PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 1, 1967

A NOTE ON DAVID HARRISON'S THEORY OF PREPRIMES

D. W. DUBOIS

A Stone ring is a partially ordered ring K with unit element 1 satisfying (1) 1 is positive; (2) for every x in K there exists a natural number n such that $n \cdot 1 - x$ belongs to K; and (3) if $1 + nx$ is positive for all natural numbers n then x is positive. Our first theorem: Every Stone ring is order-isomorphic with a subring of the ring of all continuous real functions on some compact Hausdorff space, with the usual partial order. A corollary is a theorem first proved by Harrison: Let K be a partially ordered ring satisfying conditions (1) and (2), and suppose the positive cone of K is maximal in the family of all subsets of K which exclude -1 and are closed under addition and multiplication. Then K is order-isomorphic with a subring of the reals.

The present paper is inspired by David Harrison's recently begun program of arithmetical ring theory where the basic objects are primes and preprimes; the positive cones of a ring are example of preprimes.

Throughout the paper, K will be a ring with unit element 1, and N will denote the set of positive integers. A *preprime* P in K is a nonempty subset of K excluding -1 and closed under addition and multiplication. A *prime* in K is a preprime maximal relative to set inclusion. A preprime P is *infinite* provided it contains both zero and 1, and is *conic* if $P \cap (-P) = \{0\}$. A conic preprime is simply a positive cone and induces a partial order: $x \geqq y \Leftrightarrow y \leqq x \Leftrightarrow x - y \in P$. A preprime P is *Archimedean* if for all x in K there exists a natural number n with $n - x$ in P, (condition (2) in the definition of Stone ring) and is (AC) if from $1 + nx \in P$ for all $n \in N$ follows $x \in P$ (condition (3)). We redefine a *Stone ring* as a pair $\langle K, P \rangle$ where P is an infinite conic Archimedean (AC) preprime in K. An *imbedding* of $\langle K, P \rangle$ in $\langle K', P' \rangle$ is an injective ring homomorphism $\psi: K \rightarrow K'$ such that $P = \psi^{-1}(P')$. If X is a compact Hausdorff space, $C(X)$ denotes the ring of all continuous real functions on X, $P(X)$ denotes the subset of nonnegative functions. If K is any subring of $C(X)$ then $\langle K, K \cap P(X) \rangle$ is a Stone ring. The principal tool in the proof of Theorem 1 is the Stone-Kadison ordered algebra theorem [3; Theorem 3.1], which characterizes $C(X)$ as a complete Archimedean ordered algebra. To imbed a Stone ring $\langle K, P \rangle$ in such an algebra we show that K is torsionfree, imbed it in a divisible ring K_N, put a norm on K_N and then complete it to K^*. At each step we have an imbedding of Stone rings:

$$\langle K, P\rangle \rightarrow \langle K_N, P_N\rangle \rightarrow \langle K^*, P^*\rangle \rightarrow \langle C(X), P(X)\rangle ,$$

where the last is Kadison's order-isomorphism. If P is a prime then so is P_N. [An order-isomorphism is an imbedding onto.]

In the proofs following, $\langle K, P\rangle$ is a Stone ring, N is the set of all positive integers.

PROPOSITION 1. If $n \in N, a \in K$, and $na \geqq 0$ then $a \geqq 0$.

Proof. By the unique factorization in N, it is enough to prove the proposition for the case where n is a prime number. Suppose for all primes $q < p$ and all $a \in K, qa \geqq 0$ implies $a \geqq 0$. Then for all $n < p$ and all $a \in K, na \geqq 0$ implies $a \geqq 0$. Now suppose that $pa \geqq 0$ but $a \ngeqq 0$. By the Archimedean property choose m in N with $m + a \geqq 0, x = m - 1 + \ngeqq 0$. Then $px \geqq 0, 1 + x \geqq 0$ and for all n in N, $1 + (pn + d)x \geqq 0$, if $d = 0$ or $d = 1$. [In case $p = 2$ this implies that $1 + kx \geqq 0$ for all k, so $x \geqq 0$ by (AC), a contradiction; hence $2a \geqq 0$ implies $a \geqq 0$.] Now let $1 < d < p$, with d in N. Since p is a prime there exists e in N, with $1 < e < p, ed = 1 + pn$, for some n in N. Then $e(1 + dx) = e + (1 + pn)x = (e - 1) + (1 + x) + (pnx) \geqq 0$. Since $e < p$ this implies that $1 + dx \geqq 0$. So for all k in N, $1 + (pk + d)x = 1 + dx + pkx \geqq 0, 0 \leqq d \leqq p - 1$. That is, $1 + nx \geqq 0$ for all n in N. By (AC) again, $x \geqq 0$, a contradiction. So $a \geqq 0$ and the induction is complete.

Now put

$$\begin{aligned} &K_N = Q \otimes K = \{k/n; k \in K, n \in N\} , \\ &P_N = \{p/n; p \in P, n \in N\} . \\ &\varphi: K \rightarrow K_N, \varphi(k) = k/1 . \end{aligned}$$

PROPOSITION 2. $\langle K_N, P_N\rangle$ is also a Stone ring. If P is a prime then so is P_N. φ is an imbedding.

Proof. That φ is injective follows from Proposition 1. If k/n, for k in K, n in N, belongs to P_N, then k belongs to P. For k/n in P_N implies $k/n = p/m$, for some p in P, m in N, so $mk = np \in P$. By Proposition 1, $k \in P$. Hence φ is an imbedding. The preprime, infinite, and conical properties of P_N follow easily from the corresponding properties for P. For the Archimedean property, let k/m be arbitrary in K_N (k in K, m in N) and choose n in N with $n > k$. Then $n - k/m = (nm - k)/m$ belongs to P_N since $nm > k, m \in N$. Now if $1 + n(k/m) \geqq 0$ holds in K_N, with m in N, k in K, and for all n in N, then for all $n, \varphi(1 + nk) = 1 + mn(k/m) \in P_N$. Since φ is an imbedding, $1 + nk \in P$. By the (AC) property for $P, k \in P, k/m \in P_N$.

This establishes (AC) for P_N. Finally let P' be a preprime containing P_N and let $P_1 = \varphi^{-1}(P')$. Then P_1 is a preprime containing P. If the first containment is proper so is the second. This proves that if P is a prime then P_N is a prime.

NOTE. The additive group of K_N is divisible. If K were already divisible then φ would be an order-isomorphism of $\langle K, P\rangle$ onto $\langle K_N, P_N\rangle$. The rational multiples of 1 in K_N form a field order-isomorphic with Q.

Now define t on K_N by

$$t(x) = \inf\{r; -r < x < r, r \in Q\}.$$

PROPOSITION 3. The function t is a norm on K_N:

(a) $t(x) \geqq 0; t(x) = 0$ if and only if $x = 0$.

(b) $t(x + y) \leqq t(x) + t(y)$.

(c) $t(xy) \leqq t(x)t(y)$

(d) $t(rx) = |r|\, t(x)$ for r in Q.

Put K^* equal to the completion of K_N, P^* equal to the closure of P_N in K^*. Then $\langle K^*, P^*\rangle$ is a Stone ring and an Archimedean ordered algebra as defined by Kadison.

Proof. The property (a) follows from (AC). Properties (b) and (c) follow from: if $-r < x < r$, $-s < y < s$ then $-(r + s) < x + y < r + s$, and $-rs < xy < rs$. See [1], § 2. The proofs there make no use of commutativity or of multiplicative inverses. Property (d) is a consequence of: $-r < x < r$ if and only if $-rq < qx < rq$, where q is a positive rational. It is clear that $t(-x) = t(x)$ and for rational r, $t(r) = |r|$. We now identify K_N with its injection in its completion K^* and note that $P^* \cap K_N = P_N$: for if $k \in P^* \cap K_N$ then $k = \lim p_n$, $p_n \in P_N$, and p_n may be chosen so that $-1/n < k - p_n < 1/n$ for all $n \in N$; it follows that $1 + nk > np_n > 0$ for all $n \in N$ and thence by (AC) that $k \in P_N$. The reverse inclusion is obvious. It remains to prove that P^* is an infinite conical Archimedean (AC) preprime. It is certainly closed under addition and multiplication. Let $x \in P^* \cap (-P^*)$. Then there exist positive sequences p_n and q_n with $x = \lim p_n$, $-x = \lim q_n$, $0 = \lim (p_n + q_n)$. Thus if ε is any positive real then for all large n, $0 \leqq p_n \leqq p_n + q_n < \varepsilon$, so $x = \lim p_n = 0$; P^* is therefore conical. Let $x_n \in K_N$, with $x = \lim x_n$. The Cauchy sequence $\{x_n\}$ is bounded in norm so there exists an integer m with $m > x_n$ for all n. Hence $m - x = \lim (m - x_n) \in P^*$, $m > x$. This shows P^* is Archimedean. Now let $1 + nx \in P^*$ for all n in N $(x \in K^*)$. P^*, as closure of P, is closed and hence contains $x = \lim (x + 1/n)$, since $x + 1/n$ belongs to P^*. Thus P^* is (AC). That $1 \in P^*$ and $-1 \notin P^*$ are obvious, and it has now been proved that $\langle K^*, P^*\rangle$ is a Stone ring. The closure of Q in K^* is (order-isomorphic with) the reals R. Using t for the induced norm in K^* we have

(e) $t(rx) = |r|\, t(x)$ for all r in R.

R is contained in the center of K^* and so $\langle K^*, P^*\rangle$ is an algebra over the reals. For the sake of completeness we list Kadison's axioms for an Archimedean ordered algebra. Each is obviously satisfied by $\langle K^*, P^*\rangle$ with $e = 1$.

1. K^* is a real algebra with unit e.
2. P^* is closed under addition, multiplication, and multiplication by positive reals.
3. For every x in K^* there exists a positive real r with $re > x$.
4. If $re \geqq x$ for all positive real r, then $x \leqq 0$.

An Archimedean ordered algebra is *complete* if and only if it is complete in our norm t. Thus $\langle K^*, P^*\rangle$ is a complete Archimedean ordered algebra. Collecting results of Propositions 1, 2, and 3 and applying Theorem 3.1 of Kadison we get our Theorem 1.

Now we are ready to prove the corollary. As we remarked earlier, Harrison showed that a prime P satisfying the hypotheses there is also (AC). By Proposition 2, P_N is also a prime. Now identify each of $\langle K, P\rangle, \langle K_N, P_N\rangle, \langle K^*, P^*\rangle$ with its imbedding in $\langle C(X), P(X)\rangle$, so that $P(P_N)$ is the set of all nonnegative functions in $K(K_N)$. The proof is completed by showing that X is a singleton. Suppose that x and y are distinct points of X. Since X is normal and K_N is dense in $C(X)$, Urysohn's lemma guarantees that there is a function f in K_N with $f(x) > 0, f(y) < 0$. Then $P' = \{g; g \in K_N \text{ and } g(x) \geqq 0\}$ is a preprime in K_N containing P_N and f, while f is not in P_N. This contradicts the primality of P_N and the corollary is proved.

TWO EXAMPLES. 1. Example of a ring $\langle K, P\rangle$ where all the conditions of Theorem 1 hold for P except the Archimedean condition. Let K be the ring of all 2×2 real matrices, P the set of matrices with every entry nonnegative.

2. Example of a ring $\langle K', P'\rangle$ where P' satisfies all except the condition (AC). Put K' equal to the set of all triangular 2×2 matrices over R and let P' be the subset consisting of 0 and all matrices with strictly positive diagonal entries. Thus if either of the Archimedean conditions is omitted then commutativity cannot be deduced.

References

1. D. W. Dubois, *On partly ordered fields*, Proc. Amer. Math. Soc. **7** (1956), 918-930.
2. D. K. Harrison, *Finite and infinite primes for rings and fields.* (Mimeographed preprint)
3. R. V. Kadison, *A representation theory for commutative topological algebra*, Memoirs of Amer. Math. Soc. No. **7** (1951).

Received April 28, 1966.