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HITTING TIMES FOR TRANSIENT STABLE PROCESSES

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In this paper we explicitly find the asymptotic behavior, for large t, of the probability that a transient d-dimensional stable process first (last) hits a bounded Borel set during the time interval (t, ∞).

Assume that $X(t)$ is a stable process on R^d (d-dimensional Euclidean space) having exponent $\alpha < d$ and normalized so that the paths are right continuous with left-hand limits at every point. Assume further that $[X(t) - X(0)]t^{-1/\alpha}$ is distributed like $X(1) - X(0)$, and moreover, that $X(1) - X(0)$ has a genuinely d-dimensional distribution on R^d. [In particular, every symmetric stable process on R^d with 0 mean (when it exists) satisfies these conditions.]

From these assumptions it follows that $X(t) - X(0)$ has a bounded, continuous density, $f(t, x)$, which satisfies the well-known scaling property

$$f(t, x) = t^{-d/\alpha} f(1, t^{-1/\alpha} x) \ . \tag{1.1}$$

For a Borel (more generally, analytic) set $B \subset R^d$, let

$$V_B = \inf\{t > 0 \colon X(t) \in B\}$$

denote the first hitting time of B. As usual we set $V_B = \infty$ if

$$X(t) \notin B$$

for all $t > 0$. Our main purpose in this note is to establish the following.

THEOREM 1. *Let B be a bounded Borel (or analytic) subset of R^d. Then under the above assumptions on $X(t)$,*

$$\lim_{t \to \infty} t^{(d/\alpha)-1} P_x(t < V_B < \infty) = P_x(V_B = \infty) C(B) \left[\frac{d}{\alpha} - 1\right]^{-1} f(1, 0) \ , \tag{1.2}$$

where $C(B)$ is the natural capacity of B.

Previously, (by using a different method) Joffe [2] established this result for symmetric processes with $(d/2) < \alpha < 1$ when B has a non-empty interior, and Spitzer [4] (Lemma, p. 114) established this result for arbitrary compact B in the case of 3-dimensional Brownian motion. In the case of recurrent stable processes the analogue of Theorem 1 can be found in [3].

It is interesting to compare Theorem 1 with the following, much easier

THEOREM 2. *Let*

$$T_B = \inf\{t \geqq 0 \colon X(s) \notin B, \ all \ s > t\}$$

be the last hitting time of B. Then under the same conditions as Theorem 1,

$$\lim_{t \to \infty} t^{(d/\alpha)-1} P_x(T_B > t) = C(B) \left[\frac{d}{\alpha} - 1\right]^{-1} f(1, 0) . \tag{1.3}$$

2 Proofs.

Proof of Theorem 1. A first passage decomposition yields

$$\begin{aligned} (2.1) \quad P_x(t < V_B < \infty) &= \int_{R^d} P_x(V_B > t, X(t) \in dy) P_y(V_B < \infty) \\ &= \int_{R^d} \left[f(t, y - x) - \int_0^t \int_{\bar{B}} H_B(x, ds, dz) f(t - s, y - z) \right] P_y(V_B < \infty) dy , \end{aligned}$$

where here and in the following,

$$H_B(x, ds, dz) = P_x(V_B \in ds, X(s) \in dz) ,$$

and $\bar{B}$ is the closure of B. But it is a known fact ([1] Prop. 18.4) that there is a measure, $e_B(dy)$, with support contained in $\bar{B}$ (the capacitary measure of B) and finite total mass $C(B)$ (the capacity of B), such that

$$P_y(V_B < \infty) = \int_{\bar{B}} g(u - y) e_B(du) , \tag{2.2}$$

where

$$g(x) = \int_0^\infty f(t, x) dt$$

is the potential kernel density for the process $X(t)$. Setting

$$R(t, x) = \int_t^\infty f(s, x) ds$$

and using the fact that

$$\int_{R^d} f(t, y - x) g(u - y) dy = R(t, u - x) , \tag{2.3}$$

we obtain from (2.1) that

$$(2.4)\qquad P_x(t < V_B < \infty) \\ \int_{\bar B}\Big[R(t, y - x) - \int_{\bar B}\int_0^t H_B(x, ds, dz)R(t - s, y - z)\Big]e_B(dy) .$$

From the scaling property (1.1) and the fact that $f(1, x)$ is continuous, we see that $\lim_{t\to\infty} t^{d/\alpha}f(t, x) = f(1, 0)$, uniformly in x on compacts, and thus

$$(2.5)\qquad \lim_{t\to\infty} t^{(d/\alpha)-1}R(t, x) = f(1, 0)\Big[\frac{d}{\alpha} - 1\Big]^{-1} ,$$

uniformly in x on compacts. Set

$$R(t) = t^{-(d/\alpha)+1}\Big[\frac{d}{\alpha} - 1\Big]^{-1} .$$

Then from (2.5),

$$(2.6)\qquad \lim_{t\to\infty}\int_{\bar B}\frac{R(t, y - x)}{R(t)}\, e_B(dy) = f(1, 0)C(B) ,$$

and

$$(2.7)\qquad \begin{aligned} &\lim_{T\to\infty}\lim_{t\to\infty}\int_0^T\Big[\int_{\bar B}\int_{\bar B} H_B(x, ds, dz)R(t - s, y - z)e_B(dy)\Big]R(t)^{-1} \\ &= \lim_{T\to\infty}\int_0^T H_B(x, ds, \bar B)C(B)f(1, 0) = P_x(V_B < \infty)C(B)f(1, 0) . \end{aligned}$$

From (2.4), we see that in order to complete the proof it suffices to show

$$(2.8)\qquad \lim_{T\to\infty}\limsup_{t\to\infty} R(t)^{-1}\int_T^t\int_{\bar B}\int_{\bar B} H_B(x, ds, dz)R(t - s, y - z)e_B(dy) = 0 .$$

To accomplish this, decompose $\int_T^t$ as $\int_T^{t/2} + \int_{t/2}^{t-T} + \int_{t-T}^t$. Since

$$\sup_x f(1, x) = K < \infty ,$$

it follows from the scaling property that $R(t, x) \leqq KR(t)$ for all $t > 0$. Setting $A = KC(B)$, we obtain

$$\int_T^{t/2} \leqq A\int_T^{t/2} P_x(V_B \in ds)R(t - s) \leqq AR(t/2)P_x(T < V_B < \infty) ,$$

and thus

$$\lim_T \limsup_t R(t)^{-1}\int_T^{t/2} = 0 .$$

Next observe that

$$\int_{t/2}^{t-T} \leqq A \int_{t/2}^{t-T} P_x(V_B \in ds) R(t-s) \leqq AR(T) P_x(t/2 < V_B < \infty) .$$

By (2.4) this last term is dominated by $A^2R(T)R(t/2)$, and thus

$$\lim_T \limsup_t R(t)^{-1} \int_{t/2}^{t-T} = 0 .$$

Finally, from (2.2) we see that

$$\int_{t-T}^{t} \leqq \int_{t-T}^{t} \int_{\bar{B}} H_B(x, ds, dz) \int_{\bar{B}} g(y-z) e_B(dy) \leqq \int_{t-T}^{t} P_x(V_B \in ds) .$$

But

$$P_x(t - T < V_B \leqq t) = \int_{R^d} P_x(V_B > t - T, X(t-T) \in dy) P_y(V_B \leqq T)$$

$$\leqq \int_{R^d} f(t-T, y-x) P_y(V_B \leqq T) dy \leqq K(t-T)^{-d/\alpha} \int_{R^d} P_y(V_B \leqq T) dy .$$

Since the paths $X(t)$ are bounded a.s. on $[0, T]$, we see that for each T there is a sphere $S_T \supset \bar{B}$, such that $P_y(X(t) \in S_T) \geqq 1/2$ for all $t \leqq T$ and $y \in \bar{B}$. But then

$$|S_T| = \int_{R^d} P_x(X(T) \in S_T) dx \geqq \int_{R^d} dx \int_0^T \int_{\bar{B}} H_B(x, ds, dy) P_y(X(T-s) \in S_T)$$

$$\geqq \frac{1}{2} \int_{R^d} P_x(V_B \leqq T) dx .$$

Thus

$$\lim_{t \to \infty} R(t)^{-1} \int_{t-T}^{t} = 0 .$$

This completes the proof.

Proof of Theorem 2. Clearly

$$P_x(T_B > t) = \int_{R^d} f(t, y-x) P_y(V_B < \infty) dy .$$

Using (2.2) and (2.3) we see that

$$P_x(T_B > t) = \int_{\bar{B}} R(t, y-x) e_B(dy) ,$$

from which the theorem follows.

REMARK. When $d/2 < \alpha < d$, it is possible to establish Theorem 1 by a much simpler argument. Set

$$Q_B^\lambda(x) = \int_0^\infty e^{-\lambda t} P_x(t < V_B < \infty) dt ,$$

$$H_B^\lambda(x, dy) = \int_0^\infty e^{-\lambda t} P_x(V_B \in dt, x(t) \in dy)$$

and

$$R^\lambda(x) = \int_0^\infty R(t, x) e^{-\lambda t} dt .$$

Then from (2.4) we obtain

$$(2.9) \qquad Q_B^\lambda(x) = \int_{\bar{B}} \left[R^\lambda(y - x) - \int_{\bar{B}} H_B^\lambda(x, dz) R^\lambda(y - z) \right] e_B(dy) .$$

It follows from (2.5) that uniformly in x on compacts,

$$\lim_{\lambda \downarrow 0} R^\lambda(x) \lambda^{2-d/\alpha} = f(1, 0) \left[\frac{d}{\alpha} - 1 \right]^{-1} \Gamma(2 - d/\alpha) .$$

Consequently, from (2.9), we see that

$$\lim_{\lambda \downarrow 0} Q_B^\lambda(x) \lambda^{2-d/\alpha} = f(1, 0) C(B) P_x(V_B = \infty) \left[\frac{d}{\alpha} - 1 \right]^{-1} \Gamma(2 - d/\alpha) .$$

An appeal to Karamata's theorem, and the fact that $P_x(t < V_B < \infty)$ is monotone in t, then yields (1.2).

The above argument breaks down when $\alpha < d/2$ since

$$\lim_{\lambda \downarrow 0} R^\lambda(x) < \infty ,$$

and the more complicated proof given previously is needed.

References

1. G. A. Hunt, *Markoff processes and potentials III*, Illinois J. Math. **2** (1958), 151-213.
2. A. Joffe, *Sojourn Time for Stable Processes*, Thesis, cornell U., 1959.
3, S. C. Port, *Hitting times and potentials for recurrent stable processes*, J. D' Analyse Mathematique (to appear)
4. F. Spitzer, *Electrostatic capacity, heat flow, and Brownian motion*, Z. Warscheinlichk **3**, (1964).

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