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ON THE CHARACTERIZATION OF COMPACT HAUSDORFF X FOR WHICH $C(X)$ IS ALGEBRAICALLY CLOSED

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Although the problem considered here has its origins in Functional Analysis, the viewpoint and methods of this paper are purely topological. The problem is to give a completely topological characterization of those compact Hausdorff spaces X for which the algebra $C(X)$ of all complex-valued continuous functions on X is algebraically closed, i.e. for which each polynomial over $C(X)$, whose leading coefficient is constant, has a root in $C(X)$.

A necessary condition in order that $C(X)$ be algebraically closed is obtained here and it is proven that, in the presence of first countability, the condition is also sufficient. The necessary condition requires that X be hereditarily unicoherent and that each discrete sequence of continua in X have a degenerate or empty topological limit inferior. A rather general sufficient condition is also proved. It states essentially that each component of X have an algebraically closed function algebra and that each point of X be of finite order in the sense of Whyburn.

A short history of the problem is in order. In [1], Decard and Pearcy consider matrices with entries from the algebra $C(X)$ where X is a Stonian space (compact, Hausdorff, and extremely disconnected). As a tool in the investigation, they prove that every monic polynomial with coefficients in $C(X)$ has a root in $C(X)$. With the aid of this result, they are able to prove, among other things, that every invertable $n \times n$ matrix with entries from $C(X)$ has roots of all orders. In [2] they examine this tool on its own merit. They prove that if X is either totally disconnected, compact, and Hausdorff, or linearly ordered and order complete, then $C(X)$ is algebraically closed.

Concerning the problem of giving a topological characterization of the algebraic property of closure, Decard and Pearcy point out that if X contains the homeomorphic image of the unit circle, then $C(X)$ is not algebraically closed. Also, if X is the closure of the graph of the function $y = \sin(1/x)$ $0 < x \leqq 1$, then $C(X)$ is not algebraically closed. The following obvious lemma indicates that there is a reasonable chance of finding a solution to the problem.

LEMMA 1.1. *If X is compact and Hausdorff and if M is a closed subset of X such that $C(M)$ is not algebraically closed, then*

$C(X)$ is not algebraically closed either.

Thus each time one finds a space X such that $C(X)$ is not algebraically closed, one knows a configuration which cannot be a part of any space which has a closed function algebra. It turns out that the two configurations mentioned above come very close to giving a complete answer to the problem. Another lemma which sheds a great deal of light on the problem comes from [1].

LEMMA 1.2. *If $x_0 \in X$ and z_0 is a root of $P(x_0,z)$ of multiplicity m, and if $\varepsilon > 0$ is such that $P(x_0, z)$ has no root in $0 < |z - z_0| \leqq \varepsilon$, then there is an open set V containing x_0 such that if $x \in V$ then $P(x, z)$ has exactly m roots (counting multiplicities) in $|z - z_0| < \varepsilon$.*

It is easily seen that this lemma establishes a strong connection between the behavior of the roots of $P(x, z)$ and the topology on X.

2. The necessary condition. We begin by extracting the essential features of the two configurations mentioned in the introduction. A space which contains the homeomorphic image of the unit circle is not hereditarily unicoherent i.e. contains two connected closed subsets whose intersection is not connected.

LEMMA 2.1. *Let X be a compact Hausdorff space and let M and N be connected closed subsets of X such that $M \cup N = X$ and $M \cap N$ is not connected. Then $C(X)$ is not algebraically closed.*

Proof. Let $M \cap N = A \cup B$ where A and B are disjoint, nonempty, closed sets. Let $f(x)$ be a continuous mapping of X into the closed unit interval [0.1] of real numbers such that $f(x) \equiv 0$ on A and $f(x) \equiv 1$ on B. If $x \in M - N$, let $h(x) = \exp(i\pi f(x))$ and let $h(x) = \exp(-i\pi f(x))$ otherwise. It is a simple matter to verify that $h(x) \in C(X)$. Consider the monic polynomial $P(x, z) = z^2 - h(x)$. Suppose there were an element $r(x)$ in $C(X)$ such that $P(x, r(x)) \equiv 0$. Since M is connected and $r(x)$ is continuous, it must be that $r(M) = \{\exp(i\beta): 0 \leqq \beta \leqq \pi/2\}$ or $r(M) = \{\exp(i\beta): \pi \leqq \beta \leqq 3\pi/2\}$. We may clearly assume that the first statement holds. Now the same considerations hold for $r(N)$; either $r(N) = \{\exp(i\beta): \pi/2 \leqq \beta \leqq \pi\}$ or $r(N) = \{\exp(i\beta): 3\pi/2 \leqq \beta \leqq 2\pi\}$. In the first case we get that $r(M \cap A) \equiv 1$ while $r(N \cap A) \equiv -1$, and in the second case we find $r(M \cap B) \equiv i$ while $r(N \cap B) \equiv -i$. Since $M \cap N = A \cup B$ however, we see that $M \cap A = N \cap A = A$ and $M \cap B = N \cap B = B$ and we have a contradiction in either case! Thus $C(X)$ is not algebraically closed.

Thus the first part of our necessary condition will require that X be hereditarily unicoherent. In order to specify the essential features of the closure of the graph of the function $y = \sin(1/x)$ $0 \leqq x \leqq 1$, we need the following.

DEFINITION 2.2. A topological space is *almost locally-connected* in case it does not contain sequences $\{C_n\}(n = 1, 2, \cdots)$, $\{x_n\}(n = 1, 2, \cdots)$, and $\{y_n\}(n = 1, 2, \cdots)$ such that; for each n, C_n is a connected closed set which is open in $\overline{\bigcup C_k}(k = 1, 2, \cdots)$, $C_m \cap C_n = \emptyset$ for $m \neq n$, x_n and y_n are points of C_n for each n, and $\{x_n\}$ $(n = 1, 2, \cdots)$ and $\{y_n\}$ $(n = 1, 2, \cdots)$ converge to distinct points x_0 and y_0 respectively.

The term "almost locally-connected" is motivated by the fact (to be proved later) that a compact and sequentially compact connected Hausdorff space is locally connected if it is almost locally-connected. This generalizes the well-known result (cf. for example, [3], Theorem 3-12, p. 114) that a compact connected metric space is locally connected if it contains no continuum of convergence (for the meaning of "continuum of convergence" see [5], p. 18).

LEMMA 2.3. *If X is a compact Hausdorff space which is not almost locally-connected, then $C(X)$ is not algebraically closed.*

Proof. Let $\{C_n\}(n = 1, 2, \cdots)$, $\{x_n\}(n = 1, 2, \cdots)$, $\{y_n\}(n = 1, 2, \cdots)$, x_0, and y_0 be as in (2.2). Since $\overline{\bigcup C_k}(k = 1, 2, \cdots)$ is obviously closed in X, we may (in view of (1.1)) assume without loss of generality that $X = \overline{\bigcup C_k}(k = 1, 2, \cdots)$. Let A and B be disjoint closed nbhds of x_0 and y_0 respectively. Let $f(x)$ be a continuous mapping of X into the closed unit interval $[-1, 1]$ such that $f(x) \equiv 1$ on A and $f(x) \equiv -1$ on B. Define the function $h(x)$ on X and follows. If $x \in C_n$ and n is even, let

$$h(x) = f(x) + (i/n)[1 - (f(x))^2]^{1/2} ,$$

if $x \in C_n$ and n is odd, let

$$h(x) = f(x) - (i/n)[1 - (f(x))^2]^{1/2} ,$$

and otherwise let $h(x) = f(x)$. Since the C_n are disjoint, $h(x)$ is well defined, and since the C_n are open, we see that $h(x) \in C(X)$. Now consider the monic polynomial $P(x, z) = z^2 - h(x)$. Suppose there were a function $r(x) \in C(X)$ such that $P(x, r(x)) \equiv 0$. For almost all n, C_n is a continuum from A to B with $x_n \in A$ and $y_n \in B$. Thus the image $r(C_n)$ must be a connected set from $r(x_n)$ to $r(y_n)$ which is contained in some closed quadrant of the complex plane. We may assume, since $h(x_0) = 1$, that $r(x_0) = 1$. Since $x_n \rightarrow x_0$ and $r(x_n) = \pm 1$ for almost all n, we must have that $r(x_n) = +1$ for almost all n. This then requires that for almost all even n, $r(y_n) = +i$, and for almost all odd n, $r(y_n) = -i$. But $y_n \rightarrow y_0$ so that $r(y_n) \rightarrow r(y_0)$, and we have a contradiction. Thus $C(X)$ is not algebraically closed and the lemma is established.

We have thus established the following necessary condition.

THEOREM 2.4. *If X is a compact Hausdorff space, a necessary condition that $C(X)$ be algebraically closed is that X be hereditarily unicoherent and almost locally-connected.*

One naturally wants to know whether, or to what extent, the necessary condition is also sufficient. It seems appropriate to give a partial answer to the question at this point. A more complete answer must wait until a later section of this paper. The partial answer we give here is that if X is connected and sequentially compact in addition to being compact and Hausdorff, then the necessary condition is also sufficient. The following lemma will be needed to prove this fact.

LEMMA 2.5. *Let X be a compact connected Hausdorff space which is sequentially compact, hereditarily unicoherent, and almost locally-connected, and let p and q be distinct points of X. There is continuum, $E[p, q]$, in X which is irreducible from p to q. Each point of $E(p, q)(E(p, q) = E[p, q] - (p + q))$ separates p and q in X and the order topology induced on $E[p, q]$ by the separation order is the same as the subspace topology on $E[p, q]$.*

Proof. The proof rests on showing that X is locally connected. Suppose X were not locally connected and find, therefore, a point x_0' and an open set V containing x_0' such that x_0' is not an interior point of the component of V which contains it. Find open sets V_1 and V_2 such that $x_0' \in V_2 \subseteqq \bar{V}_2 \subseteqq V_1 \subseteqq \bar{V}_1 \subseteqq V$. Since X is compact, connected, and Hausdorff, every component of $\bar{V}_1$ which intersects V_2 must contain points of $\mathrm{Bd}(V_1)$. Also, if H is a component of $\bar{V}_1$ and N is any closed set contained in $\bar{V}_1$ and disjoint from H, then there is an open set A such that $A \cap N = \emptyset$, $H \subseteqq A$, and $A \cap \bar{V}_1$ is closed.

Let H_0 be the component of $\bar{V}_1$ which contains x_0'. Since x_0' is not an interior point of H_0, V_2 is not contained in H_0. Let H_1 be a component of $\bar{V}_1$ such that $H_1 \neq H_0$ and $H_1 \cap V_2 \neq \emptyset$, and let A_1 be an open set such that $H_0 \cap A_1 = \emptyset$, $H_1 \subseteqq A_1$, and $A_1 \cap \bar{V}_1$ is closed.

Now suppose we have sequences $H_1, H_2, \cdots, H_n$ and $A_1, A_2, \cdots, A_n$

such that for each i; H_i is a component of $\bar{V}_1$, $H_i \cap V_2 \neq \emptyset$, A_i is open, $A_i \cap H_0 = \emptyset$, $A_i \supseteq \bigcup H_j (j = 1, \cdots, i)$, $H_{i+1} \cap A_i = \emptyset$, $A_{i+1} \supseteq A_i$, and $A_i \cap \bar{V}_1$ is closed. Since $A_n \cap \bar{V}_1$ is closed and $A_n \cap H_0 = \emptyset$, we see that $V_2 - A_n$ is an open set containing x_0'; hence, there is a component H_{n+1} of $\bar{V}_1$ such that $H_{n+1} \neq H_0$ and $H_{n+1} \cap (V_2 - A_n) \neq \emptyset$. Since A_n is both open and closed in $\bar{V}_1$ and H_{n+1} contains points not in A_n, it follows that $H_{n+1} \cap A_n = \emptyset$. We can find an open set B such that $B \cap H_0 = \emptyset$, $B \supseteq H_{n+1}$, and $B \cap \bar{V}_1$ is closed. Put $A_{n+1} = A_n \cup B$. It is easy to see that the pair of sequences $H_1, H_2, \cdots, H_{n+1}$ and $A_1, A_2, \cdots, A_{n+1}$ retain all the original properties. The axiom of induction thus guarantees the existence of countably infinite sequences $H_1, H_2, \cdots$ and $A_1, A_2, \cdots$ with the same properties.

Since X is sequentially compact, we can find a point $x_0 \in \bar{V}_2$ and a subsequence $H_{n(1)}, H_{n(2)}, \cdots$ such that x_0 is a sequential limit point of a sequence $a_1, a_2, \cdots$ where $a_i \in H_{n(i)}$. Each $H_{n(i)}$ must intersect $\mathrm{Bd}(V_1)$ and hence, again by sequential compactness, we can find a subsequence $s(1), s(2), \cdots$ of the integers and a point $y_0 \in \mathrm{Bd}(V_1)$ such that y_0 is a sequential limit point of a sequence $y_1, y_2, \cdots$ where $y_i \in H_{n(s(i))}$. Since $\bar{V}_2 \subseteq V_1$, $x_0 \neq y_0$. Put $C_i = H_{n(s(i))}$ and $x_i = a_{s(i)}$ and we have a violation of the definition of almost locally-connected (C_j will be open in the closure of $\bigcup C_i (i = 1, 2, \cdots)$ since $H_{n(s(j))} \subseteq (A_{n(s(j))} - A_{n(s(j))-1}) \cap \bar{V}_1$ which is an open set in $\bar{V}_1$ that does not intersect C_j for $j \neq i$). Therefore, X is locally connected.

Since X is itself a continuum from p to q, there is surely a subcontinuum which is irreducible from p to q, call it $E[p, q]$. $E[p, q]$ is unique, for two distinct, irreducible continua from p to q could not have a connected intersection (X must be hereditarily unicoherent). Let y be a point of $E(p, q)$. We must show that $X - y = A \cup B$ where A and B are disjoint open sets containing p and q respectively. This will be true if y is an element of every closed connected set which contains both p and q (see [3], Theorem 3-6). But, again because of hereditary unicoherence, this last statement is immediate. Now, since $E[p, q]$ is a compact, connected, Hausdorff space with just two noncut points, we see that the order topology induced by the separation order on $E[p, q]$ is the same as the subspace topology (see [3], Theorem 2-25).

We can now prove the following:

THEOREM 2.6. *Let X be a compact Hausdorff space which is also sequentially compact and connected. In order that $C(X)$ be algebraically closed, it is necessary and sufficient that X be hereditarily unicoherent and almost locally-connected.*

Proof. The necessity has already been shown. To prove sufficiency, let $P(x, z)$ be a monic polynomial over $C(X)$. Let $\mathscr{X} = \{(D, f): D$ is connected subset of X, $f \in C(D)$, and $P(x, f(x)) \equiv 0$ on $D\}$. If (D, f) and (D', f') are elements of $\mathscr{X}$, define $(D, f) \leqq (D', f')$ if and only if $D \subseteqq D'$ and $f'(x) \equiv f(x)$ on D. It is evident that $\mathscr{X}$ is not empty.

Suppose $\{(D_\alpha, f_\alpha)\}_{\alpha \in I}$ is a linearly ordered subset of $\mathscr{X}$. Let $D = \bigcup D_\alpha(\alpha \in I)$ and let $f = \bigcup f_\alpha(\alpha \in I)$. Since $\{(D_\alpha, f_\alpha)\}_{\alpha \in I}$ is linearly ordered, it is clear that f is a well-defined function on D. Surely D is connected and $P(x, f(x)) \equiv 0$ on D. If we can show that $f \in C(D)$, we will have found an upper bound in $\mathscr{X}$ for $\{(D_\alpha, f_\alpha)\}_{\alpha \in I}$ and thus, by Zorn's Lemma, we will know that $\mathscr{X}$ has maximal elements.

Suppose there is a point x_0 of D at which f is not continuous. Then there is a $\varepsilon_1 > 0$ such that for no nbhd V of x_0 is it true that $f(V) \subseteqq \{z: |z - f(x_0)| < \varepsilon_1\}$. Let $z_1 = f(x_0), z_2, \cdots, z_k$ be the distinct roots of $P(x_0, z)$ and let $\varepsilon_2 = (1/2) \min\{|z_i - z_j|: i \neq j\}$. Let ε be the smaller of ε_1 and ε_2. There is a nbhd $V(x_0)$ of x_0 such that if $x \in V(x_0)$ and $P(x, z) = 0$, then $|z - z_i| < \varepsilon$ for some i (apply (1.2) to each z_i and take the intersection of the resulting nbhds). Since X is locally connected (see the proof of (2.5)) we can take $V(x_0)$ to be connected. Now $f(V(x_0)) \nsubseteq \{z: |z - z_1| < \varepsilon\}$, hence there is a point y_0 such that $y_0 \in V(x_0)$ and $|f(y_0) - z_1| \geqq \varepsilon$. Find $a \in I$ so that both x_0 and y_0 are in D_α, and notice that it will then follow that $E[x_0, y_0] \subseteqq D_\alpha$ (remember, each point of $E(x_0, y_0)$ separates $E[x_0, y_0]$ in X). Now we see the contradiction, for $f(E[x_0, y_0]) = f_\alpha(E[x_0, y_0])$, and being a continuous function on D_α, f_α carries connected sets onto connected sets; however, $f(E[x_0, y_0]) \subseteqq \bigcup \{z: |z - z_i| < \varepsilon\}(i = 1, 2, \cdots, k)$ and these are disjoint open sets. Thus f must be continuous on D and $\mathscr{X}$ has maximal elements.

Let (D^*, f^*) be a maximal element of $\mathscr{X}$. If $D^* = X$, we are done, so assume $y_0 \in X - D^*$ and let $x_0 \in D^*$. Note that, in general, if $r \in E[p, q]$ then $E[p, r] \cup E[r, q] = E[p, q]$. Thus $E[x_0, y_0] \cap D^* = \bigcup E[x_0, y](y \in E[x_0, y_0] \cap D^*)$ and therefore, $E[x_0, y_0] \cap D^*$ is connected. It is thus clear that there is a point m of $E[x_0, y_0]$ such that $E[x_0, m] - m \subseteqq D^*$ and $E[m, y_0] - m \subseteqq X - D^*$. We need to show that $m \in D^*$. If $m \notin D^*$, then there can be no way of extending f^* continuously to $D^* + m((D^*, f^*)$ is maximal). This means that there is a $\varepsilon_1 > 0$ such that, if V is any nbhd of m, there are points x and y in $V \cap D^*$ such that $|f(x) - f(y)| \geqq \varepsilon_1$. Let $z_1, z_2, \cdots, z_k$ be the distinct roots of $P(m, z)$ and let $\varepsilon_2 = (1/2) \min\{|z_i - z_j|: i \neq j\}$ and finally let ε be the smaller of ε_1 and ε_2. There is a connected nbhd $V(m)$ of m such that for $x \in V(m)$, each root of $P(x, z)$ is within ε of some z_i. Since $V(m)$ is connected, $E[x, y] \subseteqq V(m)$ whenever x and y are in $V(m)$. Also, $E[x, y] \subseteqq D^*$ whenever x and y are in D^*.

Therefore, $V(m) \cap D^*$ is connected (it contains a continuum between each pair of its points).

But f^* is continuous and hence $f^*(V(m) \cap D^*)$ is connected. On the other hand, $f^*(V(m) \cap D^*)$ cannot be connected since it is contained in $\bigcup \{z: |z - z_i| < \varepsilon\}(i = 1, 2, \cdots, k)$, a union of disjoint open sets, and meets at least two of these sets. Thus f^* would be extendable to $D^* + m$, and since this cannot be, we know that $m \in D^*$.

Since $E[m, y_0]$ is a linearly ordered, order-complete space, we can find continuous functions $f_1, f_2, \cdots, f_n$ on $E[m, y_0]$ (see [2], the proof of Theorem 3) such that $P(x, z) = (z - f_1(x))(z - f_2(x)) \cdots (z - f_n(x))$ on $E[m, y_0]$. Put $D' = D^* \cup E[m, y_0]$ and put $f' = f^* \cup f_s$ where s is such that $f^*(m) = f_s(m)$. Since $D^* - m$ and $E(m, y_0]$ are open in D', it is easy to see that $f' \in C(D')$. Thus $(D^*, f^*) < (D', f')$, but since (D^*, f^*) is maximal, we have a contradiction. Hence $D^* = X$ and f^* is a continuous root for $P(x, z)$ and $C(X)$ is algebraically closed.

3. A general sufficient condition. The sufficient condition which is developed in this section was obtained by generalizing the methods used by Deckard and Pearcy to prove that $C(X)$ is algebraically closed if X is compact, Hausdorff, and totally disconnected (see [2]). Since the components of such a space are single points, it is obvious that one can find continuous root functions on each component. As a start in formulating our sufficient condition, we require that the space satisfy the following definition.

DEFINITION 3.1. A compact Hausdorff space X is a *C-space* if, given a component M of X, a point x_0 of M, a monic polynomial $P(x, z)$ over $C(X)$, and a root z_0 of $P(x_0, z)$; one can always find a function $r(x)$ in $C(M)$ such that $r(x_0) = z_0$ and $P(x, r(x)) \equiv 0$ on M.

The method of Deckard and Pearcy is to prove, by an inductive method, that local continuous solutions can be found, then to patch together the local solutions (using the fact that there is a basis for the topology consisting of open and closed sets). The analogous procedure here would be to show that continuous solutions can be found in a nbhd of each component of X, and then patch. The proof of this itself requires the patching of still smaller local solutions (as is the case with Deckard and Pearcy), and in order to carry out this plan, it seems necessary to assume that there is a base for the topology on X consisting of open sets with finite boundaries, i.e. every point is of finite order (see [5], p. 48). The form in which we use this assumption is contained in the following definition.

DEFINITION 3.2. If X is a space and V is a subset of X, V is an

A-set in case V is open and $\mathrm{Bd}(V)$ is finite or empty and is contained in a single component of $\bar{V}$. If $\mathrm{Bd}(V)$ is not empty, then the component of $\bar{V}$ which contains $\mathrm{Bd}(V)$ is the *A-component* of $\bar{V}$. If $x \in V$, then V is an *A-nbhd* of x in case $\mathrm{Bd}(V)$ is empty, or else x is a point of the A-component of $\bar{V}$. Finally, if there is a base for the topology at each point x of X consisting of A-nbhds of x, then X is an *A-space.*

Let us see that a compact Hausdorff space, in which each point is of finite order, is an A-space. Suppose that V is an open set and p is a point of V. Let V_1 be an open set with finite boundary such that $p \in V_1 \subseteqq V$. Let Q be the component of $\bar{V}_1$ which contains p, and let R be the set consisting of all the boundary points of V_1 which are not in Q. Since $\bar{V}_1$ is compact and Hausdorff, there are disjoint closed sets E and G, containing Q and R respectively, such that $E \cup G = \bar{V}_1$. Let $V_2 = E \cap V_1$. It is a routine computation to show that V_2 is an A-nbhd of p and thus see that the space is an A-space.

Note that the property that each point be of finite order is clearly hereditary so that each closed subset of a compact Hausdorff A-space is itself an A-space.

The following lemma will be invaluable when the time comes to patch together several local solutions.

LEMMA 3.3. *Let X be a compact Hausdorff space and let V_1, $V_2, \cdots, V_n$ be a finite sequence of A-sets. There is another sequence $V_1', V_2', \cdots, V_n'$ of A-sets such that $\bigcup V_i'$ $(i = 1, 2, \cdots, n) = \bigcup V_i$ $(i = 1, 2, \cdots, n)$ and if $V_i' \neq V_i$, then $V_i' \subseteqq V_i$ and V_i' is closed. Further, if $i < j$ and $V_j' \cap \mathrm{Bd}(V_i') \neq \varnothing$, then V_j' is an A-nbhd of each point of $V_j' \cap \mathrm{Bd}(V_i')$. We will call a sequence of A-sets satisfying this last condition an A-sequence.*

Proof. We use induction. If $n = 1$, put $V_1' = V_1$ and there is nothing to prove. Assume the lemma true if $n < m$ and consider a sequence $V_1, V_2, \cdots, V_m$ of A-sets. We can find an A-sequence V_2', $V_3', \cdots, V_m'$ such that $\bigcup V_i'$ $(i = 2, 3, \cdots, m) = \bigcup V_i$ $(i = 2, 3, \cdots, m)$, and if $V_i' \neq V_i$, then $V_i' \subseteqq V_i$ and V_i' is closed. If the sequence $V_1, V_2', V_3', \cdots, V_m'$ is an A-sequence, put $V_1' = V_1$ and we are done. If it is not an A-sequence, then there is an integer s and a point x_s of V_s' such that x_s is a point of $V_s' \cap \mathrm{Bd}(V_1)$ and yet V_s' is not an A-nbhd of x_s. It then follows that the component H_s of $\bar{V}_s'$ which contains x_s is contained in V_s'. Since x is compact and Hausdorff, it must be that H_s is actually a component of X. Hence there is an open and closed set V such that $H_s \subseteqq V \subseteqq V'_s$. Further, since H_s is a component of X, H_s contains the A-component of $\bar{V}_1$ and hence contains Bd(V_1). Thus, putting $V''_1 = V_1 - V$, we obtain an open and closed set. Now $V''_1, V'_2, \cdots, V'_m$ is an A-sequence, and the lemma follows by induction.

We are now ready to prove the general sufficient condition.

THEOREM 3.4. *Let X be a compact Hausdorff space. If X is an AC-space then $C(X)$ is algebraically closed.*

Proof. The proof rests on establishing the following.

Local result 3.5. If $x_0 \in X$, M is the component of X containing x_0, $P(x, z)$ is a monic polynomial over $C(X)$, $r(x) \in C(M)$ such that $P(x, r(x)) = 0$ on M; then there is a nbhd $N(x_0)$ of x_0 and a function $r^*(x) \in C(N(x_0))$ such that $P(x, r^*(x)) \equiv 0$ on $N(x_0)$ and $r^*(x) = r(x)$ if $x \in M \cap N(x_0)$.

We prove this by induction on the multiplicity of the root $r(x_0)$ of the polynomial $P(x_0, z)$. If $r(x_0)$ is a simple root, we can find, by (1.2), a nbhd $N'(x_0)$ of x_0 such that for $x \in N'(x_0)$, $P(x, z)$ has just one root satisfying $|z - r(x_0)| < \varepsilon$, where ε is half the minimum distance between distinct roots of $P(x_0, z)$. Let $N''(x_0)$ be a nbhd of x_0 such that for $x \in M \cap N''(x_0)$, $|r(x) - r(x_0)| < \varepsilon$ (remember, $r(x)$ is continuous on M). Let $N(x_0) = N'(x_0) \cap N''(x_0)$.

If $x \in N(x_0)$, define $r^*(x)$ to the root (there is only one) of $P(x, z)$ which satisfies $|z - r(x_0)| < \varepsilon$. We must show that $r^*(x) \in C(N(x_0))$. Let y_0 be any point of $N(x_0)$, and let W be an open set of complex numbers such that $r^*(y_0) \in W$. Let ε' be so small that if $|z - r^*(y_0)| \leqq \varepsilon'$, then $|z - r(x_0)| < \varepsilon$ and $z \in W$. Since $P(y_0, z)$ has only one root satisfying $|z - r(x_0)| < \varepsilon$, it is clear that there is no root satisfying $0 < |z - r^*(y_0)| \leqq \varepsilon'$; therefore, by (1.2), there is a nbhd $N(y_0)$ of y_0 such that for $x \in N(y_0)$, $P(x, z)$ has exactly one root satisfying $|z - r^*(y_0)| < \varepsilon'$. Now if $x \in N(x_0) \cap N(y_0)$, we see that $|r^*(x) - r^*(y_0)| < \varepsilon'$ (otherwise $P(x, z)$ would have two roots satisfying $|z - r(x_0)| < \varepsilon$, $r^*(x)$ and the one satisfying $|z - r^*(y_0)| < \varepsilon'$). Thus $N(x_0) \cap N(y_0)$ is contained in the inverse image of W, and it follows that the latter is open in $N(x_0)$ so that $r^*(x) \in C(N(x_0))$.

Clearly $P(x, r^*(x)) \equiv 0$ on $N(x_0)$. To see that $x \in N(x_0) \cap M$ implies that $r^*(x) = r(x)$, recall that $x \in N(x_0) \cap M$ implies that $|r(x) - r(x_0)| < \varepsilon$. Thus $r(x) \neq r^*(x)$ would give two roots of $P(x, z)$ satisfying $|z - r(x_0)| < \varepsilon$.

Now assume that the local result holds whenever the multiplicity is less than $k \geqq 2$. Let x_0 be a point of M such that the multiplicity of $r(x_0)$ is k. Again, let ε be half the minimum distance between distinct roots of $P(x_0, z)$. Let $N'(x_0)$ be a nbhd of x_0 such that for $x \in N'(x_0)$, $P(x_0, z)$ has exactly k roots satisfying $|z - r(x_0)| < \varepsilon$, and the remaining roots are each within ε of some root of $P(x_0, z)$ (apply (1.2) to each distinct root of $P(x_0, z)$ and take the intersection of the resulting nbhds). Let $N''(x_0)$ be a nbhd of x_0 such that for $x \in N''(x_0) \cap M$, $|r(x) - r(x_0)| < \varepsilon$. Let $N(x_0)$ be an A-nbhd of x_0 whose closure, say Y, is contained in $N'(x_0) \cap N''(x_0)$.

We divide Y into two parts. Let B be the subset of Y consisting of all points x such that $P(x, z)$ has at least two distinct roots satisfying $|z - r(x_0)| < \varepsilon$. A simple application of (1.2) shows that B is open in Y. If n is a positive integer, let B_n be the subset of Y consisting of all points x such that $P(x, z)$ has distinct roots which satisfy $|z - r(x_0)| < \varepsilon$ and are at least $1/n$ apart. Another application of (1.2) shows that each B_n is closed in Y and hence is compact. Obviously $B = \bigcup B_n$ $(n = 1, 2, \cdots)$. We will obtain $r^*(x)$ by first defining it on B, and then extending it to all of Y.

Let $\{H_\alpha\}$ $(\alpha \in I)$ be the collection of components of Y. If $\alpha \in I$ is such that $H_\alpha \subseteqq M$, let $F_\alpha(x)$ be the function defined on H_α such that for $x \in H_\alpha$, $F_\alpha(x) = r(x)$. If $\alpha \in I$ is such that $H_\alpha \cap M = \varnothing$ (this is the only other possibility), let $x(\alpha)$ be a point of H_α and let $z(\alpha)$ be a root of $P(x(\alpha), z)$ such that $|z(\alpha) - r(x_0)| < \varepsilon$. Since X is a C-space, there is a continuous function $r_\alpha(x)$, defined on the component of X which contains H_α, such that $r_\alpha(x(\alpha)) = z(\alpha)$ and $P(x, r_\alpha(x)) = 0$ for all relevant x. Let $F_\alpha(x)$ be the restriction of $r_\alpha(x)$ to H_α. It is clear that for $\alpha \in I$, $F_\alpha(x) \in C(H_\alpha)$ and $P(x, F_\alpha(x)) \equiv 0$ on H_α. Now, because $Y \subseteqq N''(x_0)$, we can see that if $\alpha \in I$ is such that $H_\alpha \subseteqq M$, then $|F_\alpha(x) - r(x_0)| < \varepsilon$ whenever $x \in H_\alpha$. Further, the same statement holds even if $H_\alpha \cap M = \varnothing$, for, since $Y \subseteqq N'(x_0)$, if $x \in H_\alpha$ then there is no root of $P(x, z)$ which satisfies $|z - r(x_0)| = \varepsilon$. Thus, since $|z(\alpha) - r(x_0)| < \varepsilon$ and H_α is connected, it must follow that $F_\alpha(x)$ satisfies $|z - r(x_0)| < \varepsilon$ whenever $x \in H_\alpha$. We see, therefore, that if $\alpha \in I$ and $x \in H_\alpha \cap B$, then the multiplicity of the root $F_\alpha(x)$ of $P(x, z)$ is less that k, and hence the local result holds at each point of B subject only to the restriction that the '$r(x)$' be a $F_\alpha(x)$.

We now restrict our attention to the closed subspace Y of X, and all topological terms will be relative to Y. As we have seen, Y is an A-space, and the local result, (3.5), holds at each point of B with the restriction that the '$r(x)$' be a $F_\alpha(x)$. Thus, if $x \in B$, there is a nbhd $V(x)$ of x and a function $g_x(y) \in C(V(x))$ such that $P(y, g_x(y)) \equiv 0$ on $V(x)$ and $g_x(y) = F_\alpha(y)$ if $y \in H_\alpha \cap V(x)$, where $\alpha \in I$ is such that $x \in H_\alpha$. Since $|F_\alpha(x) - r(x_0)| < \varepsilon$ and $g_x(y)$ is continuous, we can assume that $|g_x(y) - r(x_0)| < \varepsilon$ on $V(x)$. We can further assume that $V(x)$ is an A-nbhd of x, and that $g_x(y)$ is actually defined and has all the above properties even on the closure of $V(x)$. Since B_1 is compact, there is a finite number of such sets which cover B_1.

To summarize all of this, there is a sequence $V_1, V_2, \cdots, V_n$ of A-sets and a sequence $g_1, g_2, \cdots, g_n$ of functions such that

$$B_1 \subseteqq \bigcup V_i \ (i = 1, 2, \cdots, n) \subseteqq \bigcup \bar{V}_i \ (i = 1, 2, \cdots, n) \subseteqq B ,$$
$$g_i \in C(\bar{V}_i) \text{ and } | g_i(x) - r(x_0) | < \varepsilon \text{ on } \bar{V}_i \ i = 1, 2, \cdots, n ,$$
$$P(x, g_i(x)) \equiv 0 \text{ on } \bar{V}_i \ i = 1, 2, \cdots, n ,$$
$$\text{if } H \text{ is the } A\text{-component of } \bar{V}_i \text{ and } H \subseteqq H_\alpha \text{ then for } x \in H ,$$
$$g_i(x) = F_\alpha(x) \ i = 1, 2, \cdots, n .$$

In view of (3.3), it is apparent that we can take the V_i to be an A-sequence.

Now since the V_i are A-sets, the boundary of their union is finite (being contained in the union of the boundaries). Let $V_{n+1}, V_{n+2}, \cdots, V_m$ be a sequence of A-sets with disjoint closures so that $\bar{V}_j \subseteqq B$ for $n + 1 \leqq j \leqq m$ and each boundary point of $\bigcup V_i \ (i = 1, 2, \cdots, n)$ is contained in exactly one V_j with $n + 1 \leqq j \leqq m$ and that V_j is an A-nbhd of that point. It should be evident that these nbhds can be chosen in such a way that there are continuous functions $g_{n+1}, g_{n+2}, \cdots, g_m$ on the corresponding $\bar{V}_j$ such that for $n + 1 \leqq j \leqq m$; $| g_j(x) - r(x_0) | < \varepsilon$ on $\bar{V}_j$, $P(x, g_j(x)) \equiv 0$ on $\bar{V}_j$, and if H is the A-component of $\bar{V}_j$ and $H \subseteqq H_\alpha$, then $g_j(x) \equiv F_\alpha(x)$ on H. Because of the way in which $V_{n+1}, \cdots, V_m$ were chosen, $V_1, V_2, \cdots, V_m$ is an A-sequence.

Now $B_2 - \bigcup V_i \ (i = 1, 2, \cdots, m)$ is compact and is disjoint from $\bigcup \bar{V}_i \ (i = 1, 2, \cdots, n)$, therefore, we can find a sequence $V_{m+1}, V_{m+2}, \cdots, V_t$ of A-sets such that $B_2 - \bigcup V_i \ (i = 1, 2, \cdots, m)$ is contained in $\bigcup V_i$ $(i = m + 1, m + 2, \cdots, t)$, and for $m + 1 \leqq j \leqq t$, $\bar{V}_j \subseteqq B$ and does not intersect $\bigcup \bar{V}_i \ (i = 1, 2, \cdots, n)$. Again, it should be clear that $V_{m+1}, V_{m+2}, \cdots, V_t$ can be chosen so that there is a sequence $g_{m+1}, g_{m+2}, \cdots, g_t$ of continuous functions on the corresponding $\bar{V}_j$ such that for $m + 1 \leqq j \leqq t$; $P(x, g_j(x)) \equiv 0$, $| g_j(x) - r(x_0) | < \varepsilon$, and if H is the A-component of $\bar{V}_j$ and $H \subseteqq H_\alpha$, then $g_j(x) \equiv F_\alpha(x)$ on H. In view of (3.3), we may also assume that $V_{m+1}, V_{m+2}, \cdots, V_t$ is an A-sequence.

It is a rather tedious but straightforward task to verify the following facts.

$$V_1, V_2, \cdots, V_t \text{ is a } A\text{-sequence,}$$
$$g_i \in C(\bar{V}_i) \ 1 \leqq i \leqq t ,$$
$$\text{for } x \in \bar{V}_i, \ P(x, g_i(x)) = 0 \text{ and } | g_i(x) - r(x_0) | < \varepsilon \ 1 \leqq i \leqq t ,$$
$$B_2 \subseteqq \bigcup V_i \ (i = 1, 2, \cdots, t) \subseteqq \bigcup \bar{V}_i \ (i = 1, 2, \cdots, t) \subseteqq B, \text{ and}$$
$$\text{if } H \text{ is the } A\text{-component of } \bar{V}_i \text{ and } H \subseteqq H_\alpha, \text{ then } g_i(x) \equiv F_\alpha(x)$$
$$\text{on } H \ 1 \leqq i \leqq t .$$

It should now be clear how to continue the process indefinitely and so to obtain countably infinite sequences $V_1, V_2, \cdots$ and $g_1, g_2, \cdots$ such that

$V_1, V_2, \cdots, V_i$ is an A-sequence $i = 1, 2, \cdots$,
$g_i \in C(\bar{V}_i)$ $i = 1, 2, \cdots$,
for $x \in \bar{V}_i$, $P(x, g_i(x)) = 0$ and $|g_i(x) - r(x_0)| < \varepsilon$ $i = 1, 2, \cdots$,
$B = \bigcup V_i$ $(i = 1, 2, \cdots) = \bigcup \bar{V}_i$ $(i = 1, 2, \cdots)$, and
if H is the A-component of $\bar{V}_i$ and $H \subseteqq H_\alpha$, then $g_i(x) \equiv F_\alpha(x)$ on H $i = 1, 2, \cdots$.

If $x \in B$, define $r^*(x) = g_n(x)$, where n is the smallest integer such that $x \in V_n$. If $x \in Y - B$, let $r^*(x)$ be the root of $P(x, z)$ which satisfies $|z - r(x_0)| < \varepsilon$. It is obvious that $P(x, r^*(x)) \equiv 0$ on Y.

Let us see that $r^*(x) \equiv r(x)$ on $M \cap Y$. If $x \in (M \cap Y) - B$, then $r^*(x)$ is the root of $P(x, z)$ which satisfies $|z - r(x_0)| < \varepsilon$. This is true of $r(x)$, hence $r(x) = r^*(x)$. If $x \in M \cap Y \cap B$, let V_n be the first V_j which contains x. Since $M \cap Y$ is compact, connected, and Hausdorff (remember, Y is the closure in X of an A-nbhd of x_0) and $M \cap V_n$ is a proper $(V_n \subseteqq B)$ open subset of $M \cap Y$, the closure of each component of $M \cap V_n$ must meet the boundary of V_n. Thus x is in the A-component of $\bar{V}_n$, hence $r^*(x) = g_n(x) = F_\alpha(x) = r(x)$, since $H_\alpha \subseteqq M$ implies that $F_\alpha(x)$ is the restriction of $r(x)$ to H_α.

It remains to be shown that $r^*(x)$ is continuous on Y. If $y \in Y - B$, using the fact that $|r^*(x) - r(x_0)| < \varepsilon$ on Y, one can apply (1.2) to show $r^*(x)$ is continuous at y. If $y \in B$, let V_n be the first V_j which contains y. To show that $r^*(x)$ is continuous at y, it suffices to show that for $j < n$, if $y \in \bar{V}_j$ then $g_j(y) = g_n(y)$. Indeed, since $V_1, V_2, \cdots, V_n$ is an A-sequence, if $j < n$ and $y \in \bar{V}_j$, then y is in the A-component of $\bar{V}_j$ (since $y \in \mathrm{Bd}(V_j)$) and y is in the A-component of $\bar{V}_n$ (since V_n must be an A-nbhd of y); therefore, $g_j(y) = F_\alpha(y) = g_n(y)$, where $\alpha \in I$ is such that H_α is the component of Y which contains y. Recalling that Y is the closure of a nbhd of x_0, we see that (3.5) holds if the multiplicity of $r(x_0)$ is k, and thus, by induction, (3.5) holds in general.

It should now be quite evident how one obtains (3.4) from (3.5).

4. We now return our attention to the question concerning the sufficiency of the necessary condition of § 2. The answer we obtain is that if the compact Hausdorff space in question is first-countable, then the necessary condition is sufficient. This fact was discovered in a very natural way, namely, by asking under what conditions does the necessary condition of §2 imply the sufficient condition of §3. If X is sequentially compact, (2.6) applies to the components of X. A

close look at the proof of (2.6) shows that the following theorem was actually proved.

THEOREM 2.6′. *Let X be a compact Hausdorff space which is also sequentially compact and connected. In order that X be a C-space, it is necessary and sufficient that X be hereditarily unicoherent and almost locally-connected.*

Thus, with the added condition of sequential compactness, the necessary condition implies the second half of the sufficient condition.

In order to show that the necessary condition implies that the space is an A-space, it seems necessary to assume that the space is first-countable (an assumption which also implies, for compact spaces, sequential compactness).

THEOREM 4.1. *Let X be a compact Hausdorff space which is hereditarily unicoherent, almost locally-connected, and first-countable. Then X is an A-space.*

Proof. As we have seen, it will suffice to show that every point of X is of finite order. Accordingly, let x_0 be a point of X and let V be an open set containing x_0. We must show that there is an open set with finite boundary which contains x_0 and is contained in V. To that end we shall need the following fact whose proof will be delayed until the end of this section.

LEMMA 4.2. *Let X be a compact Hausdorff space which is hereditarily unicoherent, almost locally-connected, and first-countable. If a and b are distinct points of a component of X and V is an open set containing a, then there are a point c of V and disjoint open sets A and B containing a and b respectively such that $X - c = A \cup B$.*

For each point x of $X - V$; either x is not in the same component of X as x_0, in which case there are disjoint open sets $A(x)$ and $B(x)$ containing x_0 and x respectively such that $A(x) \cup B(x) = X$, or, by (4.2), there are a point $c(x)$ of V and disjoint open sets $A(x)$ and $B(x)$ containing x_0 and x respectively such that $A(x) \cup B(x) = X - c(x)$. Since $X - V$ is compact, there are finitely many points $x_1, x_2, \cdots, x_n$ of $X - V$ such that the corresponding $B(x_i)$ cover $X - V$. Putting $V' = \bigcap A(x_i)$ $(i = 1, 2, \cdots, n)$, we obtain an open set with finite boundary which contains x_0 and is contained in V. Thus x_0 is of finite order and X is an A-space.

We now have as a corollary to (2.6′) and (4.1) the answer we are seeking.

COROLLARY 4.3. *Let X be a first-countable compact Hausdorff space. A necessary and sufficient condition that $C(X)$ be algebraically closed is that X be hereditarily unicoherent and almost locally-connected.*

Proof of 4.2. Choose a point c of $E[a, b] \cap V$ distinct from a (remember, (2.5) applies to the components of X). There are separated sets A' and B' containing a and b respectively such that $(A' \cup B') + c = H$ is the component of X which contains a. Let $V_1, V_2, \cdots$ be a countable base of open sets for the topology at c such that $V_i \supseteqq \bar{V}_{i+1}$. Let $\{H_\alpha\}_{\alpha \in I}$ be the collection of components of X which are distinct from H. If $\alpha \in I$ is such that for some i, $H_\alpha \subseteqq V_i$ but $H_\alpha \not\subseteqq V_{i+1}$, let F_α be an open and closed set such that $V_i \supseteqq F_\alpha \supseteqq H_\alpha$. If $\alpha \in I$ is such that $H_\alpha \subseteqq V_i$ fails for all i, let F_α be an open and closed set such that $H_\alpha \subseteqq F_\alpha$ and $F_\alpha \cap H = \emptyset$.

Now $A' - V_1$ and $B' - V_1$ are separated in $X - V_1$, and hence, since $X - V_1$ is compact, $X - V_1 = A_1 \cup B_1$ where $A_1 \supseteqq A' - V_1$ and $B_1 \supseteqq B' - V_1$ and A_1 and B_1 are disjoint closed sets. Since X is sequentially compact, there are at most a finite number of α in I such that H_α intersects both A_1 and B_1 (recall the proof of (2.5)). Let $\alpha_1, \alpha_2, \cdots, \alpha_{n(1)}$ be all such α. Let

$$A'_1 = (A_1 \cup A') - \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(1))$$

and

$$B'_1 = (B_1 \cup B') \cup \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(1)) .$$

We note that H_{α_i} $(i = 1, 2, \cdots, n(1))$ must intersect V_1, A'_1 and B'_1 are separated in X, $A'_1 \supseteqq A'$, $B'_1 \supseteqq B'$, and $A'_1 \cup B'_1 \supseteqq X - V_1$.

There is a positive integer k_1 such that $(A'_1 \cup B'_1) \cap V_{k_1}$ is contained in $A' \cup B'$. There are disjoint closed sets A_2 and B_2 such that $A_2 \supseteqq A'_1 - V_{k_1}$ and $B_2 \supseteqq B'_1 - V_{k_1}$ and $A_2 \cup B_2 \supseteqq X - V_{k_1}$. Again, there are at most a finite number of $\alpha \in I$ such that H_α intersects both A_2 and B_2. Let $\alpha_{n(1)+1}, \alpha_{n(1)+2}, \cdots, \alpha_{n(2)}$ be the set of such α. Let

$$A'_2 = (A_2 \cup A') - \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(2))$$

and

$$B'_2 = (B_2 \cup B') \cup \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(2)) .$$

Note that H_{α_i} $(i = n(1) + 1, n(1) + 2, \cdots, n(2))$ must intersect V_{k_1}, A'_2 and B'_2 are separated in X, $A'_2 \supseteqq A'$, $B'_2 \supseteqq B'$, and $A'_2 \cup B'_2 \supseteqq X - V_{k_1}$. We will also need that $A'_2 \supseteqq A'_1 - \bigcup F_{\alpha_i}$ $(i = 1, 2, \cdots, n(2))$ and $B'_2 \supseteqq B'_1$. To see this, observe that $A_2 \supseteqq A'_1 - V_{k_1}$ and thus

$$A_2 \cup A' \supseteqq (A'_1 - V_{k_1}) \cup A' = (A'_1 - (A'_1 \cap V_{k_1})) \cap A' \supseteqq (A'_1 - A') \cup A' = A'_1 ,$$

so that $A_2' = (A_2 \cup A') - \bigcup F_{\alpha_i} \supseteqq A_1' - \bigcup F_{\alpha_i}$. Similarly, $B_2 \cup B' \supseteqq B_1'$, so that $B_2' \supseteqq B_1'$.

There is a positive integer k_2 such that $(A_2' \cup B_2') \cap V_{k2} \subseteqq A' \cup B'$. There are disjoint closed sets A_3 and B_3 such that $A_3 \supseteqq A_2' - V_{k_2}$, $B_3 \supseteqq B_2' - V_{k_2}$, and $A_3 \cup B_3 = X - V_{k_2}$. There are at most a finite number of α in I such that H_α meets both A_3 and B_3; denote these α by $\alpha_{n(2)+1}, \cdots, \alpha_{n(3)}$. Let

$$A_3' = (A_3 \cup A') - \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(3))$$

and

$$B_3' = (B_3 \cup B') \cup \bigcup F_{\alpha_i} \ (i = 1, 2, \cdots, n(3)) .$$

One can see, as before, that H_{α_i} $(i = n(2) + 1, \cdots, n(3))$ must intersect V_{k_2}, A_3' and B_3' are separated in X, $A_3' \supseteqq A'$, $B_3' \supseteqq B'$, $A_3' \cup B_3' \supseteqq X - V_{k_2}$, $A_3' \supseteqq A_2' - \bigcup F_{\alpha_i}$ $(i = 1, 2, \cdots, n(3))$, and $B_3' \supseteqq B_2'$.

Continuing this construction countably often, one obtains five sequences, $\{A_i'\}$, $\{B_i'\}$, $\{\alpha_i\}$, $\{n(i)\}$, and $\{k_i\}$, with the following properties.

(1)	A_m' and B_m' are separated sets	$m = 1, 2, \cdots$
(2)	$A_{m+1}' \supseteqq A_m' - \bigcup F_{\alpha_i}$ $(i = 1, 2, \cdots, n(m))$	$m = 1, 2, \cdots$
(3)	$B_{m+1}' \supseteqq B_m' \supseteqq \bigcup F_{\alpha_i}$ $(i = 1, 2, \cdots, n(m))$	$m = 1, 2, \cdots$
(4)	$A_m' \cup B_m' \supseteqq X - V_{k_{m-1}}$ where $k_0 = 1$	$m = 1, 2, \cdots$
(5)	$A_m' \supseteqq A'$ and $B_m' \supseteqq B'$	$m = 1, 2, \cdots$,

and

(6) H_{α_m} intersects V_{k_i} $\qquad n(i) + 1 \leqq m \leqq n(i + 1)$.

Now let $B = \bigcup B_i'$ $(i = 1, 2, \cdots)$ and $A = X - (B + c)$. We must show that A and B are open. Let $x \in B$, then surely $x \neq c$ so that there is a nbhd $V(x)$ of x and an integer i such that $V(x) \cap V_{k_{i-1}} = \emptyset$. Since A_i' and B_i' are separated, (1), there is a nbhd $V'(x)$ of x such that $V'(x) \cap A_i' = \emptyset$ ($x \in B_i'$ since (4) $\Rightarrow x \in A_i' \cup B_i'$). Thus $V(x) \cap V'(x)$ is contained in B_i' and hence in B so that B is open. Let $x \in A$, then $x \neq c$ and $x \notin B$ and there exist a nbhd $V(x)$ of x and an integer i such that $V(x) \cap V_{k_{i-1}} = \emptyset$. Since $x \notin B, x \notin B_i'$, and since $V(x) \subseteqq A_i' \cup B_i'$ (from (4) since $V(x) \cap V_{k_{i-1}} = \emptyset$), it follows that $x \in A_i'$. There is a nbhd $V'(x)$ of x such that $V'(x) \cap B_i' = \emptyset$ (A_i' and B_i' are separated), thus $V(x) \cap V(x) \subseteqq A_i'$. There are only finitely many α_j such that $F_{\alpha_j} \nsubseteqq V_{k_{i-1}}$ (see (6) and remember that $\bar{V}_{i+1} \subseteqq V_i$); subtracting these F_{α_j} from $V(x) \cap V'(x)$, we obtain a nbhd of x (by (3), no F_{α_i} contains x since $x \notin B$) contained in A_j' for $j \geqq i$ (here (2) is crucial) and hence which misses every B_i' and is therefore contained in A. Thus A is also open. Clearly $A \supseteqq A'$, $B \supseteqq B'$, and $A \cup B = X - c$, and (4.1) is established.

5. Remarks. (1) Of fundamental importance throughout this paper was the local connectivity of the components of X (all A-spaces have this property). The fact that βR (the Stone-Cech compactification of the reals) is not locally connected, but that nevertheless $C(\beta R) \cong C^*(R)$ is algebraically closed, indicates the limitations of the methods used here.

(2) It should be pointed out that the necessary condition was proved by assuming only that square roots could be taken. This leads to the conjecture: if X is compact and Hausdorff and if each element in $C(X)$ has a square root in $C(X)$, then $C(X)$ is algebraically closed. In this connection, note that the existence of all 2^nth roots for a given function need not imply the existence of all integral roots. If we identify in βR all limit points of the sequence (-2), $(-2)^2$, $(-2)^3, \cdots$, and call the resulting space αR, we can prove that the function $\exp(i\pi x)$, defined for all real x, has a continuous extension to all of αR, that this extension has continuous 2^nth roots for all n, but that no continuous fifth root exists.

(3) It has been shown that all compact and sequentially compact connected Hausdorff spaces which satisfy the necessary condition are trees (in the sense of L. E. Ward, Jr. [4]). One of the theorems in [4] states that trees are hereditarily unicoherent locally connected continua, thus we can say that in the presence of sequential compactness, a necessary and sufficient condition that $C(X)$ be algebraically closed is that X be a tree. This leads us to another conjecture: in the presence of first-countability, a necessary and sufficient condition that $C(X)$ be algebraically closed is that X be a closed subset of some first-countable tree.

REFERENCES

1. D. Deckard and C. Pearcy, *On matrices over the ring of continuous functions on a Stonian space*, Proc. Amer. Math. Soc. **14** (1963), 322-328.
2. D. Deckard and C. Pearcy, *On algebraic closure in function algebras*, Proc. Amer. Math. Soc. **15** (1964), 259-263.
3. J. G. Hocking and G. S. Young, *Topology*, Addison Wesley, Reading, 1961.
4. L. E. Ward, Jr., *Mobs trees and fixed points*, Proc. Amer. Math. Soc. **8** (1957), 798-804.
5. Whyburn, *Analytic Topology*, Amer. Math. Soc. Collog. Pub., Vol. 28.

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