$ are spectral. Moreover, the Boolean algebra of projections generated by their resolutions of the identity may be imbedded in a complete Boolean algebra of projections and thus is bounded.*

Proof. By Theorem 1 there exist projections E'_k and E''_k; $1 \leqq k \leqq N$ such that

$$A' = \sum_{k=1}^{N} \left(\int z'_k(\omega)E(d\omega)\right)E'_k + \mathscr{N}'$$
$$A'' = \sum_{k=1}^{N} \left(\int z''_k(\omega)E(d\omega)\right)E''_k + \mathscr{N}'' .$$

But from [11, p. 687] it follows that $E'_{k,p}$ commutes with $E''_{j,m}$ for every $1 \leqq k, j \leqq N$; $p, m = 1, 2, \cdots$. Since $E'_k x = \lim_{p\to\infty} E'_{k,p}x$; $x \in X$, and $E''_j x = \lim_{m\to\infty} E''_{j,m}x$; $x \in X$ we have that E'_k commutes with E''_j; $1 \leqq k, j \leqq N$. Denote

$$F_{k,j} = E'_k E''_j = E''_j E'_k$$

and

$$Z'_{k,j}(\omega) = Z'_k(\omega) \; ; \qquad Z''_{k,j}(\omega) = Z''_j(\omega) \; .$$

Then

$$A' + A'' = \sum_{k,j=1}^{N} \left(\int (z'_{k,j}(\omega) + z''_{k,j}(\omega))E(d\omega)\right)F_{k,j} + (\mathscr{N}' + \mathscr{N}'')$$

where $F_{k,j}$ are disjoint projections commuting with A' and A''. By [6, XVI-5-3] it will follow that $A' + A''$ is spectral. Therefore $(A' + A'')^2$ and $-(A' - A'')^2$ are spectral operators belonging to Ξ and by the first part of the proof we can conclude that

$$A'A'' = \frac{1}{4}[(A' + A'')^2 - (A' - A'')^2]$$

is also spectral.

In order to prove the last statement it suffices to observe that by Lemma 2 the Boolean algebra of projections generated by $\mathscr{E}$ and the projections E_k', E_k''; $1 \leqq k \leqq N$ is complete and contains the resolutions of the identity of both the operators which are explicitly given by (2.2). Using Bade [2, Th. 2.2] we get that this Boolean algebra of projections is bounded.

The next theorem will show that in most cases we are actually dealing with a separable space.

THEOREM 4. *Let A be a spectral operator on the Banach space X, $\mathscr{E} = \{E(\cdot)\}$ its resolution of the identity defined on the Borel sets of the complex plane. The underlying space X will be separable if and only if*

(a) *$\mathscr{E}$ is complete and countably decomposable.*

(b) *The multiplicity of each $E \in \mathscr{E}$ is not greater than $\aleph_0$.*

Proof. The necessity is obvious. Conversely, if (a) and (b) hold, then, by Bade [3, Def. 3.2]

$$X = \operatorname{clm}\{\mathfrak{M}(x) \mid x \in Y\}$$

where the cardinal power of the set Y is not greater than $\aleph_0$ and therefore, one can easily see that it suffices to show that $\mathfrak{M}(x)$ is a separable subspace for every $x \in Y$.

Let $\{e_n\}$; $n = 1, 2, \cdots$ be a countable base of the topology of the complex plane. Denote $\mathfrak{M} = \operatorname{clm}\{E(e_n)x; n = 1, 2, \cdots\}$. Since $\mathscr{E}$ is a complete Boolean algebra of projections $E(\delta)x \in \mathfrak{M}$ for every open set δ. Now, if e is a Borel set and $E(e)x \notin \mathfrak{M}$ then by Hahn-Banach theorem we can find a bounded linear functional $x^* \in X^*$ satisfying;

$$x^*E(e)x = 1 ; \qquad x^*z = 0 \qquad\qquad z \in \mathfrak{M}$$

and this fact contradicts the regularity of the scalar measure $x^*E(\cdot)x$. Thus $\mathfrak{M}(x) = \mathfrak{M}$ is separable.

3. Convergence in Ξ. In this section we shall study properties concerning strong convergence in Ξ. Let $A, A_n \in \Xi$; $n = 1, 2, \cdots$ be operators having the correspondent matricial representations

$$(a_{ij}(\omega)),\ (a_{ij}^{(n)}(\omega)) ; \qquad\qquad n = 1, 2, \cdots .$$

Foguel [10, Th. 2.3] has proved that on a Hilbert space the sequence of operators $\{A_n\}$ converges strongly to the operator A if and only if:

(i) the sequence of functions $\{a_{ij}^{(n)}(\omega)\}$ converges in measure to $a_{ij}(\omega)$.

(ii) the sequence $\{\|A_n\|\}$ is bounded.

In the general case of Banach spaces he has proved in [11, Th. 2.2] the sufficiency of this theorem under the following additional assumption

(iii) $\bigcup_{k=1}^{\infty}\{\omega \mid |a_{ij}^{(n)}(\omega)| \leqq k; 1 \leqq i, j \leqq N; n = 1, 2, \cdots\} = \Omega$.

We shall show that it is superflous. Indeed, in the proof of the previous theorem, Foguel has shown that (relying only on (i) and (ii))

$$\lim_{n\to\infty} A_n E(e) x_k = AE(e) x_k\ ; \qquad 1 \leqq k \leqq N$$

for every Borel set e, on which the functions $a_{ij}^{(n)}(\omega)$ are uniformly bounded. Denote

$$\Sigma_s = \{\omega \mid s - 1 \leqq |a_{ij}(\omega)| \leqq s; i, j = 1, 2, \cdots, N\} \qquad s = 1, 2, \cdots .$$

Since $a_{ij}(\omega)$, $1 \leqq i, j \leqq N$, are almost everywhere finite functions (they belong to $L(\Omega, \mu_i)$) the set $\Sigma_0 = \Omega - \bigcup_{s=1}^{\infty}\Sigma_s$ is a null set (it does not matter with respect to what measure because all the measures $\mu_k = x_k^* E(\cdot)x_k$; $1 \leqq k \leqq N$ are finite and equivalent). Now, let us remark that (i) insures the exsistence of a subsequence of the sequence $\{a_{ij}^{(n)}(\omega)\}$ converging to $a_{ij}(\omega)$ a.e. In order to simplify the notation we shall continue to denote this subsequence by $\{a_{ij}^{(n)}(\omega)\}$. By a theorem of Lusin (see [5, Corr. III-6-3]) the sequence $\{a_{ij}^{(n)}(\omega)\}$ converges μ-uniformly to $a_{ij}(\omega)$ i.e., there exists a sequence of sets $\Omega_m \subset \Omega$; $m = 1, 2, \cdots$ such that $\Omega_0 = \Omega - \bigcup_{m=1}^{\infty}\Omega_m$ is a null set and $\{a_{ij}^{(n)}(\omega)\}$ converges uniformly to $a_{ij}(\omega)$ on every set Ω_m. Also with no loss of generality we can assume that they are disjoint. Thus for every $s = 1, 2, \cdots$ and $m = 1, 2, \cdots$ we can find an integer $n(s, m)$ such that

$$|a_{ij}^{(n)}(\omega)| < s + 1\ ; \qquad n \geqq n(s, m); 1 \leqq i, j \leqq N; \omega \in \Omega_m \cap \Sigma_s .$$

Consequently, for every Borel set δ and $s, m = 1, 2, \cdots$

$$\lim_{n\to\infty} A_n E(\delta \cap \Omega_m \cap \Sigma_s) x_k = AE(\delta \cap \Omega_m \cap \Sigma_s) x_k\ ; \qquad k = 1, 2, \cdots, N .$$

Since

$$E(\delta) x_k = \sum_{s,m=1}^{\infty} E(\delta \cap \Omega_m \cap \Sigma_s) x_k\ ; \qquad k = 1, 2, \cdots, N$$

it follows from (ii) that

$$\lim_{n\to\infty} A_n E(\delta) x_k = AE(\delta) x_k\ ; \qquad k = 1, 2, \cdots, N .$$

Using (1.2) we can see that $\lim_{n\to\infty} A_n x = Ax$ for x in a dense set of X. Thus by (ii) $\lim_{n\to\infty} A_n x = Ax$; $x \in X$.

In conclusion we have proved that every sequence of operators $A_n \in \Xi$; $n = 1, 2, \cdots$ satisfying (i) and (ii) has a subsequence converging strongly to A. If the original sequence does not converge strongly to A, then we can find a vector $x \in X$, a number $\varepsilon > 0$ and a subsequence $\{A_{n_i}\}$ such that $\| A_{n_i} x - Ax \| \geqq \varepsilon$; $i = 1, 2, \cdots$. This fact contradicts the existence of a subsequence of $\{A_{n_i}\}$ which converges strongly to A.

The next theorem is a straightforward consequence of the preceding discussion and [11, Th. 2.2].

THEOREM 5. *Let $A, A_n \in \Xi$; $n = 1, 2, \cdots$ have the matricial representation $(a_{ij}(\omega))$ and $(a_{ij}^{(n)}(\omega))$; $n = 1, 2, \cdots$. Then $\{A_n\}$ converges strongly to A if and only if conditions* (i) *and* (ii) *are fulfilled.*

Let us now reproduce some unpublished results and proofs concerning measure theory (Lemma 6 and Theorem 7) which are due to M. O. Rabin.

To every $0 < u < \infty$, let us define a total order on the complex plane as follows. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then, there exist two real numbers v_1 and v_2 such that

$$y_j = ux_j + v_j \; ; \qquad j = 1, 2 \, .$$

We shall say that z_2 is greater than z_1 with respect to the order (u) if $v_1 > v_2$ or if $v_1 = v_2$ then $y_1 < y_2$. It will be easy to see that this order is a lexicographic one with respect to the straight lines having the same inclination u.

LEMMA 6. *Let μ be a positive countable additive measure on the measure space (S, Σ) and $P(s, z) = \sum_{k=0}^{N} c_k(s) z^k$; $c_N(s) \equiv 1$; $s \in S$ a polynomial whose coefficients are μ-measurable functions. Denote by $\zeta_u(s) = x_u(s) + iy_u(s)$ the greatest root of $P(s, z)$ with respect to the order (u). Assume that it lies on the straight line.*

$$y = ux + v(s) \qquad s \in S \, . \tag{3.1}$$

Then, the set G_u of all $s \in S$ for which there are at least two different roots of $P(s, z)$ on (3.1), *is μ-measurable.*

Proof. Put

$$S_j = \left\{ s \mid s \in S \begin{array}{l} (z - \zeta_u(s))^j \text{ divides } P(s, z) \text{ and} \\ (z - \zeta_u(s))^{j+1} \text{ does not divide it} \end{array} \right\} ; \qquad 1 \leqq j \leqq N \, .$$

Since $\zeta_u(s)$ is a measurable function (see the proof of [10, Lemma 3.1]) the sets $S_j; 1 \leqq j \leqq N$ will be measurable. Denote

$$Q_j(s, z) = \frac{P(s, z)}{(z - \zeta_u(s))^j} \qquad s \in S_j; 1 \leqq j \leqq N .$$

This will be a new polynomial whose coefficients are also measurable functions on S_j. If $\zeta_{u,j}(s) = x_{u,j}(s) + jy_{u,j}(s)$ is its first (greatest)root with respect to the order (u) then $\zeta_{u,j}(s)$ is, in fact, the second different root of $P(s, z)$; $s \in S_j$. Thus

$$G_u \cap S_j = \left\{s \mid s \in S_j \frac{y_u(s) - y_{u,j}(s)}{x_u(s) - x_{u,j}(s)} = \operatorname{Arctg} u\right\} \qquad 1 \leqq j \leqq N$$

are measurable sets since $\zeta_u(s)$ and $\zeta_{u,j}(s)$ are measurable functions on S_j; $1 \leqq j \leqq N$. Therefore $G_u = \bigcup_{j=1}^{N} (G_u \cap S_j)$ is measurable too.

Now, we can prove Rabin's theorem.

THEOREM 7. *Let μ be a positive σ-finite countable additive measure on the measure space (S, Σ), $P(s, z) = \sum_{k=0}^{N} c_k(s)z^k$; $P_n(s, z) = \sum_{k=0}^{N} c_k^{(n)}(s)z^k$; $c_N(s) \equiv c_N^{(n)}(s) \equiv 1$; $n = 1, 2, \cdots$ polynomials whose coefficients are μ-measurable functions such that*

$$\lim_{n\to\infty} c_k^{(n)}(s) = c_k(s) ; \qquad 0 \leqq k < N$$

for a.e. $s \in S$. Then, there exist an arrangement $\{\zeta_1(s), \zeta_2(s), \cdots, \zeta_N(s)\}$ of the roots of $P(s, z)$ and $\{\zeta_1^{(n)}(s), \zeta_2^{(n)}(s), \cdots, \zeta_N^{(n)}(s)\}$ of the roots of $P_n(s, z)$; $n = 1, 2, \cdots$ such that $\zeta_h(s), \zeta_h^{(n)}(s)$; $1 \leqq h \leqq N$ are measurable functions and

$$\lim_{n\to\infty} \zeta_h^{(n)}(s) = \zeta_h(s) ; \qquad 1 \leqq h \leqq N \tag{3.2}$$

for a.e. $s \in S$.

Proof. To every order (u); $0 < u < \infty$, define $\zeta_u(s)$ and $\zeta_u^{(n)}(s)$ be the greatest root of $P(s, z)$ and $P_n(s, z)$ respectively. As we have pointed out in the proof of the previous lemma, they are measurable functions. Denote

$$D_u = \{s \mid s \in S \lim_{n\to\infty} \zeta_u^{(n)}(s) \neq \zeta_u(s)\} .$$

Then $D_u \subset G_u$. Indeed, if $s \notin G_u$ then $\zeta_u(s)$ is the unique root of $P(s; z)$ which lies on the right most line (3.1) which contains a root of $P(s; z)$; therefore, the second different root of $P(s; z)$ will lie on a line $y = ux + v'(s)$ where $v'(s) - v(s) = d(s) > 0$. Now, describe circles around the roots of $P(s; z)$, each one of radius $0 < \varepsilon < d(s)/3$. According to [13, p. 125–126], there exists an integer $n(\varepsilon, s)$ such that for $n \geqq n(\varepsilon, s)$ the roots of $P_n(s; z)$ will enter the circles (in every circle there will be a number of roots of $P_n(s; z)$ equal to the multiplicity of the center considered as a root of $P(s, z)$). Thus $\zeta_u^{(n)}(s); n \geqq n(\varepsilon, S)$ will be necessarily in the circle around $\zeta_u(s)$ and consequently, $s \notin D_u$.

Hence, it should be observed that it suffices to prove that there exists at least one number $u; 0 < u < \infty$ such that $\mu(G_u) = 0$ since the same process can be applied to

$$\frac{P(s; z)}{z - \zeta_u(s)}$$

and continued until the N roots are found.

The measure u is σ-finite, i.e. there exists a sequence of disjoint measurable sets $\{S_r\}$ such that $S = \bigcup_{r=1}^{\infty} S_r$ and $0 < \mu(S_r) < \infty; r = 1, 2, \cdots$. If we suppose that $\mu(G_u) > 0; 0 < u < \infty$ then we can find an integer $1 \leqq r < \infty$ such that $\mu(G_u \cap S_r) > 0$ for $u \in U$ where the cardinal power of U is $\aleph$. Split U into $\aleph$ disjoint sets U_α; each of them having the cardinal power $\aleph$. One can easily see that for every α there exist two sets $G_{u'} \cap S_r$ and $G_{u''} \cap S_{r'}; u', u'' \in U_\alpha$, whose intersection has a positive measure. Since we have $\aleph$ pairs of such sets, by applying the same proceeding to the above-mentioned intersections we shall be able to get $\aleph$ sequences; each of them consisting of four sets $G_u \cap S_r$ whose intersection has again a positive measure. Repeating the process L times ($L = 1, 2, \cdots$) we shall insure the existence of a finite sequence of sets $\{G_{u_l} \cap S_r\}; 1 \leqq l \leqq 2^L$ such that $\bigcap_{l=1}^{2^L} (G_{u_l} \cap S_r) \neq \emptyset$ i.e. $\bigcap_{l=1}^{2^L} G_{u_l} \neq \emptyset$. If we choose L with $2^L > \binom{N}{2}$ and $s_0 \in \bigcap_{l=1}^{2^L} G_{u_l}$ then there are 2^L different straight lines and on each of them there exist at least two roots of $P(s_0; z)$, i.e., we have a contradiction.

THEOREM 8. *Let $A_n \in \Xi; n = 1, 2, \cdots$ be a sequence of spectral operators converging strongly to an operator A. Let $G(\cdot, A_n)$ denotes the resolution of the identity for A_n. If there is a constant K such that $\| G(\delta, A_n) \| \leqq K; n = 1, 2, \cdots, \delta \in$ Borel sets, then A is spectral (of type N). Moreover, the sequence of the scalar parts of A_n converges strongly to the scalar part of A and the same assertion holds with respect to the nilpotent parts, real parts and imaginary parts (see Foguel* [8, *Th.* 1 p. 59]).

Proof. Let $(a_{ij}(\omega)), (a_{ij}^{(n)}(\omega))$ be the respective matricial representations of A and A_n. By Theorem 5 the sequence $\{a_{ij}^{(n)}(\omega)\}$ converges in measure to $a_{ij}(\omega)$ for all $i, j = 1, 2, \cdots, N$. Hence we can find a subsequence which converges almost everywhere to $a_{ij}(\omega)$. For simplicity of the notation we shall continue to denote this subsequence by $\{a_{ij}^{(n)}(\omega)\}$.

Now, for $\omega \in \Omega$, let

$$P(\omega, \zeta) = \sum_{h=0}^{N} c_h(\omega)\zeta^h$$

$$P_n(\omega, \zeta) = \sum_{h=0}^{N} c_h^{(n)}(\omega)\zeta^h \qquad n = 1, 2, \cdots$$

be the characteristic polynomials of the above mentioned matrices One can easily see that

$$\lim_{n\to\infty} c_h^{(n)}(\omega) = c_h(\omega) ; \qquad 0 \leqq h \leqq N \tag{3.3}$$

for a.e. $\omega \in \Omega$. Thus, according to Theorem 7, there is an arrangement of the roots $\{\zeta_1(\omega), \zeta_2(\omega), \cdots, \zeta_N(\omega)\}$ of $P(\omega, \zeta)$ and one of the roots $\{\zeta_1^{(n)}(\omega), \zeta_2^{(n)}(\omega), \cdots, \zeta_N^{(n)}(\omega)\}$ of $P_n(\omega, \zeta)$ such that

$$\lim_{n\to\infty} \zeta_h^{(n)}(\omega) = \zeta_h(\omega) ; \qquad 1 \leqq h \leqq N \tag{3.4}$$

for a.e. $\omega \in \Omega$. In order to prove the theorem we shall need to find a decomposition of the type (1.3) for $(a_{ij}(\omega))$ and $(a_{ij}^{(n)}(\omega))$ that will have some additional properties.

Consider the (finite) class of all partitions π of the set of integers $\{1, \cdots, N\}$; thus π is a collection $\Delta_1, \cdots, \Delta_{n(\pi)}$ of nonempty disjoint subsets of $\{1, \cdots, N\}$ whose union is $\{1, \cdots, N\}$. We write $j = k(\mathrm{mod}\, \pi)$ if j and k are both in the same subset of $\{1, \cdots, N\}$ of π.

Define $W_\pi = \{\omega \mid \zeta_k(\omega) = \zeta_l(\omega)$ whenever $k = l(\mathrm{mod}\, \pi)\}$. For each π, W_π is measurable: it is simply the zero set of

$$\sum_{\Delta\in\pi} \sum_{k,l\in\Delta} | \zeta_k(\omega) - \zeta_l(\omega) |$$

minus the zero set of

$$\prod_{\Delta\in\pi} \prod_{\substack{k\in\Delta \\ l\notin\Delta}} | \zeta_k(\omega) - \zeta_l(\omega) | .$$

Further, the various W_π are disjoint. Now define

$$W_{\pi,m} = \left\{\omega \in W_\pi \,\middle|\, | \zeta_k(\omega) - \zeta_l(\omega) | > \frac{1}{m} \text{ if } k \neq l(\mathrm{mod}\, \pi)\right\} .$$

Then $W_{\pi,m}$ is increasing in m, is measurable and $\bigcup_{m=1}^{\infty} W_{\pi,m} = W_\pi$.

Now let S be a subset of W_π on which the convergence of $\{\zeta_h^{(n)}(\omega)\}$ to $\zeta_h(\omega)$; $1 \leqq h \leqq N$ is uniform. Then there is an integer $n(S, \pi, m)$ such that $| \zeta_h^{(n)}(\omega) - \zeta_h(\omega) | < 1/10m$; $h = 1, \cdots, N$; $\omega \in S \cap W_{\pi,m}$; $n \geqq n(S, \pi, m)$, so that for $\omega \in S \cap W_{\pi,m}$, $| \zeta_k^{(n)}(\omega) - \zeta_l^{(n)}(\omega) | < 7/10m$ implies that $k = l(\mathrm{mod}\, \pi)$. Since $W_{\pi,m}$ are increasing in m we have

$$(3.5) \qquad |\zeta_k^{(n)}(\omega) - \zeta_l^{(n)}(\omega)|_{n\to\infty} \longrightarrow 0$$

uniformly for $\omega \in S \cap W_{\pi,m}$ whenever $k = l (\operatorname{mod} \pi)$. For each class $\Delta_s \in \pi; 1 \leqq s \leqq n(\pi)$ choose the first integer h_s and call $\zeta_{h_s}(\omega) = z_s(\omega)$; $\omega \in W_\pi$. Then

$$(a_{ij}(\omega)) = \sum_{s=1}^{n(\pi)} z_s(\omega)\varepsilon_s(\omega) + \mathscr{N}(\omega) ; \qquad \omega \in W_\pi$$

as in Lemma 3.2 of [10].

For $n < n(S, \pi, m)$ the decomposition of $(a_{ij}^{(n)}(\omega))$ will be arbitrary while for any given $n \geqq n(S, \pi, m)$ the class $\Delta_s \in \pi$ will be decomposed further to get all the different eigenvalues of $(a_{ij}^{(n)}(\omega))$. Then

$$(a_{ij}^{(n)}(\omega)) = \sum_{s=1}^{n(\pi)} \sum_{r\in\Delta_s} z_r^{(n)}(\omega)\varepsilon_r^{(n)}(\omega) + \mathscr{N}^{(n)}(\omega) ; \qquad \omega \in S \cap W_{\pi,m}$$

where $z_r^{(n)}(\omega); r \in \Delta_s$ are $\zeta_k^{(n)}(\omega); k \in \Delta_s$ without repetitions. According to (3.5) we get

$$(3.6) \qquad |z_k^{(n)}(\omega) - z_l^{(n)}(\omega)|_{n\to\infty} \longrightarrow 0$$

uniformly on $S \cap W_{\pi,m}$, whenever $k = l (\operatorname{mod} \pi)$.

Now, define a new sequence of matrices by

$$(3.7) \qquad (\tilde{a}_{ij}^{(n)}(\omega)) = \sum_{s=1}^{n(\pi)} \tilde{z}_s^{(n)}(\omega)\tilde{\varepsilon}_s^{(n)}(\omega) + \mathscr{N}^{(n)}(\omega) ; \qquad \omega \in S \cap W_{\pi,m}$$

in which $\tilde{z}_s^{(n)}(\omega) = z_{h_s}^{(n)}(\omega)$ where h_s is the first integer of the class Δ_s and $\tilde{\varepsilon}_s^{(n)}(\omega) = \sum_{r\in\Delta_s} \varepsilon_r^{(n)}(\omega)$.

Since $A_n; n = 1, 2, \cdots$ are spectral operators, by Theorem 1 there exist projections $E_s^{(n)} \in \Xi$ corresponding to the matrices $\varepsilon_s^{(n)}(\omega)$ and satisfying

$$(3.8) \qquad \| E_s^{(n)} \| \leqq 32\sqrt{N\pi} M_0 K ; \qquad n = 1, 2, \cdots .$$

Hence, we can define the operators

$$\tilde{A}_n = \sum_{s=1}^{n(\pi)} \left(\int \tilde{z}_s^{(n)}(\omega)E(d\omega)\right)\tilde{E}_s^{(n)} E(S \cap W_{\pi,m}) + \mathscr{N}_n E(S \cap W_{\pi,m}) ; \qquad n = 1, 2, \cdots$$

where $\tilde{E}_s^{(n)} = \sum_{r\in\Delta_s} E_r^{(n)}$ and it is easy to see that $\chi_{S\cap W_{\pi,m}}(\omega)\tilde{a}_{ij}^{(n)}(\omega)$ is the matricial representation of $\tilde{A}_n$ and by (3.8)

$$(3.9) \qquad \sup \| \tilde{A}_n \| < \infty .$$

Moreover, in view of (3.6), [5, Th. IV-10-10] and (3.8)

$$(3.10) \qquad \lim_{n\to\infty} (\tilde{A}_n - A_n)x = 0 ; \qquad x \in E(S \cap W_{\pi,m})X$$

thus,

$$\lim_{n\to\infty} \widetilde{A}_n x = Ax \; ; \qquad x \in E(S \cap W_{\pi,m})X \, . \tag{3.11}$$

On the set $S \cap W_{\pi,m}$ the matrices $\varepsilon_1(\omega)$ and $\tilde{\varepsilon}_1^{(n)}(\omega)$ can be calculated as follows (see [11, p. 687]). Let $Q(z)$ and $Q_n(z)$ be the polynomials

$$Q(z) = d_0 + d_1 z + \cdots + d_{n(\pi)(N+1)-1} z^{n(\pi)(N+1)-1}$$
$$Q_n(z) = d_0^{(n)} + d_1^{(n)} z + \cdots + d_{n(\pi)(N+1)-1}^{(n)} z^{n(\pi)(N+1)-1}$$

such that

$$\begin{aligned} Q(z_1(\omega)) &= 1 \; ; \\ Q_n(\tilde{z}_1^{(n)}(\omega)) &= 1 \; ; \\ Q(z_j(\omega)) &= 0 \; ; \\ Q_n(\tilde{z}_j^{(n)}(\omega)) &= 0 \; ; & 2 \leqq j \leqq n(\pi) \\ Q^{(p)}(z_j(\omega)) &= 0 \; ; \\ Q_n^{(p)}(\tilde{z}_j^{(n)}(\omega)) &= 0 \; ; & 1 \leqq j \leqq n(\pi); 1 \leqq p \leqq N \, . \end{aligned}$$

Then

$$\begin{aligned} \varepsilon_1(\omega) &= Q((a_{ij}(\omega))) \; ; \\ \tilde{\varepsilon}_1^{(n)}(\omega) &= Q_n((\tilde{a}_{ij}^{(n)}(\omega))) \; ; \qquad n = 1, 2, \cdots \, . \end{aligned} \tag{3.12}$$

Let us observe that the coefficients d_i and $d_i^{(n)}$ are given by the same rational function of $z_j(\omega)$ and $\tilde{z}_j^{(n)}(\omega)$ respectively; thus by (3.6)

$$\lim_{n\to\infty} d_i^{(n)}(\omega) = d_i(\omega) \qquad 0 \leqq i \leqq n(\pi)(N+1) - 1 \tag{3.13}$$

for a.e. $\omega \in S \cap W_{\pi,m}$. Using (3.11) and Theorem 5 we can see that the sequence $(\tilde{a}_{ij}^{(n)}(\omega))$ converges in measure to $a_{ij}(\omega)$ (on the set $S \cap W_{\pi,m}$). Thus, by (3.12) and (3.13) we get

$$\lim_{n\to\infty} \tilde{\varepsilon}_1^{(n)}(\omega) = \varepsilon_1(\omega) \qquad \text{a.e. } \omega \in S \cap W_{\pi,m} \, . \tag{3.14}$$

By (3.8) $\| \widetilde{E}_1^{(n)} E(S \cap W_{\pi,m}) \| \leqq 32N\sqrt{N\pi} M_0^2 K$, therefore, using Theorem 5 for the sequence $\{\widetilde{E}_1^{(n)} E(S \cap W_{\pi,m}) - \widetilde{E}_1^{(n+p)} E(S \cap W_{\pi,m})\}$; $p \geqq 1$ we can see that $\{\widetilde{E}_1^{(n)} E(S \cap W_{\pi,m})\}$ is a strongly convergent sequence. Define

$$L_1 x = \lim_{n\to\infty} \widetilde{E}_1^{(n)} E(S \cup W_{\pi,m}) x \; ; \qquad x \in X \, .$$

Then $\chi_{S \cap W_{\pi,m}}(\omega)\varepsilon_1(\omega)$ is the matricial representation of L_1.

Now, let us observe that in view of [5, Cor. III-6-3] one can choose among the various sets $S \cap W_{\pi,m}$ a sequence of disjoint sets, denoted by $\{\Omega_r\}$, such that $\bigcup_{r=1} \Omega_r = \Omega$. As we have seen, for every

r there is an operator $E_{1,r}$ (noted before by L_1) such that

$$E_{1,r}x = \lim_{n\to\infty} \tilde{E}_1^{(n)} E(\Omega_r)x\ ; \qquad x \in X$$

and $\chi_{\Omega_r}(\omega)\varepsilon_1(\omega)$ is a matricial representation of $E_{1,r}$. Put

$$F_{1,R} = \sum_{r=1}^{R} E_{1,r}\ ; \qquad R = 1, 2, \cdots .$$

Evidently, by (3.8)

$$\begin{aligned} \| F_{1,R} \| &\leqq \sup_{1\leqq n<\infty} \left\| \sum_{r=1}^{R} \tilde{E}_1^{(n)} E(\Omega_r) \right\| \\ &= \sup_{1\leqq n<\infty} \left\| \sum_{r=1}^{R} \Big(\sum_{s\in\Delta_1} E_s^{(n)} \Big) E(\Omega_r) \right\| \\ &= \sup_{1\leqq n<\infty} \left\| \sum_{s=1}^{R} E_s^{(n)} E(\delta_s) \right\| \\ &\leqq 32N\sqrt{N\pi} M_0^2 K \end{aligned}$$

where δ_s are measurable sets. Put

$$E_1 x = F_{1,R}x\ , \qquad x \in E\Big(\bigcup_{r=1}^{R} \Omega_r\Big)X\ .$$

Then, E_1 is a bounded linear operator defined on a dense set of X and it has an unique continuation on the whole space X; clearly, its matricial representation is $\varepsilon_1(\omega)$ i.e. by Theorem 1, A is spectral.

Now, in order to prove the last statement of the theorem, let us see that according to (3.6) and (3.13)

$$\text{(3.15)} \qquad \lim_{n\to\infty} \sum_{s=1}^{n(\pi)} \tilde{z}_s^{(n)}(\omega)\tilde{\varepsilon}_s^{(n)}(\omega) = \sum_{s=1}^{n(\pi)} z_s(\omega)\varepsilon_s(\omega) \qquad \text{a.e. } \omega \in S \cap W_{\pi,m}\ .$$

But (3.8) implies

$$\sup_{n,s,S,\pi,m} \left\| \Big(\int \tilde{z}_s^{(n)}(\omega) E(d\omega) \Big) E_s^{(n)} E(S \cap W_{\pi,m}) \right\| < \infty\ .$$

Hence, the sequence of the scalar parts of the operators $\tilde{A}_n$ converges strongly to the scalar part of the operator $AE(S \cap W_{\pi,m})$ when $n \to \infty$. In (3.10), it was show in essence that the sequence of the scalar parts of $A_n E(S \cap W_{\pi,m})$ and $\tilde{A}_n$ have the same strong limit. Thus

$$\lim_{n\to\infty} \mathscr{S}_n E(S \cap W_{\pi,m})x = \mathscr{S}(S \cap W_{\pi,m})x \qquad x \in X\ ;$$

where $A_n = \mathscr{S}_n + \mathscr{N}_n$; $n = 1, 2, \cdots$, $A = \mathscr{S} + \mathscr{N}$ and $\mathscr{S}_n$, $\mathscr{S}$ are the respective scalar parts and $\mathscr{N}_n$, $\mathscr{N}$ the nilpotent parts. It follows that $\{\mathscr{S}_n x\}$ converges to $\mathscr{S}x$ for x in $E(\bigcup_{r=1}^{\infty} \Omega_r)X$ (which is a dense subset of X) and further, since $\{\| \mathscr{S}_n \|\}$ is a bounded sequence (as a consequence of the uniform boundedness of the resolutions of the identity), the sequence $\{\mathscr{S}_n\}$ converges strongly to $\mathscr{S}$. It should be mentioned that in fact we have proved only the existence of a subsequence of $\{\mathscr{S}_n\}$ which converges strongly to $\mathscr{S}$. Repeating the arguments used in the last part of the proof of Theorem 5 it may be shown that the whole sequence $\{\mathscr{S}_n\}$ converges strongly to $\mathscr{S}$. Consequently, $\{\mathscr{N}_n\}$ converges strongly to $\mathscr{N}$ too.

Now, we remark that if an operator is given by

$$\mathscr{S} = \sum_{s=1}^{N} \left(\int z_s(\omega)E(d\omega)\right)E_s ,$$

then it is spectral and its real part in the sense of Foguel [8, Th. 1, p. 59] is given by

$$\mathscr{R} = \sum_{s=1}^{N} \left(\int \mathrm{Re}\ z_s(\omega)E(d\omega)\right)E_s ,$$

and we can finish the proof by the same method used for scalar parts.

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