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DEDEKIND GROUPS

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A Dedekind group is a group in which every subgroup is normal. The author gives two characterizations of such groups, one in terms of sequences of subgroups and the other in terms of factors in the group.

In a 1940 paper by R. Baer [1], two characterizations of the class of Dedekind groups are given. However, Lemma 5.2 of that paper is in error. The results which follow Lemma 5.2 depend on the validity of that Lemma. Hence these results, which constitute Baer's characterizations of Dedekind groups, are also in error. A portion of this paper is devoted to new characterizations of Dedekind groups which are similar to those attempted by Baer. In §7 there are some further related results.

2. Notation. As usual $o(G)$ and $Z(G)$ will denote the order of G and the center of G, respectively. If S is a subset of a group G, then $\langle S\rangle$ is defined to be the subgroup generated by S. G' will denote the commutator subgroup of G. If H is a subgroup of G, then Core (H) is the maximum subgroup of H which is normal in G.

3. Definitions. Let $G_1(p, n)$, where p is a prime and n is a nonnegative integer, be defined as the group generated by the two elements a and b, subject to the following relations:

(1) b and $c = b^{-1}a^{-1}ba$ are both of order p,

(2) $ac = ca$, $bc = cb$, $a^p = c^n$.

It is noted that $G_1(p, n)$ is isomorphic to $G_1(p, n')$ if neither n nor n' is 0. Also, for $p = 2$, $G_1(p, n)$ is the dihedral group of order 8, and for $p > 2$, $G_1(p, 0)$ and $G_1(p, 1)$ are the two non-Abelian groups of order p^3[2, pp. 51–52].

Let $G_k(p)$, for $k > 1$ and p a prime, denote the group which is generated by the two elements a and b, subject to the following relations:

$$a^{p^{k+1}} = e,\ b^p = e,\ ba = a^{1+p^k}b\ .$$

Then $o(G_k(p)) = p^{k+2}$ and is the unique group of that order with cyclic subgroup of index p, and a commutator subgroup of order p [2, p. 187].

A group G is said to satisfy condition:

(N_k) if $S \subset T \subseteqq G$, implies $S \lhd T$, whenever the complete lattice of subgroups properly between S and T consists only of a chain of at

most k groups.
(N) if G satisfies condition (N_k) for all $k \geqq 0$.
(N^*) if every subgroup of G is normal in G.
(F_k) if $S \lhd T \subseteqq G$ implies T/S is not isomorphic to $G_1(p, n)$ for all primes p and all integers $n \geqq 0$, or to $G_j(p)$ for all $j \leqq k$ and for all primes p.
(F) if G satisfies condition (F_k) for all $k > 0$.

4. Counterexamples. In the paper of R. Baer [1] it is stated that for groups G where $G/Z(G)$ is Abelian, the three conditions (N_1), (N^*), and (F_1) are equivalent. The following is a counterexample to this statement:

Consider the group $G = G_k(p)$ for any $k > 1$. Then $o(G) = p^{k+2}$. Since $a^{-1}ba = a^{p^k}b$, it follows that $\langle b\rangle \ntriangleleft G$. Therefore, G does not satisfy condition (N^*). Since

$$ba^pb^{-1} = (bab^{-1})^p = (a^{1+p^k})^p = a^p ,$$

it follows that a^p is in $Z(G)$ and $o(Z(G)) \geqq p^k$. But G is non-Abelian and hence $o(Z(G)) = p^k$. Therefore, $Z(G) = \langle a^p\rangle$ and $o(G/Z(G)) = p^2$. Thus $G/Z(G)$ is Abelian. And we have the commutator subgroup $G' \subseteqq Z(G)$ and $G' = \langle a^{p^k}\rangle$.

Now suppose $T < G$, where $o(T) = p^{k+2-n}$, $0 < n < k$, and T is non-Abelian. Then

$$E \neq T' \subseteqq G' < Z(G) = \langle a^p\rangle .$$

Since $o(G') = p$, $T' = \langle a^{p^k}\rangle$. $a^{p^n} \notin T$ since, otherwise, $o(Z(T)) \geqq p^{k-n+1}$ and, hence $Z(T) = T$, a contradiction. However, $[G : T] = p^n$ and, therefore, $a^{p^n} \in T$, a contradiction. Hence T is Abelian. Thus every proper subgroup of G is Abelian. To show that G satisfies condition (F_1) it is sufficient to consider only the case where $T = G$, $o(S) = p^{k-1}$, and $S \lhd T$. In this case $S \cap Z(G) \neq E$. Therefore, S contains a subgroup of order p which is contained in $Z(G)$. But $Z(G)$ is cyclic and G' is the only subgroup of $Z(G)$ or order p. Hence $G' \subseteqq S$. Then T/S is Abelian and, therefore, satisfies condition (F_1).

Thus, for every natural number $k > 1$ and every prime number p, $G_k(p)$ is a counterexample to Lemma 5.2 in [1].

5. Preliminary lemmas.

LEMMA 1. *The group $G_k(p)$ has an elementary Abelian subgroup of order p^2, and no such subgroup of larger order.*

Proof. The subgroup $H = \langle a^{p^k}, b\rangle$ is elementary Abelian since $a^p \in Z(G_k(p))$. Also, $o(H) = p^2$. Suppose L is an elementary Abelian subgroup of $G_k(p)$, then $[L : L \cap \langle a \rangle] \leqq [G: \langle a \rangle] = p$. Since $L \cap \langle a \rangle$ is elementary Abelian and $\langle a \rangle$ is cyclic, $o(L \cap \langle a \rangle) \leqq p$ and $o(L) \leqq p^2$.

LEMMA 2. *$G_1(p, n)$ does not satisfy condition (N_1) and $G_k(p)$ does not satisfy condition (N_k) for $k > 1$.*

Proof. In $G_1(p, n)$ let $B = \langle b \rangle$. Then $B \ntriangleleft G_1(p, n)$ and $\langle b, c \rangle$ is the only group of order p^2 which contains B. Hence $G_1(p, n)$ does not satisfy condition (N_1).

In $G_k(p)$ let $B = \langle b \rangle$. Since $a^{-1}ba = ap^kb$, it follows that $BG_k \ntriangleleft (p)$. It remains to show that there exist at most k subgroups between B and $G_k(p)$. Suppose $B \subset H \subset G_k(p)$ and $B \subset K \subset G_k(p)$ where $H \neq K$ and $o(H) = o(K) = p^{k+1}$. Then $H \triangleleft G_k(p)$, $K \triangleleft G_k(p)$ and $G_k(p) = HK$. Also, $B \triangleleft H$, $B \triangleleft K$ since every proper subgroup of $G_k(p)$ is Abelian. Hence $B \triangleleft G_k(p)$, a contradiction. Therefore $\langle a^p, b \rangle$ is the only subgroup of order p^{k+1} which contains B. But there are only $k - 1$ proper subgroups of the Abelian group $\langle a^p, b \rangle$ which contain B properly. Hence there are exactly k subgroups between B and $G_k(p)$. Therefore, $G_k(p)$ does not satisfy condition (N_k).

COROLLARY 3. *$G_1(p, n)$ does not satisfy (N_j) for $j > 1$, and $G_k(p)$ does not satisfy (N_j) for $j > k$.*

Proof. In the proof of Lemma 2 it was shown that there are k subgroups between $\langle b \rangle$ and $G_k(p)$ and $\langle b \rangle \ntriangleleft G_k(p)$. But $k < j$ and, therefore, $G_k(p)$ does not satisfy condition (N_j).

There is only one subgroup between $\langle b \rangle$ and $G_1(p, n)$, hence $G_1(p, n)$ does not satisfy condition (N_j) for $j > 1$.

COROLLARY 4. *$G_1(p, n)$ satisfies condition (N_0) and $G_k(p)$ satisfies condition (N_j) for $j < k$.*

Proof. Suppose $S \subset T \subseteqq G_1(p, n)$. Since $G_1(p, n)$ is a p-group, it follows that $N_T(S) = T$ since $[T : S] = p$. Hence $S \triangleleft T$ and $G_1(p, n)$ satisfies (N_0).

Now suppose $S \subset T \subseteqq G_k(p)$ and that the complete lattice of subgroups between S and T consists of a chain of at most j subgroups. We may assume $S \neq E$, since $E \triangleleft T$ for all subgroups T. Further, we may assume that $T = G_k(p)$, for if $T < G_k(p)$, then T is Abelian and $S \triangleleft T$.

Suppose $S \subset T = G_k(p)$. Then $[T : S] \leqq p^k$. Therefore, $a^{p^k} \in S$. Thus $G_k(p)' \subseteqq S$ and $S \triangleleft G_k(p) = T$.

LEMMA 5. *If the group G satisfies condition (N_k), then all* *subgroups and factor groups of G satisfy condition (N_k).*

Proof. This follows immediately from the definitions.

LEMMA 6. *If G satisfies condition (N_k) then G satisfies condition (F_k).*

Proof. Let $S \lhd T \subseteqq G$. By Lemma 5, T/S satisfies condition (N_k). But $G_1(p, n)$ does not satisfy condition (N_1) and $G_k(p)$ does not satisfy condition (N_k), by Lemma 2. Therefore, T/S cannot be isomorphic to $G_1(p, n)$, or isomorphic to $G_k(p)$ for $k > 1$. Hence G satisfies condition (F_k).

6. Corrected theorem. Using the new definitions which are given in §2, we obtain the following revised form of Lemma 5.2 in [1].

THEOREM 7. *If the group G is such that $G/Z(G)$ is Abelian, then the three conditions (N), (N^*) and (F) are equivalent.*

Proof. If G satisfies condition (N^*), then G satisfies condition (N_k) for all $k \geqq 0$, and therefore, G satisfies condition (N). If G satisfies condition (N), then G satisfies condition (N_k) for all $k \geqq 0$. By Lemma 6, G satisfies condition (F_k) for all $k \geqq 0$. Hence G satisfies condition (F). It remains to show that condition (F) implies condition (N^*).

We will assume that G does not satisfy condition (N^*), but $G/Z(G)$ is Abelian. Then there exists $u \in G$ such that $\langle u \rangle \ntriangleleft G$, for otherwise every subgroup would be normal in G. Hence there exists an element $v \in G$ such that

$$c = [u, v] \notin \langle u \rangle,\ c \in G' \subseteqq Z(G)\ . \tag{$*$}$$

Now if

$$H \lhd G \quad \text{and} \quad c \notin \langle u, H \rangle\ , \tag{$**$}$$

then we assert that G/H has all the properties which G possesses. Since G is nilpotent of class 2, then G/H is nilpotent of class at most 2. $[uH, vH] = cH \notin \langle uH \rangle$ since $c \notin \langle u, H \rangle$ and $cH \in (G/H)' \subseteqq Z(G/H)$. Hence G/H does not satisfy condition (N^*). Thus to show that G does not satisfy (F) it is sufficient to show that G/H does not satisfy (F).

Assume $H = \operatorname{Core}(\langle u \rangle)$. Then H satisfies $(**)$ and without loss of generality it can be assumed that $\operatorname{Core}(\langle u \rangle) = E$. Hence $\langle u \rangle \cap Z(G) = E$.

If $o(c)$ is infinite, let p be any prime and if $o(c)$ is finite, let p be a prime such that $p \mid o(c)$. Then $\langle c^p \rangle < \langle c \rangle$. Since $\langle u \rangle \cap Z(G) = E$

and $c \in Z(G)$, the group $\langle c\rangle \times \langle u\rangle$ exists. Therefore, $c \notin \langle u, c^p\rangle$. Thus $(**)$ holds with $H = \langle c^p\rangle$. Thus it can be assumed that $c^p = e$. Also, it can be assumed that $G = \langle u, v\rangle$. Since G is nilpotent of class 2,

$$[u, v^p] = [u^p, v] = [u, v]^p = c^p = e .$$

Then $u^p, v^p \in Z(G)$. Since Core $(\langle u\rangle) = E$, $u^p = e$. Every $x \in G$ can be written as $u^a v^b c^d$ because G is nilpotent of class 2. Since $u^p, v^p, c \in Z(G)$ and $G/Z(G)$ is Abelian of type (p, p), then $x \in Z(G)$ if and only if $p \mid a$ and $p \mid b$. Therefore, $Z(G) = \langle u^p, v^p, c\rangle = \langle v^p, c\rangle$. If $c \notin \langle v^p\rangle$, then $G/\langle v^p\rangle$ is non-Abelian of order p^3. This group has at least two subgroups of order p and hence is not the quaternion group. Thus $G/\langle v^p\rangle \cong G_1(p, n)$ for some n. If $c \in \langle v^p\rangle$, then $\langle v^p\rangle$ is finite and G is a finite nilpotent group. Let L be a p-complement in $\langle v^p\rangle$. Then $c \notin L$ since L is a p-complement. Then G/L is a non-Abelian p-group. The subgroup $\langle vL\rangle$ is cyclic of index p in G/L. Then G/L is isomorphic to some $G_k(p)$ or $G_1(p, n)$, [2, p. 187]. Then G does not satisfy condition (F), a contradiction.

7. **Related results.** We note that, for an arbitrary group, condition (N^*) implies condition (N), and condition (N) implies condition (F). Using the new definitions for condition (N) and (F) in the theorems which follow Lemma 5.2 in the paper of Baer [1], the theorems are true. The proof are the same as given in [1] by using Theorem 7 of this paper in the place of Lemma 5.2. Therefore, these results will not be stated here.

In Lemma 6 it was shown that condition (N_k) implies condition (F_k). In the remainder of this paper, the converse of Lemma 6 will be considered.

LEMMA 8. *If the group G satisfies condition (F_k) and $N \lhd G$ such that $[G : N] = q^t$ where $t \leqq k$, $o(N) = p^n$, p and q are primes, and N is elementary Abelian, then G satisfies condition (F).*

Proof. Case 1. Suppose $p = q$. Let $H \lhd K \subseteqq G$ where K/H is isomorphic to $G_i(p)$. Since N is elementary Abelian it follows that $(K \cap NH)/H$ is an elementary Abelian subgroup of K/H. Then

$$[K \cap NH : H] \leqq p^2$$

by Lemma 1. Then

$$\begin{aligned}[K : H] &= [K : K \cap NH]\cdot[K \cap NH : H] \\ &\leqq [G : NH]\cdot[K \cap NH : H] \leqq p^{t+2} \leqq p^{k+2} .\end{aligned}$$

Therefore, K/H is isomorphic to $G_i(p)$ where $i \leqq k$. But G satisfies condition (F_k), a contradiction. Hence G satisfies condition (F).

Case 2. Assume $p \neq q$. Let $H \lhd K \subseteqq G$ where K/H is a group whose order has only one prime factor. If K/H is a q-group, then its order is at most $q^t \leqq q^k$. But G satisfies condition (F_k), so in this case G satisfies condition (F). In the case where K/H is a p-group, it is then elementary Abelian and hence satisfies condition (F).

THEOREM 9. *Let G be a group satisfying condition $(F_k + N_0)$, then G satisfies condition (N_k).*

Proof. Deny and let k be the minimal integer for which G satisfies condition $(F_k + N_0)$, but not (N_k). Then there exist subgroups $S \subset T \subseteqq G$ such that there is a chain

$$S = S_0 < S_1 < \cdots < S_n < S_{n+1} = T ,$$

$n \leqq k$, and $S \ntriangleleft T$ where this chain is the complete lattice of subgroups between S and T. Since T satisfies condition (N_0), we have $n \geqq 1$. By the minimality of k, we get $n = k$.

But we have $S_i \lhd S_j$ for $0 \leqq i < j \leqq k + 1$, except for the one case where $i = 0$ and $j = k + 1$, since k was chosen to be minimal.

Then S_{i+1}/S_i has no nontrivial subgroups and, therefore, $[S_{i+1} : S_i] = p_i$ for some prime p_i. Hence S_{i+1}/S_i is Abelian.

Let $p_0 = p$. If $y \in T$, then $S_1^y = S_1$ since $S_1 \lhd T$, but S_0^y may or may not be S_0. Let $W = \bigcap_{y \in T} S_0^y$. Then $W \lhd T$. Since $[S_1 : S_0^y] = [S_1 : S_0] = p$, we have $x^p \in S_0^y$ for all $x \in S_1$ and all $y \in T$. Hence $x^p \in W$ for all $x \in S_1$. Thus S_1/W is a p-group.

Since S/S_0 is Abelian, $S_1' \subseteqq S_0$. But $S_1' \lhd T$, so $S_1' \subseteqq S_0^y$ for all $y \in T$. Hence $S_1' \subseteqq W$ and S_1/W is Abelian. Then S_1/W is an elementary Abelian p-subgroup of T/W.

Since $S_0 \lhd S_k$ and $S_0 \ntriangleleft T$, we have $S_k \subseteqq N_T(S_0) < T$. But there are no subgroups between S_k and T so $N_T(S_0) = S_k$. Hence

$$[S_1 : W] = [S_1 : \bigcap S_0^y] \leqq \pi[S_1 : S_0^y] = p^{p_k} .$$

Since S_1/W is a finite p-group, $[S_1 : W]$ is a power of p, say p^m, where $2 \leqq m \leqq p_k$.

Now T/W is a finite group satisfying condition (N_0). Then T/W is nilpotent [3, p. 216].

Case 1. Suppose $k = 1$ and let $q = p_1$. Then $[T/W : S_1/W] = q$, T/W satisfies condition (F_1), and S_1/W is elementary Abelian. By Lemma 8, T/W satisfies condition (F). And by the revised Corollary 5.3 in [1], T/W satisfies condition (N). In particular, T/W satisfies condition (N_1). Hence $S_0/W \ntriangleleft T/W$ and $S_0 \ntriangleleft T$, a contradiction.

Case 2. Hence $k > 1$. Then $S_i \lhd S_{i+2}$ and $o(S_{i+2}/S_i) = p_i p_{i+1}$. Hence

S_{i+2}/S_i has proper subgroups of order p_i and p_{i+1}. By assumption, S_{i+2}/S_i has exactly one proper subgroup. Hence $p_i = p_{i+1}$. Thus

$$p = p_0 = p_1 = \cdots = p_k .$$

Thus $[T/W : S_1/W] = p^k$ and S_1/W is elementary Abelian. By Lemma 8, T/W satisfies condition (F). Then T/W satisfies condition (N) by the revised Corollary 5.3. in [1]. In particular, T/W satisfies condition (N_k) and $S_0/W \lhd T/W$. Then $S_0 \lhd T$, a contradiction.

COROLLARY 10. *If* $G = \bigcup_{n=1}^{\infty} Z_n(G)$ *and* G *satisfies condition* (F_k), *then* G *satisfies condition* (N_k).

Proof. Let $S \subset T \subsetneqq G$ be such that S is maximal in T. But T is nilpotent and hence $S \lhd T$ [3, p. 220]. Therefore, G satisfies condition (N_0). By Theorem 9, G satisfies condition (N_k).

REMARK. In Corollary 10, if the hypothesis of nilpotency is replaced by solvable, the theorem is not true. As an example consider the symmetric group S_3. A subgroup of order 2 is maximal in S_3, but not normal. Hence S_3 is solvable and satisfies condition (F_k) for every integer $k>0$, but does not satisfy condition (N_0).

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