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POWER-SERIES AND HAUSDORFF MATRICES

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The purpose of this paper is to pair classes of continuous functions from $[0, 1]$ to the complex numbers with classes of complex sequences. If f is a function from $[0, 1]$ to the complex numbers and c is a complex sequence, a sequence $L(f, c)$ is defined:

$$L(f, c)_n = \sum_{p=0}^{n} f(p/n)\binom{n}{p}\sum_{q=0}^{n-p}(-1)^q\binom{n-p}{q}c_{p+q}\ .$$

A class A of continuous functions is paired with a class B of sequences provided that

(1) if f is in A and c is in B then $L(f, c)$ converges,

(2) if f is a continuous function and $L(f, c)$ converges for each c in B then f is in A, and

(3) if c is a sequence and $L(f, c)$ converges for each f in A then c is in B.

We establish the following pairings:

CONTINUOUS	SEQUENCES
all continuous functions	Hausdorff moment sequences
power-series absolutely convergent at 1	bounded sequences
power-series absolutely convergent at r ($r < 1$)	sequences dominated by geometric sequences having ratio r
entire functions	all sequences dominated by geometric sequences
polynomials	all sequences

Felix Hausdorff's work [2] (see also T. H. Hildebrandt [3]) on the moment problem for [0, 1] has been continued by J. S. Mac Nerney [5, p. 368] to provide the first pairing on the table (see Theorem B). Theorem A, also due to Mac Nerney [6, p. 56], helps establish the last pairing.

THEOREM A. *If f is a polynomial and is c is a complex sequence, then the sequence $L(f, c)$ converges. Furthermore, if $f = \sum_{p=0}^{n} A_p I^p$, where I is the identity function on the complex plane, then $L(f, c)$ has limit $\sum_{p=0}^{n} A_p c_p$.*

THEOREM B. *Suppose that c is a complex sequence. Then these are equivalent:*

(1) *There is a function g of bounded variation from $[0, 1]$ to the complex numbers such that, for each non-negative integer n, $c_n = \int_0^1 I^n dg$.*

(2) *For each* f *in* $C[0, 1]$, *the class of continuous functions from* $[0, 1]$ *to the complex numbers*, $L(f, c)$ *converges.*

Furthermore, if (1) *holds and* f *is in* $C[0, 1]$, *then* $L(f, c)$ *has limit* $\int_0^1 f dg$.

DEFINITION. If each of p and n is a nonnegative integer and c is a complex sequence, then $\Delta^0 c_p = c_p$ and $\Delta^{n+1} c_p = \Delta^n c_p - \Delta^n c_{p+1}$.

The following notes are helpful.

Note 1. If each of m and p is a nonnegative integer and c is a complex sequence,

$$\Delta^m c_p = \sum_{q=0}^{m} (-1)^q \binom{m}{q} c_{p+q} ,$$

so that if f is a function from $[0, 1]$ to the complex numbers then

$$L(f, c)_n = \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p} c_p f(p/n) .$$

DEFINITION. If each of p and k is a nonnegative integer, $Y_{pk} = \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} q^k$, where we interpret 0^0 as 1.

Note 2.

$$Y_{p+1,k+1} = (p + 1)(Y_{pk} + Y_{p+1,k}); \quad Y_{pp} = p\,!; \quad Y_{pk} \geqq 0; \quad Y_{pk} = 0$$

for $p > k$.

Note 3. If f is a function from $[0, 1]$ to the complex numbers and c is a complex sequence and n is a positive integer, then

$$\begin{aligned} L(f, c)_n &= \sum_{p=0}^{n} \binom{n}{p} \sum_{q=0}^{n-p} (-1)^q \binom{n-p}{q} c_{p+q} f(p/n) \\ &= \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n) , \end{aligned}$$

and, in case there is a complex sequence A such that, for each number x in $[0, 1]$, $f(x) = \sum_{k=0}^{\infty} A_k x^k$, then

$$\begin{aligned} L(f, c)_n &= \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{k=0}^{\infty} A_k n^{-k} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} q^k \\ &= \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{k=p}^{\infty} A_k n^{-k} Y_{pk} . \end{aligned}$$

The following theorem is useful in a later argument and is stated here for purposes of introduction.

THEOREM 0. *Let p be a nonnegative integer and let z be the sequence whose value at p is 1 and whose value elsewhere is 0. Let k be a function from $[0, 1]$ to the complex numbers which is continuous at 0. Then $L(k\cdot I^p, z)$ has limit $k(0)$.*

Indication of proof. If n is an integer greater than p, then

$$L(k\cdot I^p, z)_n = \binom{n}{p}\sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}k(q/n)q^p n^{-p}$$
$$= \binom{n}{p}n^{-p}\sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}q^p k(q/n);$$
$$\lim_{n\to\infty}\binom{n}{p}n^{-p} = 1/p!;\ p! = Y_{pp} = \sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}q^p;$$

and

$$\left|\sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}q^p k(q/n) - p!\,k(0)\right|$$
$$\leqq \sum_{q=0}^{p}\binom{p}{q}q^p\,|\,k(q/n) - k(0)\,|\,.$$

1. Radius of Convergence $\geqq 1$. In this chapter, for each r not less than 1, we pair functions having power-series expansions about 0 which are absolutely convergent at r with sequences which are dominated by geometric sequences with ratio r. In particular we match functions with power-series expansions about 0 which are absolutely convergent at 1 with the class of bounded sequences.

THEOREM 1. *Suppose that $r \geqq 1$, each of A and c is a complex sequence, $\sum_{p=0}^{\infty}|A_p|r^p$ converges, there is a number t such that if p is a nonnegative integer then $|c_p| \leqq t\cdot r^p$, and $f = \sum_{p=0}^{\infty}A_p I^p$. Then $L(f, c)$ converges to $\sum_{p=0}^{\infty}A_p c_p$.*

LEMMA 1. *If g is a function from $[0, 1]$ to the complex numbers and u is a constant sequence, then, for each positive integer n, $L(g, u)_n = u_0\cdot g(1)$.*

Proof.

$$L(g, u)_n = \sum_{p=0}^{n} g(p/n)\binom{n}{p}\Delta^{n-p}u_p = g(n/n)\binom{n}{n}u_n\,.$$

LEMMA 2. *Suppose that $d > 0$ and b is a complex sequence such that $\sum_{p=0}^{\infty}|b_p|$ converges. Then there is a positive integer N such that if n is an integer greater than N then*

$$\left|\sum_{p=0}^{n} b_p\left[1 - \binom{n}{p}n^{-p}p!\right]\right| < d\,.$$

Proof. We note that if p is a nonnegative integer and s is a sequence such that, for each positive integer n, $s_n = \binom{n}{p} n^{-p} p!$, then s is nondecreasing with limit 1.

Let m be a positive integer such that $\sum_{p=m}^{\infty} |b_p| < d/2$. There is an integer N greater than m such that if k is an integer in $[0, m]$ then, for each integer n greater than N,

$$1 - \binom{n}{k} n^{-k} k! < d/[2(m+1)(|b_k|+1)] .$$

If n is an integer greater than N,

$$\left| \sum_{p=0}^{n} b_p \left[1 - \binom{n}{p} n^{-p} p! \right] \right| \\ \leqq \sum_{p=0}^{m} |b_p| \left[1 - \binom{n}{p} n^{-p} p! \right] + \sum_{p=m}^{n} |b_p| < d .$$

LEMMA 3. *Let b be a positive number. Then there is a positive integer N such that if n is an integer greater than N then*

$$\sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| r^k n^{-k} Y_{pk} < b .$$

Proof. There is a positive integer m such that $\sum_{p=m}^{\infty} |A_p| r^p < b/2$. Let g be $\sum_{p=0}^{\infty} |A_p| r^p I^p$. Let N be an integer greater than m such that if n is an integer greater than N then

$$\sum_{p=0}^{n} |A_p| r^p \left[1 - \binom{n}{p} n^{-p} p! \right] < b/2 .$$

Then, if n is an integer greater than N,

$$\begin{aligned} b/2 &> g(1) - \sum_{p=0}^{n} |A_p| r^p \\ &= L(g, 1)_n - \sum_{p=0}^{n} |A_p| r^p \\ &= \sum_{p=0}^{n} \left[\binom{n}{p} \sum_{k=p}^{\infty} |A_k| r^k n^{-k} Y_{pk} - |A_p| r^p \right] \\ &= \sum_{p=0}^{n} \left[\binom{n}{p} n^{-p} p! - 1 \right] |A_p| r^p \\ &\qquad + \sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| r^k n^{-k} Y_{pk}. \end{aligned}$$

so

$$\sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| r^k n^{-k} Y_{pk} \\ \leqq b/2 + \sum_{p=0}^{n} |A_p| r^p \left[1 - \binom{n}{p} n^{-p} p! \right] < b .$$

Proof of Theorem 1. Let ε be a positive number. There is a positive integer N such that if n is an integer greater than N then

$$|\sum_{p=0}^{\infty} A_p c_p - \sum_{p=0}^{n} A_p c_p| < \epsilon/2,$$

$$\sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| r^k n^{-k} Y_{pk} < \varepsilon/(4t),$$

and

$$\left|\sum_{p=0}^{n} A_p c_p \left[1 - \binom{n}{p} n^{-p} p!\right]\right| < \varepsilon/4 .$$

Hence, if n is an integer greater than N,

$$\begin{aligned}
&\left|L(f, c)_n - \sum_{p=0}^{\infty} A_p c_p\right| \\
&< \varepsilon/2 + \left|\sum_{p=0}^{n}\left[c_p \binom{n}{p} \sum_{k=p}^{\infty} A_k n^{-k} Y_{pk} - A_p c_p\right]\right| \\
&\qquad \leqq \varepsilon/2 + \left|\sum_{p=0}^{n} c_p A_p \left[1 - \binom{n}{p} n^{-p} p!\right]\right| \\
&\qquad\qquad + t\cdot \sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| r^k n^{-k} Y_{pk} \\
&< \varepsilon .
\end{aligned}$$

THEOREM 2. *Suppose $r \geqq 1$ and S is the set to which f belongs only if there is a complex sequence A such that $\sum_{p=0}^{\infty} |A_p| r^p$ converges and $f = \sum_{p=0}^{\infty} A_p I^p$. Suppose that c is an infinite complex sequence and, for each f is S, $L(f, c)$ converges. Then c is bounded by a geometric sequence with ratio r.*

Proof. For each nonnegative integer p let g_p be r^{-p}, and suppose that the sequence $c\cdot g$ is not bounded; that is, suppose that c is not bounded by a geometric sequence with ratio r.

For each f in S, let $N(f)$ be $\sum_{p=0}^{\infty} r^p |f^{(p)}(0)|/p!$. Then (S, N) is a complete, normed, linear space.

For each positive integer n, let T_n be a function from S to the complex numbers such that if f is in S then $T_n(f) = L(f, c)_n$. If f is in S and n is a positive integer and $|f|_{[0,1]}$ denotes the maximum modulus of f on $[0, 1]$, then

$$\begin{aligned}
|T_n(f)| &= \left|\sum_{p=0}^{n} c_p \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n)\right| \\
&\leqq |f|_{[0,1]} \sum_{p=0}^{n} |c_p| \binom{n}{p} \sum_{q=0}^{p} \binom{p}{q} \\
&\leqq N(f) \sum_{p=0}^{n} |c_p| \binom{n}{p} 2^p ,
\end{aligned}$$

so that T_n is a continuous linear transformation from (S, N) to the complex numbers.

Let m be a positive integer. $N(r^{-m}I^m) = 1$. By Theorem A, $L(r^{-m}I^m, c)$ has limit $g_m c_m$. Hence, there is a positive integer n such that $|T_n(r^{-m}I^m)| > |g_m c_m| - 1$, so that the sequence $N'[T]$—where, for each positive integer n, $N'(T_n)$ is the least number b such that if F is in S and $N(F) \leqq 1$ then $|T_n(F)| \leqq b$—is not bounded. So, by the "principle of uniform boundedness," there is a member f of S such that the sequence $T(f)$ is not bounded, but $L(f, c)$ converges and $T(f) = L(f, c)$, so the theorem is proved.

THEOREM 3. *Suppose that $r > 0$, f is in $C[0, 1]$, and, for each complex sequence c which is dominated by a geometric sequence with ratio r, $L(f, c)$ converges. Then there is a complex sequence A such that $\sum_{p=0}^{\infty} |A_p| r^p$ converges and, if x is in $[0, 1]$ and $x \leqq r$, then $f(x) = \sum_{p=0}^{\infty} A_p x^p$.*

Proof. For each nonnegative-integer pair (n, p), let g_n be r^n and let M_{np} be $\binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n)$. Then, for each bounded complex sequence c and each positive integer n, $L(f, c \cdot g)_n = \sum_{p=0}^{n} c_n M_{np} r^p$, so that, by the "principle of uniform boundedness", there is a number D such that, for each positive integer n, $\sum_{p=0}^{n} |M_{np}| r^p < D$.

For each nonnegative integer p, let $z(p)$ be the sequence whose value at p is 1 and whose value elsewhere is 0. Then the sequence $M[\ , p] = L(f, z(p))$, which, by hypothesis, has limit, say A_p, and from the preceding paragraph we see that $\sum_{p=0}^{\infty} |A_p| r^p$ converges.

For each positive integer n let $B_n(f)$ be the *Bernstein polynomial for f of order n*; *i.e.*, let $B_n(f)$ be $\sum_{p=0}^{n} M_{np} I^p$. $B(f)$ converges to f on $[0, 1]$ [see 1, pp. 1–2; also 4, pp. 5–7]. Now, if x is a complex number and $|x| \leqq r$, then

$$|B_n(f)(x)| \leqq \sum_{p=0}^{n} |x|^p |M_{np}| < D .$$

By the convergence of the Bernstein polynomials on $[0, 1]$ and the convergence continuation theorem, a subsequence of $B(f)$ has limit, say h, on $[0, 1]$ and the disc with center 0 and radius r. h is analytic at 0. By Theorems 0 and A we see that if p is a non-negative integer then

$$\begin{aligned} h^{(p)}(0)/p! &= \lim L(h, z(p)) \\ &= \lim L(f, z(p)) = A_p , \end{aligned}$$

and, if x is in $[0, 1]$ and $x \leqq r$, then $f(x) = h(x) = \sum_{p=0}^{\infty} A_p x^p$.

The following theorem parts somewhat from the main stream of our study but sheds additional light on the problem at hand.

THEOREM 4. *Suppose that A is a complex sequence, $\sum_{p=0}^{\infty} |A_p|$ converges, and $f = \sum_{p=0}^{\infty} A_p I^p$. Then there is an unbounded number-sequence c such that $L(f, c)$ converges.*

Proof. For each positive integer n, let s_n be

$$\sum_{p=0}^{n} \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| n^{-k} Y_{pk} .$$

By Lemma 3, s has limit 0. A has limit 0. So there is an increasing, nonnegative-integer sequence u such that, for each nonnegative-integer pair (p, q), $s(u_p + q) < 4^{-p}$ and $|A(u_p + q)| < 4^{-p}$.

For each nonnegative integer p, let c_p be 2^m if m is a nonnegative integer such that $p = u_m$, otherwise let c_p be 0.

If k is a nonnegative integer and $n = u_k$, $|c_n A_n| < 2^{-k}$, so that $\sum_{p=0}^{\infty} |A_p c_p|$ converges.

Let b be a positive number. By Lemma 2, there is a positive integer N such that $2^{-N} < b/2$ and if n is an integer greater than N then

$$\sum_{p=0}^{n} |A_p c_p| \left[1 - \binom{n}{p} n^{-p} p!\right] < b/4$$

and

$$\left| \sum_{p=n+1}^{\infty} A_p c_p \right| < b/4 .$$

Let n be an integer not less than u_N and let m be the greatest integer k such that $u_k \leqq n$. Then

$$\begin{aligned}
&\left| \sum_{p=0}^{\infty} A_p c_p - L(f, c)_n \right| \\
&\quad < b/4 + \left| \sum_{p=0}^{n} A_p c_p - \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{k=p}^{\infty} A_k n^{-k} Y_{pk} \right| \\
&\qquad \leqq b/4 + \sum_{p=0}^{n} |A_p c_p| \left[1 - \binom{n}{p} n^{-p} p!\right] \\
&\qquad\qquad + \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| n^{-k} Y_{pk} \\
&\quad < b/2 + \sum_{p=0}^{n} 2^m \binom{n}{p} \sum_{k=p+1}^{\infty} |A_k| n^{-k} Y_{pk} \\
&\quad < b/2 + 2^m s_n < b/2 + 2^m \cdot 4^{-m} < b ,
\end{aligned}$$

so $L(f, c)$ converges to $\sum_{p=0}^{\infty} A_p c_p$.

2. Entire functions. Following from Theorems 1 and 3 we have:

THEOREM 5. *Suppose that f is in $C[0, 1]$. Then the following statements are equivalent:*

(1) *f is a subset of an entire function.*

(2) *If c is a complex sequence which is dominated by a geometric sequence, then $L(f, c)$ converges.*

Furthermore, if (2) *holds, $L(f, c)$ converges to $\sum_{p=0}^{\infty}(f^{(p)}(0)/p!)c_p$.*

THEOREM 6. *Suppose that c is a complex sequence such that $L(f, c)$ converges for each entire function f. Then c is dominated by a geometric sequence.*

Proof. Suppose that c is not dominated by a geometric sequence.

LEMMA. *If each of m and r is a nonnegative integer, then there is a positive integer q such that $|c_{m+q}| > r^{m+q+1}$ and $|c_{m+q}| > 2^m|c_p|$ for each nonnegative integer p less than $m+q$.*

Proof of lemma. Let R be $r + 2^m + \sum_{p=0}^{m}|c_p|$. Since no geometric sequence dominates c, there is a positive integer k such that $|c_{m+k}| > R^{m+k+1}$. Let q be the least positive integer n such that $|c_{m+n}| > R^{m+n+1}$.

Suppose that p is a nonnegative integer.

If $p \leqq m$, then $|c_{m+q}| > R^{m+q+1} \geqq R^2 > 2^m|c_p|$.

If $m < p < m+q$, then

$$|c_{m+q}| > R^{m+q+1} \geqq R\cdot R^{p+1} \geqq R\cdot|c_p| > 2^m|c_p| \,.$$

Continuation of proof of Theorem 6. By the lemma, there is an increasing interger-valued sequence u such that $u_0 = 0$ and, if p is a positive integer, then $|c(u_p)| > p^{u(p)+1}$ and $|c(u_p)| \geqq 2^{u(p-1)}|c_n|$ for each nonnegative integer n less than u_p.

Let f be $\sum_{p=1}^{\infty}(1/c(u_p))I^{u(p)}$.

If N is a positive integer then, for each integer p greater than N,

$$\left|\frac{1}{c(u_p)}\right|^{1/u_p} < p^{-(u(p)+1)/(u(p))} < p^{-1} < 1/N \,,$$

so that f is an entire function.

For each nonnegative integer k let A_k be $f^{(k)}(0)/k!$. Now, if p and k are integers and $0 \leqq p < k$, then $|c_pA_k| < 2^{-k}$.

Suppose that $B > 0$. By Lemma 3 and the note at the beginning of the proof of Lemma 2, there is a positive-integer pair (n, m) such that

$$\sum_{p=0}^{n}\binom{n}{p}\sum_{k=p+1}^{\infty}2^{-k}n^{-k}Y_{pk} < 1 \,,$$

$$\sum_{p=0}^{m}\binom{n}{u_p}n^{-u(p)}(u_p)! > B \,,$$

and $n \geqq u_m$. Then $|L(f, c)_n|$

$$= \left|\sum_{p=0}^{n} c_p\binom{n}{p}\sum_{k=p}^{\infty} A_k n^{-k} Y_{pk}\right|$$

$$\geqq \sum_{p=0}^{n} c_p\binom{n}{p}A_p n^{-p}p! - \sum_{p=0}^{n}\binom{n}{p}\sum_{k=p+1}^{\infty}|c_p A_k|\, n^{-k} Y_{pk}$$

$$\geqq \sum_{p=0}^{m} c(u_p)\binom{n}{u_p}A(u_p)n^{-u(p)}(u_p)!$$

$$- \sum_{p=0}^{n}\binom{n}{p}\sum_{k=p+1}^{\infty} 2^{-k}n^{-k}Y_{pk}$$

$> B - 1$, so that $L(f, c)$ does not converge. Hence, c is dominated by a geometric sequence.

3. A converse to Theorem A. The following theorem, together with Theorem A, shows that the last pair on our table belongs there.

THEOREM 7. *Suppose that f is in $C[0, 1]$ and, for each complex sequence c, $L(f, c)$ converges. Then f is a subset of a polynomial.*

Proof. By Theorem 3 there is a complex sequence A such that if x is in $[0, 1]$ then $f(x) = \sum_{p=0}^{\infty} A_p x^p$.

For each nonnegative-integer pair (n, p), let M_{np} be

$$\binom{n}{p}\sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}f(q/n)\ ,$$

let w_p be $1/A_p$ if $A_p \neq 0$ and w_p be 0 if $A_p = 0$, and let Q_{np} be $w_p M_{np}$. Now, if x is an infinite complex sequence and $T(x)_n = \sum_{p=0}^{n} Q_{np}x_p$ for each positive integer n, then $T(x) = L(f, w\cdot x)$, so that $T(x)$ converges. Therefore, by the "principle of uniform boundedness," there is a number B such that, for each positive integer n, $\sum_{p=0}^{n}|Q_{np}| < B$. Now, if p is a nonnegative integer such that $A_p \neq 0$, the sequence $Q[\ , p]$ has limit 1. Hence, there is a positive integer N such that if p is an integer greater than N then $A_p = 0$, so f is a subset of a polynomial.

4. Radius of convergence less than 1. Lemma 1 tells us that constant sequences prevent us from altering Theorem 2 to allow r to be less than 1.

Theorem 3, as it is, not restricted in this way.

This leaves the question: Can we find anything like Theorem 1 with the radius of convergence for our power-series expansions about 0 less than 1?

THEOREM 8. *Suppose that $0 < r < 1$, f is a function analytic on the disc with center 1 and radius $1 + r$, $\sum_{p=0}^{\infty}(|f^{(p)}(1)|/p!)(1 + r)^p$*

converges, c is a complex sequence, $t > 0$, and, for each nonnegative integer n, $|c_n| \leqq t\cdot r^n$. Then $L(f, c)$ converges to $\sum_{p=0}^{\infty}(f^{(p)}(0)/p!)c_p$.

Indication of proof. For each nonnegative integer n let B_n be $f^{(n)}(1)/n!$ and let d_n be $\Delta^n c_0$. Then

$$|d_n| = \left|\sum_{q=0}^{n}(-1)_q\binom{n}{q}c^q\right| \leqq t\cdot\sum_{q=0}^{n}\binom{n}{q}r^q = t\cdot(1+r)^n .$$

For each complex number z such that $|z| < 1 + r$. let $g(z)$ be $f(1-z)$. Then for each positive integer n,

$$\begin{aligned} L(f, c)_n &= \sum_{p=0}^{n} f(p/n)\binom{n}{p}\Delta^{n-p}c_p \\ &= \sum_{p=0}^{n} f(1-p/n)\binom{n}{n-p}\Delta^p c_{n-p} \\ &= \sum_{p=0}^{n} g(p/n)\binom{n}{p}\Delta^{n-p}d_p \\ &= L(g, d)_n , \end{aligned}$$

so that, by Theorem 1, $L(f, c) = L(g, d)$ converges to

$$\begin{aligned} \sum_{p=0}^{\infty}\frac{g^{(p)}(0)}{p!}d_p &= \sum_{p=0}^{\infty}(-1)^p B_p d_p \\ &= \sum_{p=0}^{\infty}(-1)^p B_p \sum_{q=0}^{p}(-1)^q\binom{p}{q}c_q \\ &= \sum_{q=0}^{\infty} c_q \sum_{p=q}^{\infty}(-1)^{p+q}\binom{p}{q}B_p \\ &= \sum_{q=0}^{\infty} c_q f^{(q)}(0)/q! . \end{aligned}$$

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