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DECOMPOSITIONS OF E^3 WHICH YIELD E^3

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In recent years interest has been focused on the following two questions.

If G is an upper semi-continuous decomposition of E^3 whose decomposition space G' is homeomorphic to E^3, under what conditions can we conclude that

(1) each element of G is point-like?

(2) there is a pseudo-isotopy $F: E^3 \times [0, 1] \to E^3$ such that $F \mid E^3 \times 0$ is the identity and $F \mid E^3 \times 1$ is equivalent to the projection map $\Pi: E^3 \to G'$?

An example of Bing of a decomposition of E^3 into points, circles, and figure-eights shows that some additional hypotheses must be inserted. The theorem presented here gives such hypotheses, namely that the nondegenerate elements form the intersection of a decreasing sequence of finite disjoint unions of cells-with-handles, and project into a Cantor set.

For definitions and notation see [1]. In the example of Bing previously mentioned, the image of the union of the nondegenerate elements H^* under the projection map Π is an arc. Thus, the first condition one might impose in an attempt to answer the above questions is that $\Pi(H^*)$ be a Cantor set. I suspect that this is sufficient, however, that is still unknown. We use an additional hypothesis here.

THEOREM. *Let G be an upper semi-continuous decomposition of E^3 whose decomposition space G' is E^3 and let the image $\Pi(H^*)$ of the union of all the nondegenerate elements be a Cantor set. Suppose also that G is definable by cells-with-handles, that is*

$$H^* = \bigcap_{i=1}^{\infty} \left(\bigcap_{j=1}^{N_i} C_{ij} \right)$$

where each C_{ij} is a cell-with-handles, $C_{ij} \cap C_{ik} = \emptyset$ for $j \neq k$, and $\bigcup_{i=1}^{N_i} C_{ij}$ is contained in the point-set interior of $\bigcup_{j=1}^{N_{i-1}} C_{i-1,j}$ for $i = 2, 3, \cdots$. Then each element of G is point-like and there is a pseudo-isotopy $F: E^3 \times [0, 1] \to E^3$ such that $G = \{F^{-1}(x, 1)\}_{x \in E^3}$.

Proof of the theorem. By Bing's approximation theorem, we can assume that each C_{ij} is polyhedral. We will rely on the following theorem of Hempel [3].

THEOREM (*Hempel*). *Suppose C and C' are polyhedral 3-manifolds with boundary in S^3 such that C is a cell-with-handles and such* *that there is a map f of C onto C' which takes* $\mathrm{Bd}(C)$ *homeomorphically onto* $\mathrm{Bd}(C')$. *Then C and C' are homeomorphic; in particular, $f|_{\mathrm{Bd}(C)}$ can be extended to a homeomorphism of C onto C'.*

We will first show that if $g \in G$, then g is point-like. Let U be some neighborhood of g. Then some C_{ij} of the theorem is such that $g \subset \mathrm{Int}\, C_{ij} \subset C_{ij} \subset U$. We will find a cell C such that $g \subset \mathrm{Int}\, C \subset C_{ij}$. $\Pi(g)$ is a point and $\Pi(g) \in \mathrm{Int}\, \Pi(C_{ij})$ so there is a cell C' such that $\Pi(g) \in \mathrm{Int}\, C' \subset \Pi(C_{ij})$. For this fixed C' there must be an i' and a j' such that $\Pi(g) \in \Pi(C_{i'j'}) \subset \mathrm{Int}\, C'$. For each $k = 1, 2, \cdots, \hat{j'}, \cdots, N_{i'}$ we will modify Π on $C_{i',k}$ so that the new map Π' is a homeomorphism except on $C_{i',j'}$. We can do this because of Hempel's theorem. It is then easy to show that $\Pi'^{-1}(C')$ is the cell C we are seeking, since Π'^{-1} is a homeomorphism on $\mathrm{Bd}\, C'$.

In order to prove that G' may be realized by pseudo-isotopy we need only show the following lemma is true. The theorem will then follow [2].

LEMMA. *If G is as in the theorem, $\varepsilon > 0$ is given, and U is any neighborhood of H^*, then there is an isotopy $F: E^3 \times [0, 1] \to E^3$ such that $F|_{E^3 \times 0} = 1$, $F(x, t) = x$ for all $x \in E^3 - U$, $t \in [0, 1]$, and for each $g \in G$, $F(g, 1)$ has diameter less that ε.*

Proof. There is an i such that $C_{ij} \subset U$ for $j = 1, \cdots, N_i$. We will take $F(x, t) = x$ for all $x \in E^3 - \bigcup_{j=1}^{N_i} C_{ij}$. For each j there is a homeomorphism $h_j: \Pi(C_{ij}) \to C_{ij}$ which agrees with Π^{-1} on the boundary and we will define a map $\Pi': E^3 \to E^3$ as follows. For all $x \in E^3 - \bigcup_{j=1}^{N_i} C_{ij}$ let $\Pi'(x) = x$. For $x \in C_{ij}$ let $\Pi'(x) = h_j \Pi(x)$. There is an integer k such that $\Pi'(C_{kl})$ has diameter less than ε for each $l = 1, 2, \cdots, N_k$. We may also assume that Π' is piecewise linear on $E^3 - \bigcup_{l=1}^{N_k} \mathrm{Int}\, C_{kl}$. Using Hempel's result again we modify Π' on each C_{kl} so that the new map Π'' is a piecewise linear homeomorphism agreeing with Π' everywhere except in $\bigcup_{l=1}^{N_k} \mathrm{Int}\, C_{kl}$. Note that for each $g \in G$, $\mathrm{diam}\, \Pi''(g) < \varepsilon$. The proof is completed by the following lemma.

LEMMA. *Let C be a polyhedral cell-with-handles in E^3 and let h be a piecewise linear homeomorphism of E^3 onto itself such that $h|_{\mathrm{Bd}\, C}$ is the identity. Then $h|_C$ is isotopic to the identity.*

Proof of lemma. This lemma appears to be well known, however, an outline of the proof is included for completeness. Since C is a polyhedral cell will-handles, there is a collection of mutually disjoint polyhedral disks $D_i \cdots, D_n$ with $D_i \cap \mathrm{Bd}\, C = \mathrm{Bd}\, D_i$, $\mathrm{Int}\, D_i \subset \mathrm{Int}\, C$ and such that C is the union of two cells C_1 and C_2 whose intersection is $\bigcup_{i=1}^{n} D_i$. Since $h(D_i)$ is polyhedral and $h(D_i) \cap \mathrm{Bd}\, C = D_i \cap \mathrm{Bd}\, C$ there is an isotopy $H: C \times [0, 1] \to C$ with $H(x, 0) = h(x)$ $H(x, t) = x$ for all $x \in \mathrm{Bd}\, C$ and $t \in [0, 1]$ and $H(x, 1) = x$ for $x \in \bigcup_{i=1}^{n} D_i$. Then $H: C \times 1 \to C$ is a homeomorphism of C onto itself which is the identity on $\mathrm{Bd}\, C_1 \cup \mathrm{Bd}\, C_2$ and we may find the appropriate isotopy returning $H: C \times 1 \to C$ to the identity.

Question. In the theorem is the requirement that each C_{ij} is a cell-with-handles necessary? Certainly since the image of the union of the nondegenerate elements is a Cantor set in E^3, it has this cell-with-handles intersection property. It is true that a 3-manifold-with-boundary need not be a cell-with-handles in order to map onto a cell-with-handles with a map which is a homeomorphism on the boundary; however, I believe that these maps would have to have a continuum of nondegenerate elements. The maps we are considering have only a Cantor set of nondegenerate elements.

References

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