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ON THE CONVERGENCE OF RESOLVENTS OF OPERATORS

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Let a family of linear operators $\{A_n\}(n = 1, 2, \cdots)$ in a Banach space X have the resolvents $\{R(\lambda; A_n)\}$ which is equicontinuous in n. Suppose that $\{A_n\}$ is a Cauchy sequence on a dense set. Then the question of convergence arises; when will $\{R(\lambda; A_n)x\}$ be a Cauchy sequence for all $x \in X$?

This problem is treated in some special cases and an application to the following theorem is presented.

Let A be the generator of a positive contraction semi-group $\sum$ and let B be a linear operator with domain $\mathscr{D}(B) \supset \mathscr{D}(A)$ in a weakly complete Banach lattice X.

Then $A + B$ or its closed extension generates a positive contraction semi-group $\sum'$ which dominates $\sum$ if and only if $A + B$ is dissipative and B is positive.

In this section we consider the above convergence problem in a Banach space X (cf. [9], [1], [11]).

Let a family of linear operators $\{A_n\}(n = 1, 2, \cdots)$ satisfy the following conditions:

(1) for some fixed number λ, the resolvent $R(\lambda; A_n) = (\lambda - A_n)^{-1}$ of A_n exists which acts on X to the domain $\mathscr{D}(A_n)$ of A_n and satisfies the norm condition $\| R(\lambda; A_n) \| \leqq K_\lambda$, where K_λ is a positive number independent of n,

(2) there is a dense subspace $\mathscr{M}$ on which $A = \lim A_n$ exists.

PROPOSITION 1. The limit operator $R_0(\lambda; A) = \lim R(\lambda; A_n)$ exists on $\overline{\mathscr{N}}$ and satisfies the norm condition $\| R_0(\lambda; A) \|_{\overline{\mathscr{N}}} \leqq K_\lambda$ where $\mathscr{N} = (\lambda - A)\mathscr{M}$ and $\overline{\mathscr{N}}$ is its closure.

Proof. For any $x \in \mathscr{M}$ we have

$$\| (\lambda - A_n)x \| \geqq K_\lambda^{-1} \| x \|$$

and thus obtain

$$\| (\lambda - A)x \| \geqq K_\lambda^{-1} \| x \| - \| A_n x - Ax \| .$$

Letting $n \rightarrow \infty$, we have

$$\| (\lambda - A)x \| \geqq K_\lambda^{-1} \| x \| .$$

It also follows that we can extend $(\lambda - A)^{-1}$ to the bounded linear operator $R_0(\lambda; A)$ on $\overline{\mathscr{N}}$ which satisfies

$$\| R_0(\lambda; A) \|_{\overline{\mathcal{N}}} = \sup \{\| R_0(\lambda; A)x \|; \| x \| = 1, x \in \overline{\mathcal{N}}\} \leqq K_\lambda .$$

Further, it is easy to see that, for any $x \in \mathcal{M}$,

$$\| R(\lambda; A_n)(\lambda - A)x - x \| \leqq K_\lambda \| A_n x - Ax \|$$

which implies that $R_0(\lambda; A) = \lim R(\lambda; A_n)$ on $\overline{\mathcal{N}}$.

REMARK 1. This proof shows that if $(\lambda - A)\mathcal{M}$ is dense in X then the convergence problem is solved.

We next remark some modification of the basic lemma in [1].

PROPOSITION 2. The following conditions are equivalent.

(1) $\lim_{n, n' \to \infty} \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| = 0 \qquad (x \in X)$,

(2) $\lim_{n, n' \to \infty} \| R(\lambda; A_n)R(\lambda; A_{n'})x - R(\lambda; A_{n'})R(\lambda; A_n)x \| = 0 \qquad (x \in (\lambda - A)\mathcal{M})$.

Proof. For any $x \in \mathcal{M}$, n and n', we have

$$\begin{aligned} R(\lambda; A_n)x - R(\lambda; A_{n'})x \\ &= R(\lambda; A_n)R(\lambda; A_{n'})(\lambda - A_{n'})x - R(\lambda; A_{n'})R(\lambda; A_n)(\lambda - A_n)x \\ &= R(\lambda; A_n)R(\lambda; A_{n'})(\lambda - A)x - R(\lambda; A_{n'})R(\lambda; A_n)(\lambda - A)x \\ &\quad + R(\lambda; A_n)R(\lambda; A_{n'})(A - A_{n'})x \\ &\quad + R(\lambda; A_{n'})R(\lambda; A_n)(A_n - A)x . \end{aligned}$$

From this relation and $\overline{\mathcal{M}} = X$, the assertion is readily verified.

PROPOSITION 3. If, for some positive integer m,

$$\lim_{n \to \infty} \| \{(A_n - A)R(\lambda; A_n)\}^m x \| = 0 \qquad (x \in \mathcal{M}_1)$$

is satisfied, where $\mathcal{M}_1$ is dense in X, then $(\lambda - A)\mathcal{M}$ is dense in X.

Proof. By virtue of the Hahn-Banach extension theorem, if there exists $x_0 \in \mathcal{M}_1 - \overline{\mathcal{N}}$, then so does a bounded linear functional F_0 acting on X which satisfies the following conditions:

$$F_0(x_0) \neq 0 , \qquad F_0(x) = 0 \qquad (x \in \overline{\mathcal{N}} = \overline{(\lambda - A)\mathcal{M}}) .$$

For this x_0 and any n, we have

$$\begin{aligned} x_0 &= (\lambda - A_n)R(\lambda; A_n)x_0 \\ &= (\lambda - A)R(\lambda; A_n)x_0 - (A_n - A)R(\lambda; A_n)x_0 \\ &= \cdots \\ &= (\lambda - A)R(\lambda; A_n)x_0 - (\lambda - A)R(\lambda; A_n)(A_n - A)R(\lambda; A_n)x_0 \\ &\quad + - \cdots \\ &\quad + (-1)^m\{(A_n - A)R(\lambda; A_n)\}^m x_0 . \end{aligned}$$

This relation implies that

$$F_0(x_0) = (-1)^m F_0(\{(A_n - A)R(\lambda; A_n)\}^m x_0)$$

and for any n

$$0 < |F_0(x_0)| \leqq \|F_0\| \, \|\{(A_n - A)R(\lambda; A_n)\}^m x_0\|$$

which is a contradiction. Consequently we have $\mathscr{M}_1 \subset \mathscr{N}$ and the assertion is proved.

We now concern with a theorem on the perturbation of operators which will be required in the sequel.

PROPOSITION 4. Suppose that linear operators A and B satisfy the following conditions:

(1) for some number λ, the equation

$$(\lambda - A)y = x \qquad (x \in X)$$

has a unique solution $y = R(\lambda; A)x$,

(2) there is a dense subspace $\mathscr{M}$ such that $BR(\lambda; A)\mathscr{M} \subset \mathscr{M}$ and

$$\lim_{k\to\infty} \|\{BR(\lambda; A)\}^k x\| = 0 \qquad (x \in \mathscr{M}) . \tag{$*$}$$

Then $(\lambda - A - B)R(\lambda; A)\mathscr{M}$ is dense in X.

The proof of this proposition is similar as that of Proposition 3 and is omitted.

REMARK 2. Suppose that for some positive integer k

$$(**) \qquad \|\{BR(\lambda; A)\}^k\|_{\mathscr{M}} < 1$$

is satisfied, then the condition $(*)$ in Proposition 4 is satisfied.

REMARK 3. Suppose that $R(\lambda; A)$ satisfies the norm condition $\|R(\lambda; A)\| \leqq K_\lambda$ in Proposition 4 and that there exist positive constants a and b such that for any $x \in \mathscr{M}_1 = R(\lambda; A)\mathscr{M}$

$$\|Bx\| \leqq a\|Ax\| + b\|x\|$$

and

$$a|\lambda|K_\lambda + a + bK_\lambda < 1 .$$

Then the condition $(**)$ in Remark 2 is satisfied.

Proof. For any $x \in \mathscr{M}$, we have

$$\begin{aligned} \| BR(\lambda; A)x \| &\leqq a \| AR(\lambda; A)x \| + b \| R(\lambda; A)x \| \\ &\leqq a \| \lambda R(\lambda; A)x - x \| + bK_\lambda \| x \| \end{aligned}$$

and

$$\| BR(\lambda; A)x \| \leqq (a |\lambda| K_\lambda + a + bK_\lambda) \| x \| < \| x \| .$$

Thus the assertion is proved.

THEOREM 1. *Suppose that a family of linear operators* $\{A_\varepsilon\}(\varepsilon > 0)$ *and a closed linear operator* A *from a Banach space* X *to* X *satisfy the following conditions*:

(1) *for some fixed number* λ, *the equation*

$$(\lambda - A_\varepsilon)y = x \qquad (x \in X)$$

has a unique solution $y = R(\lambda; A_\varepsilon)x \in \mathscr{D}(A_\varepsilon)$ *and* $\| R(\lambda; A_\varepsilon) \| \leqq K_\lambda$, *where* K_λ *is a positive number independent of* ε,

(2) $\mathscr{D}(A_\varepsilon) \supset \mathscr{D}(A), \qquad \overline{\mathscr{D}(A)} = X,$

(3) $A_\varepsilon x = Ax + \varepsilon B_\varepsilon x \qquad (x \in \mathscr{D}(A)),$

$$\| B_\varepsilon x \| \leqq K(x) \qquad (x \in \mathscr{D}(A)) ,$$

where $K(x)$ *is a positive number independent of* ε.

Then we have $\mathscr{R}(\lambda - A) = (\lambda - A)\mathscr{D}(A) = X$.

Proof. It follows from Proposition 1 that the limit operator $R_0(\lambda; A)$ exists and bounded on $\overline{\mathscr{R}(\lambda - A)}$.

Let $(\lambda - A)x_n \to y$ as $n \to \infty$. Then it follows from the boundedness of $R_0(\lambda; A)$ that $x_n \to R_0(\lambda; A)y$ and so that

$$Ax_n \to \lambda R_0(\lambda; A)y - y$$

as $n \to \infty$. Since A is closed, $R_0(\lambda; A)y \in \mathscr{D}(A)$ and $y \in \mathscr{R}(\lambda - A)$. Thus we have $\overline{\mathscr{R}(\lambda - A)} = \mathscr{R}(\lambda - A)$. It is easy to see that $\lambda - A_\varepsilon$ is closed and

$$\begin{aligned} (\lambda - A_\varepsilon)R_0(\lambda; A)x &= (\lambda - A)R_0(\lambda; A)x \\ &\quad - \varepsilon B_\varepsilon R_0(\lambda; A)x \qquad (x \in \mathscr{R}(\lambda - A)) . \end{aligned}$$

Hence, from the closed graph theorem it follows that $B_\varepsilon R_0(\lambda; A)$ is a bounded linear operator on $\mathscr{R}(\lambda - A)$. Moreover we have, for any $x \in \mathscr{D}(A)$,

$$\| B_\varepsilon R_0(\lambda; A)(\lambda - A)x \| = \| B_\varepsilon x \| \leqq K(x) < \infty .$$

Using the resonance theorem it follows that there exists a positive number L_λ which is independent of ε such that

$$\| B_\varepsilon R_0(\lambda; A) \|_{\mathscr{B}(\lambda - A)} \leqq L_\lambda .$$

Consequently we obtain the basic relation, for any $x \in \mathscr{D}(A)$,

$$\begin{aligned} \| \varepsilon B_\varepsilon x \| &= \| \varepsilon B_\varepsilon R_0(\lambda; A)(\lambda - A)x \| \\ &\leqq \varepsilon L_\lambda \| (\lambda - A)x \| \leqq \varepsilon L_\lambda \| Ax \| + \varepsilon |\lambda| L_\lambda \| x \| . \end{aligned}$$

Thus the assertion follows from Remark 3.

REMARK 4. Let A be a closed linear operator with dense domain $\mathscr{D}(A)$. Suppose that $A_\varepsilon = A + \varepsilon B$ generates a strongly continuous semi-group of linear contraction operators for every small $\varepsilon (0 < \varepsilon < \varepsilon_0)$ and $\mathscr{D}(A_\varepsilon) \supset \mathscr{D}(A)$.

Then A generates a strongly continuous semi-group of linear contraction operators.

Proof. Using Theorem 1 and Proposition 1, it follows from the Hille-Yosida theorem. (cf. [3], [11]).

2. The object of this section is to show that some special family of linear operators $\{A_n\} (n = 1, 2, \cdots)$ from a weakly complete Banach lattice X to X satisfies the convergence condition and to solve the problem on the perturbation theory for semi-groups of operators which is sited in the introductory part.

Let X be a Banach lattice with a semi-order $\geqq$ and $[x, y] (x, y \in X)$ denote a complex-valued (real-valued) function defined on $X \times X$ called a semi-inner product having the following properties (cf. [4], [6], [7]):

(1) $[x + y, z] = [x, z] + [y, z]$,

(2) $[\lambda x, y] = \lambda [x, y]$,

(3) $[x, x] = \| x \|^2$,

(4) $| [x, y] | \leqq \| x \| \, \| y \|$,

(5) if $y \geqq 0$, then $[x, y] \geqq 0$ for all $x \geqq 0$,

(6) $[x, x^+] = \| x^+ \|^2$,

where $x^+ = \sup (x, 0)$, $x^- = \sup (-x, 0)$, and $|x| = \sup (x, -x)$.

The following theorem is essentially due to Reuter [8].

PROPOSITION 5. Suppose that linear operators A_0 and A_1 on a Banach lattice X satisfy the following conditions:

(1) for $n = 0, 1$ and some $\lambda > 0$, the equation

$$(\lambda - A_n) y = x \qquad (x \in X)$$

has a unique solution $y = R(\lambda; A_n) x \in \mathscr{D}(A_n)$ and

$$R(\lambda; A_n)x \geqq 0 \qquad (x \geqq 0)\ ,$$

(2) there exist dense subspaces $\mathscr{M}$ and $\mathscr{M}_1$ such that

$$A_1 x \geqq A_0 x \qquad (x \geqq 0, x \in \mathscr{M})\ ,$$

$$R(\lambda; A_1)\mathscr{M}_1 \subset \mathscr{M}\ .$$

Then the following inequality holds:

$$R(\lambda; A_1)x \geqq R(\lambda; A_0)x \quad (x \geqq 0, x \in \mathscr{M}_1)$$

Proof. If $x \geqq 0$ and $x \in \mathscr{M}_1$, then $R(\lambda; A_1)\, x \geqq 0$ and $R(\lambda; A_1)x \in \mathscr{M}$ and thus we have

$$A_1 R(\lambda; A_1)x \geqq A_0 R(\lambda; A_1)x\ ,$$

$$(\lambda - A_0)R(\lambda; A_1)x \geqq (\lambda - A_1)R(\lambda; A_1)x = x\ .$$

Operating $R(\lambda; A_0)$, we obtain

$$R(\lambda; A_1)x \geqq R(\lambda; A_0)x\ .$$

Let $\sum = \{T_t; t \geqq 0\}$ be a one-parameter semi-group of linear operators from a Banach lattice X to X satisfying the following conditions:

(1) $T_0 x = x,\ T_{t+s}x = T_t T_s x \qquad (x \in X, t, s \geqq 0)$,

(2) $\| T_t x \| \leqq \| x \| \qquad (x \in X, t \geqq 0)$,

(3) $\lim_{t \to 0+} T_t x = x \qquad (x \in X)$,

(4) $T_t x \geqq 0 \qquad (x \geqq 0, t \geqq 0)$.

Such a semi-group is called a strongly continuous semi-group of positive contraction operators.

The following theorem is due to Phillips and is a variant of the Hille-Yosida theorem which will be convenient for purpose. (cf. [7]).

THEOREM. (*Phillips*). *A necessary and sufficient condition for a linear operator A with dense domain to generate a strongly continuous semi-group of positive contraction operators is that $\mathscr{R}(I - A) = X$ and that A is dispersive, that is,*

$$[Ax, x^+] \leqq 0 \qquad (x \in \mathscr{D}(A))\ .$$

THEOREM 2. *Suppose that a family of linear operators $\{A_n\}$ $(n = 1, 2, \cdots)$ which generate strongly continuous semi-groups $\{\sum_n\}$ of positive contraction operators on a weakly complete Banach lattice X satisfies the following conditions: there exist dense subspaces $\mathscr{M}$, $\mathscr{M}_0$ and $\{\mathscr{M}_n\}$ such that*

(1) $\lim_{n,n'\to\infty} \| A_n x - A_{n'} x \| = 0 \qquad (x \in \mathscr{M})$,

(2) $A_{n+1} x \geqq A_n x \qquad (x \geqq 0,\, x \in \mathscr{M}_n)$,

(3) $R(\lambda; A_n)\mathscr{M}_0 \subset \mathscr{M}_n$,

(4) $\mathscr{M}_0^+ = \{x^+;\, x \in \mathscr{M}_0\} \subset \mathscr{M}_0$.

Then the limit operator $A = \lim A_n$ *on* $\mathscr{M}$ *has a closed extension* $\tilde{A}$ *which generates a strongly continuous semi-group* $\sum$ *of positive contraction operators.*

Moreover we have

$$T_t x = \lim_{n\to\infty} T_t^{(n)} x \qquad (x \in X,\, t \geqq 0),$$

where $\sum_n = \{T_t^{(n)};\, t \geqq 0\}$ *and* $\sum = \{T_t;\, t \geqq 0\}$.

Proof. By the Hille-Yosida theorem (cf. [3], [11]) we find that the conditions (1) and (2) in Proposition 5 and the following norm condition are satisfied for any pair $\{A_n, A_{n+1}\}$.

$$\| R(\lambda; A_n) \| \leqq \lambda^{-1} \tag{*}$$

Thus we have, for any n,

$$R(\lambda; A_{n+1})x \geqq R(\lambda; A_n)x \qquad (x \geqq 0,\, x \in \mathscr{M}_0)\ .$$

Since X is weakly complete, the norm condition and this inequality imply that there exists $y \geqq 0$ such that

$$\lim_{n\to\infty} \| R(\lambda; A_n)x - y \| = 0\ .$$

From a representation of x: $x = x^+ - x^-$, we have, for any $x \in \mathscr{M}_0$, using the condition (4),

$$\lim_{n,n'\to\infty} \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| = 0 \tag{**}$$

and we have this convergence relation for all $x \in X$ by the condition $\overline{\mathscr{M}_0} = X$. We denote $\tilde{R}(\lambda; A) = \lim R(\lambda; A_n)$. Then $\tilde{R}(\lambda; A)$ is positive and satisfies the norm condition (*). The assertion is now proved by Theorem 2 in [1]. We sketch the proof of this theorem.

Since $R(\lambda; A_n)$ satisfies the resolvent equation

$$R(\lambda; A_n) - R(\lambda'; A_n) = -(\lambda - \lambda')R(\lambda; A_n)R(\lambda'; A_n)$$

$\tilde{R}(\lambda; A)$ also does. Then we find that $\tilde{R}(\lambda; A)$ is a one-to-one transformation from X to $\mathscr{R}(\tilde{R}(\lambda; A))$ and $\tilde{A}_\lambda = \lambda - \tilde{R}(\lambda; A)^{-1}$ is independent of λ, that is,

$$\tilde{A}x = \tilde{A}_\lambda x = \tilde{A}_{\lambda'} x \qquad (x \in \mathscr{R})\ ,$$

where $\mathscr{R} = \mathscr{R}(\tilde{R}(\lambda; A)) = \mathscr{R}(\tilde{R}(\lambda'; A))$.

Then, by the Hille-Yosida theorem, we find that $\tilde{A}$ generates a strongly continuous semi-group of contraction operators. The positivity and the convergence of semi-groups are verified by the condition (**). It is readily verified that $\tilde{A}$ is a closed extension of A.

REMARK 5. Suppose that a family of linear operators $\{A_n\}$ $(n = 1, 2, \cdots)$ which generate strongly continuous semi-groups of positive contraction operators on a weakly complete Banach lattice X satisfies the following conditions:

(1) $\lim_{n,n'\to\infty} \|A_n x - A_{n'} x\| = 0 \qquad (x \in \mathscr{M})$,

where $\mathscr{M}$ is a dense subspace in X,

(2) $A_{n+1}x \geqq A_n x \qquad (x \geqq 0, x \in \mathscr{D}(A_n))$,

(3) $\mathscr{D}(A_{n+1}) \supset \mathscr{D}(A_n)$.

Then the assertion in Theorem 2 is true.

REMARK 6. In Theorem 2, the condition (1) can be replaced by the following condition:

(1′) $$\| A_n^2 x \| \leqq K(x) \qquad (x \in \mathscr{M}_2),$$

where $K(x)$ is a positive number independent of n and $\mathscr{M}_2$ is dense in X.

Proof. We remark that the convergence of the family of resolvents in Theorem 2 does not depend on (1). Then we have, for any $x \in \mathscr{M}_2$,

$$\begin{aligned} \| A_n x - A_{n'} x \| &\leqq \lambda \| R(\lambda; A_n)A_n x - R(\lambda; A_{n'})A_{n'} x \| \\ &\quad + \| A_n x - \lambda R(\lambda; A_n)A_n x \| \\ &\quad + \| A_{n'} x - \lambda R(\lambda; A_{n'})A_{n'} x \| \\ &\leqq \lambda^2 \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| \\ &\quad + \| R(\lambda; A_n)A_n^2 x \| + \| R(\lambda; A_{n'})A_{n'}^2 x \| \\ &\leqq \lambda^2 \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| + 2\lambda^{-1}K(x) . \end{aligned}$$

Letting $\lambda \to \infty$, we have, for any $\varepsilon > 0$,

$$\| A_n x - A_{n'} x \| \leqq \lambda^2 \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| + \varepsilon$$

and the assertion is proved by (**).

From Remark 4 in [1] it follows that

REMARK 7. Suppose that there exists a dense subspace $\mathscr{M}_2$ such that $\tilde{R}(\lambda; A)\mathscr{M}_2 \subset \mathscr{M}$ in Theorem 2, then $\tilde{A}$ is the closure of A.

We next concern with the generation of contraction semi-groups which dominate a given semi-group and give an alternative form of a theorem of Reuter, Miyadera and Olubummo (cf. [8], [5], [6], [7]).

Given a one-parameter semi-group $\sum = \{T_t;\ t \geqq 0\}$ of positive contraction operators, if $\sum' = \{T'_t;\ t \geqq 0\}$ is another one, we say that $\sum'$ dominates $\sum$, if

$$T'_t x \geqq T_t x \qquad (x \geqq 0, t \geqq 0)\ .$$

In applications, it is important to know whether a given semi-group $\sum$ is dominated by any other semi-group $\sum'$.

The following lemmas in a Banach space will be required.

LEMMA. (*Lumer and Phillips*). *If A with dense domain is a dissipative operator, that is,*

$$\mathrm{Re}\,[Ax, x] \leqq 0 \qquad (x \in \mathscr{D}(A))\ ,$$

then A has a closed extension.

PROPOSITION 6. Suppose that a linear operator A which generates a strongly continuous semi-group of contraction operators on a Banach space X and a linear operator B with domain $\mathscr{D}(B) \supset \mathscr{D}(A)$ satisfy the following condition: $A + B$ has a closed extension. Then

$$\|\, BR(\lambda; A)\,\| \leqq K$$

where K is a positive number independent of $\lambda > 1$ and

$$\lim_{\lambda\to\infty} \|\, BR(\lambda; A)x\,\| = 0 \qquad (x \in X)\ .$$

The proof of Proposition 6 is readily verified by using the resolvent equation and is omitted.

THEOREM 3. *In a weakly complete Banach lattice X let A be the generator of a positive contraction semi-group $\sum$ and let B be a lenear operator with domain $\mathscr{D}(B) \supset \mathscr{D}(A)$. Then $A_1 = A + B$ or its closed extension generates a positive contraction semi-group $\sum'$ which dominates $\sum$ if and only if*

(1) $$\mathrm{Re}\,[A_1 x, x] \leqq 0 \qquad (x \in \mathscr{D}(A)),$$

(2) $$Bx \geqq 0 \qquad (x \geqq 0, x \in \mathscr{D}(A)).$$

Proof. To prove the sufficiency of the conditions (1) and (2), we approximate A_1 by a sequence of linear operators $\{A_{n,\lambda}\}$ in the following way. Define a sequence of linear operators $\{A_{n,\lambda}\}$ by

$$A_{n,\lambda} = A + (n - \lambda)BR(n; A) \qquad (n \geqq \lambda)$$

and $\{B_{n,\lambda}\}$ by

$$\begin{aligned} B_{n,\lambda} &= A_{n+1,\lambda} - A_{n,\lambda} \\ &= BR(n+1; A)(\lambda - A)R(n; A) \qquad (n \geqq \lambda) . \end{aligned}$$

Then it follows from Lemma (Lumer and Phillips) and Proposition 6 that there is a positive integer L independent of n and λ such that $\| B_{n,\lambda} \| \leqq L$.

If we assume that the resolvent $R(\lambda; A_{n,\lambda})$ exists which acts on X and is positive for some λ and n ($n \geqq \lambda$), then we have, for any $x \geqq 0$,

$$\begin{aligned} \lambda \| R(\lambda; A_{n,\lambda})x \|^2 &= [\lambda R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\leqq [\lambda R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\quad - \operatorname{Re} [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] . \end{aligned}$$

Using Theorem (Phillips), we remark that A is a dispersive operator. Thus we have

$$\begin{aligned} &\operatorname{Re} [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\quad = \operatorname{Re} [AR(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\qquad + \operatorname{Re} [BR(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\quad = [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] . \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\lambda \| R(\lambda; A_{n,\lambda})x \|^2 \\ &\qquad \leqq [\lambda R[\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] - [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\qquad = [x, R(\lambda; A_{n,\lambda})x] - [BR(n; A)(\lambda - A)R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\qquad \leqq [x, R(\lambda; A_{n,\lambda})x] , \end{aligned}$$

where the last inequality holds by virtue of the formula

$$\begin{aligned} &(\lambda - A)R(\lambda; A_{n,\lambda})x \\ &\qquad = x + (n - \lambda)BR(n; A)R(\lambda; A_{n,\lambda})x . \end{aligned}$$

Thus we obtain, for any $x \geqq 0$ and then for any $x \in X$,

$$\lambda \| R(\lambda; A_{n,\lambda})x \| \leqq \| x \| .$$

By induction on n we next show that the resolvent $R(\lambda; A_{n,\lambda})$ exists which acts on X and is positive for any $\lambda > L$ and any $n \geqq \lambda$. It is clear that $R(\lambda; A_{\lambda,\lambda}) = R(\lambda; A)$ is a positive operator for any $\lambda > L$. Suppose that $R(\lambda; A_{n,\lambda})$ is positive for any $\lambda > L$ and some n, then we have $\| B_{n,\lambda} R(\lambda; A_{n,\lambda}) \| < 1$. It follows from this norm condition that $R(\lambda; A_{n+1,\lambda})$ exists which acts on X and is given by the following formula (cf. [3], [11]):

$$R(\lambda; A_{n+1}, \lambda) = \sum_{k=0}^{\infty} R(\lambda; A_{n,\lambda})[B_{n,\lambda}R(\lambda; A_{n,\lambda})]^k .$$

Since $B_{n,\lambda}R(\lambda; A_{n,\lambda})$ is positive, it follows that

$$R(\lambda; A_{n+1,\lambda})x \geqq R(\lambda; A_{n,\lambda})x \geqq 0 \qquad (x \geqq 0) .$$

Hence, using the weakly completeness of X, we have for any $x \geqq 0$ and then $x \in X$,

$$\lim_{n,n'\to\infty} \| R(\lambda; A_{n,\lambda})x - R(\lambda; A_{n',\lambda})x \| = 0 .$$

To show that $\{R(\lambda'; A_{n,\lambda})x\}(0 < \lambda' < \lambda)$ is also a Cauchy sequence for any $x \in X$, we make use of the relation

$$R(\lambda - \mu; A_{n,\lambda}) = \sum_{k=1}^{\infty} \mu^{k-1} R(\lambda; A_{n,\lambda})^k ,$$

where, provided that $|\mu| < \lambda$, the right hand side converges uniformly in n (cf. [3], [11]). It also follows from this formula that $\lambda' R(\lambda'; A_{n,\lambda})$ is positive and is a contraction operator for any $\lambda'(0 < \lambda' < \lambda)$.

Let k be a positive integer such that $k > L$. We define, for any $\lambda \leqq k$,

$$\tilde{R}(\lambda; A_k)x = \lim_{n\to\infty} R(\lambda; A_{n,k})x \qquad (x \in X) .$$

Then it is easy to see that $\{\tilde{R}(\lambda; A_k); \lambda \leqq k\}$ satisfies the resolvent equation and the norm condition $\lambda \| \tilde{R}(\lambda; A_k) \| \leqq 1$.

Moreover $\{\tilde{R}(\lambda; A_k)\}_k$ is a consistent family of resolvents in the following sense:

$$\tilde{R}(\lambda; A_{k'})x = \tilde{R}(\lambda; A_k)x \qquad (x \in X, \lambda < k < k') .$$

In fact, we have the inequality

$$\begin{aligned} &\| \tilde{R}(\lambda; A_{k'})x - \tilde{R}(\lambda; A_k)x \| \\ &\quad \leqq \| \tilde{R}(\lambda; A_{k'})x - R(\lambda; A_{n,k'})x \| \\ &\qquad + [1 + \lambda^{-1}(k' - k)L] \| R(\lambda; A_{n,k})x - \tilde{R}(\lambda; A_k)x \| \\ &\qquad + \lambda^{-1}(k' - k) \| BR(n; A)\tilde{R}(\lambda; A_k)x \| \end{aligned}$$

and letting $n \to \infty$, we obtain the desired result.

Since $\{\tilde{R}(\lambda; A_k)\}_k$ is consistent, we have a family of resolvents

$$\{\tilde{R}(\lambda; A_1)\}; \tilde{R}(\lambda; A_1) = \tilde{R}(\lambda; A_k) \qquad (\lambda \leqq k)$$

which satisfies the norm condition $\lambda \| \tilde{R}(\lambda; A_1) \| \leqq 1$.

Then, using the same method as that in the proof of Theorem 2, we find that $\tilde{A}_1 = \lambda - \tilde{R}(\lambda; A_1)^{-1}$ generates a strongly continuous semi-group $\sum'$ of positive contraction operators which dominates $\sum$ and that $\tilde{A}_1$ is a closed extension of A_1.

We now prove the inverse part. Let $\sum = \{T_t; t \geqq 0\}$ and $\sum' = \{T'_t; t \geqq 0\}$. Then the condition (1) follows from

$$\begin{aligned} \mathrm{Re}\,[A_1x, x] &= \lim_{t\to 0+} \mathrm{Re}\,[t^{-1}(T'_t x - x), x] \\ &= \lim_{t\to 0+} t^{-1}\,\mathrm{Re}\,\{[T'_t x, x] - [x, x]\} \\ &\leqq 0 \qquad (x \in \mathscr{D}(A)) \,, \end{aligned}$$

and (2) follows from

$$A_1x = \lim_{t\to 0+} t^{-1}(T'_t x - x) \geqq \lim_{t\to 0+} t^{-1}(T_t x - x) = Ax \qquad (x \geqq 0, x \in \mathscr{D}(A)) \,.$$

Thus the assertion is proved.

REMARK 8. In Theorem 3 any one of the following conditions can take the place of the condition (1).

$$(1') \qquad [A_1x, x] \leqq 0 \qquad (x \geqq 0, x \in \mathscr{D}(A))$$

and A_1 has a closed extension,

$$(1'') \qquad [A_1x, x] \leqq 0 \qquad (x \geqq 0, x \in \mathscr{D}(A))$$

and $BR(\lambda; A)$ is a bounded linear operator for any $\lambda > 0$.

The contents of this section will be discussed in [2] by virtue of the notation of Gâteaux differentials.

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