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EQUILIBRIUM SYSTEMS OF STABLE PROCESSES

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We investigate several phenomena connected with the movement of particles through a compact subset B of d-dimensional Eucledian space in a system of infinitely many particles in statistical equilibrium, where each particle moves independently of the other particles according to the laws of the same symmetric stable process. In particular, we show that the volume of B governs the rate of flow of particles through B, and that on the one hand, for transient processes, the Riesz capacity of B governs the rate at which new particles hit B and at which particles permanently depart from B, while on the other hand, for recurrent processes, the rate at which new particles hit B is independent of B.

In this paper we will consider a system of denumerably many independent particles, all moving about in d-dimensional Euclidean space according to the laws of the same stable process in such a manner as to maintain statistical equilibrium of the system as a whole. For simplicity, and for the sake of obtaining explicit formulas, we restrict our discussion to systems of symmetric stable processes—i.e., those stable processes having transition density

$$f(t, x) = (2\pi)^{-d}\int_{R^d} e^{-i(\theta,x)}e^{-t|\theta|^\alpha}d\theta \ .$$

However, all of the results we obtain here, appropriately modified, remain true for general (not necessarily symmetric) stable processes. We will always assume that the sample paths of our processes are right continuous and have left-hand limits at every point.

Following Doob ([1], p. 404*ff*), the system we have in mind may be described as follows. Let $X_j(t)$ be the location of the jth particle at time t. At time 0 the particles are Poisson distributed over R^d with rate 1, and are numbered so that $|X_1(0)| < |X_2(0)| < \cdots$ (Details on Poisson processes on R^d can be found in [8].) We assume that $X_j(t) - X_j(0), 1 \leqq j < \infty$, are mutually independent stable processes having the common transition density $f(t, x)$. Moreover, the collections

$$\{X_j(t) - X_j(0), 1 \leqq j < \infty\} \ , \qquad \{X_j(0), 1 \leqq j < \infty\}$$

are independent. Then, as was shown by Doob, at any subsequent time $t > 0$, the distribution of particles over R^d is again Poisson with rate 1. Consequently, this system maintains itself in macroscopic equilibrium, and for this reason we refer to such a system as an *equilibrium system* of stable processes.

In applied studies, equilibriums systems (for Brownian motion, of course) usually arise by a suitable passage to the limit in a system of N particles. An interesting example of this situation is given in [7] (p. 132*ff*).

Our purpose in this paper will be to investigate the following (for compact subsets of R^d): (1) the number of distinct particles, $L_B(t)$, that enter B by the time t; (2) the number of particles, $D_B(t)$, that by time t have left B, never to return; and (3) $S_B(t) = \int_0^t A_B(s)ds$, where $A_B(s)$ is the number of particles in B at time s. (The quantity of $S_B(t)$ is just the total occupation time in B by time t of all particles in the system.) Principally, we shall be interested in establishing the strong law of large numbers and the central limit theorem for these quantities[1], but some additional facts will be developed along the way. In the Appendix we establish bounds for the rate of convergence in the central limit theorems for the above quantities.

Previously, these results were established by the author [10] for equilibrium systems of transient, discrete time, Markov processes. For equilibrium systems of recurrent Markov chains, the results for $L_B(t)$ may be found in [9], while those for $S_B(t)$ will appear in [11].

Our results for equilibrium systems of symmetric stable processes are as follows.

THEOREM 1. *Let B be a compact set in R^d having nonzero volume $|B|$. If $A_B(t)$ denotes the number of particles in B at time t, and if $S_B(t) = \int_0^t A_B(s)ds$, then*

$$P\Big(\lim_{t\to\infty} \frac{S_B(t)}{t} = |B|\Big) = 1 \,, \tag{1.1}$$

and

$$\lim_{t\to\infty} P\Big(\frac{S_B(t) - t|B|}{[\operatorname{Var} S_B(t)]^{1/2}} \leqq x\Big) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du \,. \tag{1.2}$$

Moreover, if $\alpha < d$, then

$$\operatorname{Var} S_B(t) \sim 2t\int_B\int_B g(y-x)dy\,dx \,, \tag{1.3}$$

where

$$g(x) = \Gamma\Big(\frac{d-\alpha}{2}\Big)\Big[4^{\alpha/2}\pi^{d/2}\Gamma\Big(\frac{\alpha}{2}\Big)\Big]^{-1} |x|^{\alpha-d} \,. \tag{1.4}$$

[1] It is really the explicit norming which yields these results (rather than just existence) which is of major interest.

If $d = \alpha$, *then*

$$\operatorname{Var} S_B(t) \sim 2f(1, 0) \mid B \mid^2 t \log t \,, \tag{1.5}$$

while for $d < \alpha \leqq 2$, *we must have* $d = 1$, *and*

$$\operatorname{Var} S_B(t) \sim 2f(1, 0) \mid B \mid^2 \left[1 - \frac{1}{\alpha}\right]^{-1} \left[2 - \frac{1}{\alpha}\right]^{-1} t^{2-1/\alpha} \,. \tag{1.6}$$

In all cases (*see Eq.* (3.2) *of* [3]),

$$f(1, 0) = 2\Gamma\left(\frac{d}{\alpha}\right)\left[\alpha\Gamma\left(\frac{d}{2}\right)(4\pi)^{d/2}\right]^{-1} .$$

REMARK 1. The results in this theorem show that in every equilibrium system of stable processes, the asymptotic number of particles per unit time in B is (with probability one) just the volume of B, when the mean number of particles per unit volume is 1. In other words, the process $A_B(t)$ is ergodic. However, the fluctuations around the mean depend on the processes in question and on the dimension. In every case when the processes are transient ($\alpha < d$), the variance, $\operatorname{Var} S_B(t)$, is asymptotically proportional to t, while when the processes are recurrent ($\alpha \geqq d$), $[\operatorname{Var} S_B(t)]t^{-1}$ increases to infinity at a rate proportional to the mean sojourn time by time t of a single particle in B. This, of course, is to be expected since, after a long time, most of the particles in B will be particles that have previously been there. In the transient case, particles can permanently wander away from B, so in order to maintain an equilibrium condition we must have a larger input of new particles per unit time than in the recurrent case. More precisely, we have the following result.

THEOREM 2. *Let B be a compact subset of R^d. If $\alpha < d$, assume the $d - \alpha$ dimensional Riesz capacity, $C(B) > 0$.* (*The Riesz capacity is defined in the proof.*) *When $\alpha > d$ assume that B is nonempty, while for $\alpha = d$ assume $P_x(V_B < \infty) = 1$ for all $x \in R^d$*[2], *where for a stable path $X(t)$,*

$$V_B = \inf\{t > 0 \colon X(t) \in B\} (= \infty \;\; \textit{if} \;\; X(t) \notin B, \;\textit{all}\; t > 0) \,. \tag{1.7}$$

Let $L_B(t)$ denote the number of distinct particles to visit B by time t. Then $L_B(t)$ is a nonhomogeneous Poisson process on $(0, \infty)$, *and*

$$EL_B(t) = \int_{R^d} P_x(V_B \leqq t)dx \,. \tag{1.8}$$

[2] It is known that for the processes with $\alpha = d$, $P_x(V_B < \infty) = 1$ for all x if $P_x(V_B < \infty) > 0$ for some x. [A proof of this fact can be found in a forthcoming book on Markov processes by Blumenthal and Getoor.]

Moreover,

$$P\Big(\lim_{t\to\infty} \frac{L_B(t)}{EL_B(t)} = 1\Big) = 1 \,, \tag{1.9}$$

and

$$\lim_{t\to\infty} P\Big(\frac{L_B(t) - EL_B(t)}{[EL_B(t)]^{1/2}} \leqq x\Big) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du \,. \tag{1.10}$$

Finally, if $\alpha < d$,

$$EL_B(t) \sim tC(B) \,, \tag{1.11}$$

while for $d = \alpha$,

$$EL_B(t) \sim [f(1, 0)]^{-1}t(\log t)^{-1} \,, \tag{1.12}$$

and for $1 = d < \alpha \leqq 2$,

$$EL_B(t) \sim \Big[f(1, 0)\Gamma\Big(1 - \frac{1}{\alpha}\Big)\Gamma\Big(1 + \frac{1}{\alpha}\Big)\Big]^{-1} t^{1/\alpha} \,. \tag{1.13}$$

REMARK 2. We note first of all that whenever the processes are transient, the number of new particles that enters B is asymptotically proportional to t, while in the recurrent case, $EL_B(t)t^{-1} \to 0$. Thus, as previously anticipated, the rate at which new particles enter a compact set is greater for transient processes than for recurrent processes. Observe next that since $L_B(t)$ is a Poisson process, the probability that no new particle enters B by time t is just $e^{-EL_B(t)}$. Finally, we note in passing that the capacity inequalities ([6] p. 250) can easily be derived from (1.8) and (1.11).

THEOREM 3. *Assume that* $\alpha < d$, *and let* B *be a compact set having nonzero* $d - \alpha$ *dimensional Riesz capacity* $C(B)$. *If* $D_B(t)$ *denotes the number of particles that during the time interval* $(0, t]$ *have been in* B *and left, never to return, then* $D_B(t)$ *is a homogeneous Poisson process on* $(0, \infty)$ *with rate* $C(B)$. *Moreover*

$$P[\lim_{t\to\infty} D_B(t)/t = C(B)] = 1 \tag{1.14}$$

and

$$\lim_{t\to\infty} P\Big(\frac{D_B(t) - tC(B)}{[tC(B)]^{1/2}} \leqq x\Big) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du \,. \tag{1.15}$$

REMARK 3. Using well-known facts about Poisson processes we at once obtain the probability that every particle which has been in

B during $(0, t]$ will return to B at least once during (t, ∞) is $e^{-C(B)t}$. Also for any time t, the waiting time from t till the time that the next particle leaves B, never to return, is exponentially distributed with mean $C(B)^{-1}$. Since the waiting time between two successive events in the $D_B(t)$ process is exponential with mean $C(B)^{-1}$, we see that the mean time between successive permanent departures is $C(B)^{-1}$. Finally the capacity inequalities yield facts about $ED_B(t)$ as a *set* function that are not at all apparent. For example, if $A \subset B$ then $ED_A(t) \leqq ED_B(t)$.

REMARK 4. The results of Theorems 2 and 3 show that the electrostatic (Riesz) capacity of a compact set B can be interpreted as the (asymptotic) rate per unit time at which new particles enter B and also as the rate per unit time at which particles wander off of B, never to return, in the appropriate equilibrium system of stable processes. The number of particles, $R_B(t)$, that will reenter B after time t is just $L_B(t) - D_B(t)$. Proceeding as in the proofs of Theorems 2 and 3 we may show that for each fixed t, $R_B(t)$ is Poisson distributed with mean

$$ER_B(t) = EL_B(t) - ED_B(t) = \int_{R^d} P_x(V_B \leqq t)dx - tC(B) . \tag{1.16}$$

(However, $R_B(t)$ is not a Poisson process, since $R_B(t)$ clearly doesn't have independent increments.) Now the asymptotic behavior of the expression on the right in (1.16) was investigated by Getoor in [3]. In our terminology Getoors' results show that $ER_B(t)$ is a nondecreasing function of t, and that if $\alpha < d/2$,

$$ER_B(t) \uparrow \int_{R^d} E_x(V_A < \infty)^2 dx + |B| . \tag{1.17}$$

On the other hand, if $\alpha = d/2$, then

$$ER_B(t) \sim \frac{f(1, 0)C(B)^2}{(d/\alpha) - 1} \log t , \tag{1.18}$$

while for $d/2 < \alpha < d$,

$$ER_B(t) \sim \frac{C(B)^2 f(1, 0)}{[(d/\alpha) - 1][2-(d/\alpha)]} t^{2-(d/\alpha)} . \tag{1.19}$$

Consequently, we see that if $\alpha < d/2$, $R_B(t)$ converges in law to a Poisson random variable R_B, while for $d/2 \leqq \alpha < d$, $R_B(t) \uparrow \infty$ with probability one. (However, $[R_B(t) - ER_B(t)][ER_B(t)]^{-1/2}$ is asymptotically normally distributed.)

2. Proofs.

Proof of Theorem 1. It is a known fact (which can be verified directly by a slightly tedious computation) that the process $A_B(t)$ is strictly stationary. Hence by the pointwise ergodic theorem, there is a random variable S_B^* such that

$$P\left(\lim_{t\to\infty}\frac{S_B(t)}{t} = S_B^*\right) = 1 .$$

To conclude that $P(S_B^* = |B|) = 1$, it suffices to show that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} P\left(\left|\frac{S_B(t)}{t} - |B|\right| > \varepsilon\right) = 0 .$$

By Chebechev's inequality, this will be the case provided

$$\lim_{t\to\infty}[\operatorname{Var} S_B(t)]t^{-2} = 0 . \tag{2.1}$$

At time 0 the particles are Poisson distributed over R^d with rate 1. Then for a Borel set $D \subset R^d$, the number of particles in D at time 0 is $A_0(D)$, where $A_0(dx)$ is the random counting measure with generating functional

$$E\exp\left[\int_{R^d}\log s(x)A_0(dx)\right] = \exp\left[\int_{R^d}[s(x) - 1]dx\right],$$

and $s(x)$ is a bounded complex valued function. (A discussion of generating functionals can be found in [8]). For a stable process $X(t)$, let $N_B(t)$ be the occupation time in B by time t, i.e.,

$$N_B(t) = \int_0^t 1_B[X(s)]ds ,$$

where $1_B(x)$ is the indicator function of B. Then

$$\begin{aligned} E(e^{i\theta S_B(t)}) &= E\exp\left[\int_{R^d}\log E_x(e^{i\theta N_B(t)})A_0(dx)\right] \\ &= \exp\int_{R^d}[E_x(e^{i\theta N_B(t)}) - 1]dx . \end{aligned} \tag{2.2}$$

From this we readily obtain that

$$\operatorname{Var} S_B(t) = \int_{R^d} E_x N_B(t)^2 dx .$$

Setting $g(t, x, B) = \int_B\int_0^t f(s, y - x)ds\,dy$, we obtain

$$
\begin{aligned}
\operatorname{Var} S_B(t) &= 2\int_{R^d} dx \int_B \int_B \int_0^t \int_0^{t_2} f(t_1, y-x) f(t_2 - t_1, z-y) dt_1 dt_2 dy dz \\
&= 2\int_0^t \int_B g(s, x, B) dx ds \,.
\end{aligned} \tag{2.3}
$$

Now $f(t, x)$ is a bounded continuous function in x, and satisfies the well-known scaling property (see, e.g., [3]) $f(t, x) = t^{-d/\alpha} f(1, t^{-1/\alpha}x)$. Consequently,

$$f(t, x) \sim t^{-d/\alpha} f(1, 0) \,, \tag{2.4}$$

uniformly in x on compacts. A simple computation now shows that, uniformly in x on compacts,

$$g(t, x, B) \sim t^{1-(d/\alpha)} f(1, 0) \left[1 - \frac{d}{\alpha}\right]^{-1} |B| \,, \qquad d < \alpha \leqq 2 \,, \tag{2.5}$$

$$g(t, x, B) \sim (\log t) f(1, 0) |B| , \qquad \alpha = d \,, \tag{2.6}$$

$$g(t, x, B) \uparrow \int_B g(y - x) dy < \infty \,, \qquad \alpha < d \,, \tag{2.7}$$

where by Eq. (1.2) of [3],

$$g(x) = \Gamma\left(\frac{d-\alpha}{2}\right)\left[4^{\alpha/2}\pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1} |x|^{\alpha - d} \,.$$

In all cases then, we see by (2.3) that (2.1) holds.

From (2.2) we see that

$$E \exp(i\theta[S_B(t) - t|B|]) = \exp\left\{-\frac{\theta^2}{2} \operatorname{Var} S_B(t) + R_t(\theta)\right\} , \tag{2.8}$$

where

$$|R_t(\theta)| \leqq (3!)^{-1} |\theta|^3 \int_{R^d} E_x N_B(t)^3 dx \,. \tag{2.9}$$

Hence, to establish (1.2) it suffices to show

$$\lim_{t\to\infty} \frac{\int_{R^d} E_x N_B(t)^3 dx}{[\operatorname{Var} S_B(t)]^{3/2}} = 0 \,.$$

In order to do this we need the asymptotic behavior, for large t, of the quantities in the numerator and denominator of the above expression. As for the denominator, we see at once from (2.3) and eqs. (2.5) – (2.7) that (1.6) holds, for $d < \alpha \leqq 2$, that (1.5) holds for $\alpha = d$, and that (1.3) holds for $\alpha < d$. As to the numerator, an easy computation shows

$$\int_{R^d} E_x N_B(t)^3 dx = 3!\int_0^t\int_B g(s, x, B)g(t-s, x, B)dxds .$$

In view of Eqs. (2.5) – (2.7), an appeal to a well-known Abelian theorem on convolutions (Theorem 42 of [4]) shows at once that that for $\alpha < d$,

$$\int_0^t g(s, x, B)g(t-s, x, B)ds \sim t\int_B\int_B g(y-x)g(z-x)dydz , \tag{2.10}$$

while for $\alpha = d$,

$$\int_0^t g(s, x, B)g(t-s, x, B)ds \sim t(\log t)^2 f(1, 0)^2 | B |^2 , \tag{2.11}$$

and for $d < \alpha \leqq 2$ (i.e., $d = 1$, and $1 < \alpha \leqq 2$),

$$\int_0^t g(s, x, B)g(t-s, x, B)ds \sim \frac{t^{2(1-1/\alpha)+1} f(1, 0)^2 \Gamma\left(1-\frac{1}{\alpha}\right)^2 | B |^2}{\Gamma\left(4-\frac{2}{\alpha}\right)} . \tag{2.12}$$

Consequently,

$$\begin{aligned} \operatorname{Var} S_B(t)^{-3/2}\int_{R^d} E_x N_B(t)^3 dx &= 0(t^{-1/2}) , \qquad \alpha < d , \\ &= 0\left(\left[\frac{t}{\log t}\right]^{-1/2}\right) , \qquad \alpha = d , \\ &= 0(t^{-1/2\alpha}), \qquad d < \alpha \leqq 2 . \end{aligned} \tag{2.13}$$

This completes the proof.

Proof of Theorem 2. Let $0 = t_0 < t_1 < \cdots < t_n$ be any n points on $(0, \infty)$, and let $0 < s_i < 1$. Then it is not hard to see that

$$\begin{aligned} &E\left(\prod_{i=1}^n s_i^{[L_B(t_i)-L_B(t_{i-1})]}\right) \\ &= E\left\{\exp\int_{R^d}\log\left[1+\sum_{i=1}^n (s_i-1)P_x(t_{i-1} < V_B \leqq t_i)\right]A_0(dx)\right\} \\ &= \exp\left[\sum_{i=1}^n (s_i-1)\int_{R^d} P_x(t_{i-1} < V_B \leqq t_i)dx\right] , \end{aligned}$$

where $A_0(dx)$ is the Poisson process on R^d with rate 1. Thus $L_B(t)$ is a nonhomogeneous Poisson process on $(0, \infty)$, and

$$EL_B(t) = \int_{R^d} P_x(V_B \leqq t)dx .$$

For $\alpha < d$ it was shown in [3] that

$$\int_{R^d} P_x(V_B \leqq t)dx \sim tC(B) ,$$

where $C(B)$ is the natural capacity of B, this being just the $d - \alpha$ dimensional Riesz capacity determined by the kernel (1.4). (A discussion of Riesz capacity may be found in [6], p. 290*ff*.) This establishes (1.11).

Next we establish (1.12) and (1.13). For $\lambda \geqq 0$ let

$$H_B^\lambda(x, A) = E_x[e^{-\lambda V_B} 1_A(X(V_B)); \; V_B < \infty]$$

denote the λ-hitting measure of B, and for $\lambda > 0$ let $U^\lambda(x)$ denote the Laplace transform of $f(t, x)$. The dual process to $X(t)$ is the process $-X(t)$. Since $X(t)$ is symmetric, a fundamental relation of Hunt ([5] Eqn. (18.3)), connecting a process to its dual, becomes in this case the following identity:

$$\int_B H_B^\lambda(x, dz) U^\lambda(y - z) = \int_B H_B^\lambda(y, dz) U^\lambda(x - z) \; . \tag{2.14}$$

Integrating x over R^d in (2.14) then yields the relation

$$\int_B E_B^\lambda(dz) U^\lambda(y - z) = \lambda^{-1} E_y(e^{-\lambda V_B}; \; V_B < \infty) \; , \tag{2.15}$$

where

$$E_B^\lambda(dz) = \int_{R^d} dx H_B^\lambda(x, dz) \; .$$

Consider the case $\alpha > d$. It is a known fact about these processes (see, e.g., sec. 2 of [12]) that $P_y(V_B < \infty) = 1$ for all y whenever B is nonempty. From (2.4) we readily obtain, by standard Abelian arguments, that

$$U^\lambda(y - z) \sim f(1, 0) \Gamma\Big(1 - \frac{1}{\alpha}\Big) \lambda^{-1+1/\alpha} \; , \qquad \alpha > d$$

uniformly in $z \in B$. Using this, we easily obtain from (2.15) that

$$E_B^\lambda(B) \sim \Big[f(1, 0) \Gamma\Big(1 - \frac{1}{\alpha}\Big)\Big]^{-1} \lambda^{-1/\alpha} \; , \tag{2.16}$$

from which (1.13) follows by Karamatas' Tauberian theorem.

Now consider the case $\alpha = d$. Then it easily follows from (2.4) that for $y \notin B$,

$$U^\lambda(y - z) \sim f(1, 0) \log\Big(\frac{1}{\lambda}\Big)$$

uniformly in $z \in B$. Using (2.15), we may conclude that

$$E_B^\lambda(B) \sim \Big[f(1, 0) \lambda \log\Big(\frac{1}{\lambda}\Big)\Big]^{-1} \; , \tag{2.17}$$

from which equation (1.12) follows by Karamatas' theorem.

We will now establish a simple result on Poisson processes which will be used to finish the proof of Theorem 2 and will also be used in the proof of Theorem 3.

LEMMA 2.1. *Let $Z(t)$ be a (not necessarily homogeneous) Poisson process such that $EZ(t) \to \infty$, $t \to \infty$. Then*

$$\lim_{t\to\infty} P\left(\frac{Z(t) - EZ(t)}{[EZ(t)]^{1/2}} \leqq x\right) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du . \tag{2.18}$$

If, in addition, $EZ([t]) \sim EZ([t] + 1)$, then

$$P\left(\lim_{t\to\infty} \frac{Z(t)}{EZ(t)} = 1\right) = 1 . \tag{2.19}$$

Proof. The first assertion follows at once from the well-known fact that a normalized Poisson variate is asymptotically normally distributed when its mean tends to infinity. For each integer $n \geqq 1$, let $I_n = Z(n) - Z(n-1)$. Then $\{I_n\}$ are independent Poisson variates. Since $EZ(n) \to \infty$, we obtain by the well-known theorem of Abel-Dini that

$$\sum_n \frac{EI_n}{[EZ(n)]^2} = \sum_n \frac{\operatorname{Var} I_n}{[EZ(n)]^2} < \infty .$$

The strong law of large numbers now yields the conclusion that

$$P\left(\lim_{n\to\infty} \frac{Z(n)}{EZ(n)} = 1\right) = 1 ,$$

from which (2.19) follows since

$$Z([t]) \leqq Z(t) \leqq Z([t] + 1) .$$

To complete the proof of Theorem 2 we need now only apply the above lemma to the process $L_B(t)$.

Proof of Theorem 3. Let $T_B = \inf\{t > 0 : X(s) \notin B, \text{ all } s > t\}$, where $X(s)$ is a stable process. Then for any $0 = t_0 < t_1 < \cdots < t_n$, and $0 < s_i < 1$,

$$\begin{aligned} &E\left(\prod_{i=1}^{n} s_i^{[D_B(t_i) - D_B(t_{i-1})]}\right) \\ &= E\left\{\exp\int_{R^d} \log\left[1 + \sum_{i=1}^{n}(s_i - 1)P_x(t_{i-1} < T_B \leqq t_i)\right]A_0(dx)\right\} \\ &= \exp\left[\sum_{i=1}^{n}(s_i - 1)\int_{R^d} P_x(t_{i-1} < T_B \leqq t_i)dx\right] , \end{aligned}$$

where, as before, $A_0(dx)$ is the Poisson process on R^d with rate 1. Consequently, $D_B(t)$ has independent Poisson increments. Thus for any $t, h > 0$,

$$\begin{aligned} E[D_B(t+h) - D_B(t)] &= \int_{R^d} P_x(t < T_B \leqq t+h)dx \\ &= \int_{R^d} dx \int_{R^d} f(t, y-x) P_y(V_B \leqq h, X(s) \notin B, \quad \text{all} \quad s > h)dy \\ &= \int_{R^d} P_y(V_B \leqq h, X(s) \notin B, \quad \text{all} \quad s > h)dy \,, \end{aligned} \tag{2.20}$$

which shows that the increments are stationary as well. To compute the rate, we observe that from the above it is evident that

$$\int_{R^d} P_y(V_B \leqq h, X(s) \notin B, \quad \text{all} \quad s > h)dy$$

is a linear function of h, and thus the quantities

$$e_B^h(R^d) = \frac{1}{h}\int_{R^d} P_y(V_B \leqq h, X(s) \notin B, \quad \text{all} \quad s > h)dy$$

are independent of h. Now in [5] (p. 290) it is shown that there is a subsequence of the measures $\{e_B^h(dy)\}$ that converges weakly to the $d - \alpha$ dimensional Riesz capacitory measure $e_B(dy)$ of B. This, of course, is the equilibrium distribution on B producing the potential $P_x(V_B < \infty)$ and having total charge $e_B(R^d) = C(B)$. Thus the rate of $D_B(t)$ is $C(B)$.

The remainder of the theorem now follows by applying Lemma 2.1 to the process $D_B(t)$.

Appendix. The following well-known result on distributions and their corresponding characteristic functions will play a key role in obtaining the error terms for the central limit theorems in the preceeding section. A proof may be found in [2], p. 512.

LEMMA A.1. *Let F be a probability distribution having characteristic function $\varphi(\theta)$, and let $\Phi(x)$ be the standard normal distribution, having characteristic function $e^{-\theta^2/2}$. Then for any x, and $T > 0$,*

$$|F(x) - \Phi(x)| \leqq \pi^{-1}\int_T^T |\theta|^{-1} |\varphi(\theta) - e^{-\theta^2/2}| \, d\theta + \frac{24}{\pi\sqrt{2\pi}} T^{-1} \,. \tag{A.1}$$

Let

$$F_t(x) = P\left(\frac{S_B(t) - |B|t}{[\operatorname{Var} S_B(t)]^{1/2}} \leqq x\right), \tag{A.2}$$

and let $\varphi_t(\theta)$ be the corresponding characteristic function. Setting

$$[\operatorname{Var} S_B(t)]^{-3/2}\int_{R^d} E_x N_B(t)^3 dx = \gamma_B(t)^{-1} ,$$

we see from (A.1) and (2.8) that

$$\text{(A.3)} \qquad | F_t(x) - \Phi(x) | \leqq \pi^{-1}\int_T^T e^{-\theta^2/2} | e^{R_B(t)} - 1 | | \theta |^{-1} + \frac{24}{\pi\sqrt{2\pi}} T^{-1} ,$$

where by (2.9)

$$\text{(A.4)} \qquad | R_B(t) | \leqq \frac{| \theta |^3}{3!} \gamma_B(t)^{-1} .$$

Let $T = \gamma_B(t)$, and remembering that $| e^{R_B(t)} - 1 | \leqq | R_B(t) | e^{|R_B(t)|}$, we see that the integrand (A.3) is dominated by

$$\left(\frac{1}{\pi}\right) \frac{| \theta |^2}{3!\gamma_B(t)} e^{-\theta^2/2} e^{\theta^2/6} .$$

Since

$$\int_{-\infty}^{\infty} \theta^2 e^{-\theta^2/3} d\theta = \frac{3\sqrt{3\pi}}{2} ,$$

we see that

$$| F_t(x) - \Phi(x) | \leqq \left[\frac{\sqrt{3/\pi}}{4} + \frac{24}{\pi\sqrt{2\pi}}\right]\gamma_B(t)^{-1} = 0(\gamma_B(t)^{-1}) .$$

From (2.13) we now obtain the following.

THEOREM A.1. *If $F_t(x)$ is as in* (A.2), *then*

$$\text{(A.5)} \qquad \begin{aligned} | F_t(x) - \Phi(x) | &\leqq 0(t^{-1/2}) , && \alpha < d \\ &\leqq 0\left(\left[\frac{t}{\log t}\right]^{-1/2}\right) , && \alpha = d \\ &\leqq 0(t^{-d/2\alpha}) , && d = 1 < \alpha \leqq 2 . \end{aligned}$$

To find the rate of convergence to the normal distributions in Theorems 2 and 3 we make use of the following.

LEMMA A.2. *Let $Z(t)$ be a (not necessarily homogeneous) Poisson process, and let $EZ(t) = \lambda(t)$. Then if $\lambda(t) \uparrow \infty$, and*

$$\text{(A.6)} \qquad F_t(x) = P\left(\frac{Z(t) - \lambda(t)}{\sqrt{\lambda(t)}} \leqq x\right) ,$$

we have

$$(\text{A.7}) \qquad | F_t(t) - \Phi(x) | \leqq \left[\frac{\sqrt{3/\pi}}{4} + \frac{24}{\pi\sqrt{2\pi}}\right]\lambda(t)^{-1/2} \leqq 5\lambda(t)^{-1/2} .$$

Proof. If $\varphi_t(\theta)$ is the characteristic function corresponding to the distribution in (A.6), then

$$| \varphi_t(\theta) - e^{-\theta^2/2} | \leqq e^{-\theta^2/2}[e^{R_t(\theta)} - 1] ,$$

where

$$| R_t(\theta) | \leqq \frac{|\theta|^3}{3!} \lambda(t)^{-1/2} .$$

Setting $T = \lambda(t)^{1/2}$ in Lemma A.1 and proceeding as before, we see that (A.7) holds.

From the above lemma and the results in Theorems 2 and 3 we may easily obtain an estimate of the rate of convergence to the normal in Theorems 2 and 3. The results are as follows.

THEOREM A.2. *Let*

$$F_t(x) = P\left(\frac{L_B(t) - EL_B(t)}{[EL_B(t)]^{1/2}} \leqq x\right)$$

and let

$$G_t(x) = P\left(\frac{D_B(t) - tC(B)}{[tC(B)]^{1/2}} \leqq x\right) .$$

Then if $\alpha < d$,

$$| F_t(x) - \Phi(x) | \leqq 5[EL_B(t)]^{-1/2} \leqq 0(t^{-1/2}) ,$$

and

$$| G_t(x) - \Phi(x) | \leqq 5C(B)^{-1/2}t^{-1/2} .$$

For $\alpha = d$,

$$| F_t(x) - \Phi(x) | \leqq 5[EL_B(t)]^{-1/2} \leqq 0\left[\left(\frac{t}{\log t}\right)^{-1/2}\right] ,$$

while for $d < \alpha \leqq 2$,

$$| F_t(x) - \Phi(x) | \leqq 5[EL_B(t)]^{-1/2} \leqq 0(t^{-1/2\alpha}) .$$

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