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# MACDONALD'S THEOREM WITH INVERSES

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**One of the fundamental theorems in the theory of Jordan algebras is that of I. G. Macdonald which says that any identity in three variables $x, y, z$ of degree zero or one in $z$ will be valid in all Jordan algebras if it is valid in the special Jordan algebras.**

**In this paper we will extend this result to identities which also involve the inverses of $x$ and $y$.**

Following the method and notation of N. Jacobson [3] we have the

THEOREM. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are respectively the free Jordan algebra and free special Jordan algebra on three free generators $x, y, z$ and the inverses $x^{-1}, y^{-1}$, with $\mathfrak{E}$ and $\mathfrak{E}_s$ the associative algebras of linear transformations in $\mathfrak{J}$ and $\mathfrak{J}_s$ respectively generated by the multiplications by elements of the subalgebra generated by $x, y, x^{-1}, y^{-1}$, then the canonical homomorphism $\nu$ of $\mathfrak{E}$ onto $\mathfrak{E}_s$ is an isomorphism. If $\mathfrak{F}$ is the free associative algebra with free generators $f_{i,j} (i, j \in Z)$ and $\pi$ the homomorphism of $\mathfrak{F}$ onto $\mathfrak{E}$ determined by $f_{i,j} \rightarrow U_{x^i, y^j}$ then the kernel of $\pi$ is the ideal $\mathfrak{R}$ generated by the elements*

$$(1)\quad \begin{array}{ll} \text{(i)} & f_{0,0} - 1 \\ \text{(ii)} & 2f_{i,0}f_{j,k} - (2f_{i,0}^2 - f_{2i,0})f_{j-i,k} - f_{i+j,k} \\ & 2f_{0,i}f_{k,j} - (2f_{0,i}^2 - f_{0,2i})f_{k,j-i} - f_{k,i+j} \\ \text{(iii)} & 2f_{j,k}f_{i,0} - f_{j-i,k}(2f_{i,0}^2 - f_{2i,0}) - f_{i+j,k} \\ & 2f_{k,j}f_{0,i} - f_{k,j-i}(2f_{0,i}^2 - f_{0,2i}) - f_{k,i+j} . \end{array}$$

From this as immediate corollaries we have

MACDONALD'S THEOREM WITH INVERSES [4]. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are the free Jordan algebra and free special Jordan algebra on three free generators $x, y, z$ and the inverses $x^{-1}, y^{-1}$ then the kernel of the canonical homomorphism $\nu$ of $\mathfrak{J}$ onto $\mathfrak{J}_s$ contains no elements of degree zero or one in $z$.*

SHIRSHOV'S THEOREM WITH INVERSES [6]. *The free Jordan algebra on two free generators $x, y$ and their inverses $x^{-1}, y^{-1}$ is special.*

More generally, we have the

SHIRSHOV-COHN THEOREM WITH INVERSES [1]. *Any Jordan algebra* *generated by two elements and their inverses is special.*

Indeed, such an algebra $\mathfrak{K}$ is a homomorphic image of the free Jordan algebra $\mathfrak{H}$ generated by $x, y, x^{-1}, y^{-1}$; by Shirshov's Theorem with Inverses $\mathfrak{H} = \mathfrak{H}_s$, the free special Jordan algebra generated by $x, y, x^{-1}, y^{-1}$; thus for some ideal $\mathfrak{K}_s$ we have $\mathfrak{K} = \mathfrak{H}_s/\mathfrak{K}_s$. By a result of P. M. Cohn, $\mathfrak{K}$ is special if and only if

$$\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s \subset \mathfrak{K}_s$$

where $\mathfrak{H}_s$ is imbedded in the free associative algebra $\mathfrak{A}$ generated by $x, y, x^{-1}, y^{-1}$ (we are following the argument of [1, p. 307]). Noting that $\mathfrak{K}_s \subset \mathfrak{H}_s$ and the elements of $\mathfrak{H}_s$ are symmetric under the reversal involution * of $\mathfrak{A}$, we see $\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s$ is contained in the linear span of the

$$f(x, y, x^{-1}, y^{-1}, k) = akb + b^*ka^*$$

where $a, b \in \mathfrak{A}$, $k \in \mathfrak{K}_s$, and $f(x, y, x^{-1}, y^{-1}, z)$ is a symmetric element of the free associative algebra $\mathfrak{B}$ generated by $x, y, z, x^{-1}, y^{-1}$. Ordering the generators of $\mathfrak{B}$ by $z < x < x^{-1} < y < y^{-1}$ we see that the tetrads

$$\begin{aligned} \{xx^{-1}yy^{-1}\} &= 1 \\ \{zx^{-1}yy^{-1}\} &= z\cdot x^{-1} \\ \{zxyy^{-1}\} &= z\cdot x \\ \{zxx^{-1}y^{-1}\} &= z\cdot y^{-1} \\ \{zxx^{-1}y\} &= z\cdot y \end{aligned}$$

are Jordan elements of $\mathfrak{B}$, hence by Cohn's Theorem [1, p. 306] $f(x, y, x^{-1}, y^{-1}, z)$ is a Jordan element of $\mathfrak{B}$. As a Jordan product of $x, y, x^{-1}, y^{-1}$ and the element $k$ of the Jordan ideal $\mathfrak{K}_s$, the element $f(x, y, x^{-1}, y^{-1}, k) \in \mathfrak{K}_s$. Thus $\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s \subset \mathfrak{K}_s$ as desired.

**1. Preliminaries.** By "algebra" we will mean algebra *with identity* over a field $\Phi$ *of characteristic* $\neq 2$; associativity and finite-dimensionality are not assumed.

Recall [5, p. 18] that an element $a$ of a Jordan algebra is *invertible* with (Jordan) inverse $b$ if

$$a\cdot b = 1,\ a^2\cdot b = a\ .$$

In this case $b$ is invertible with inverse $a$, and $a, b$ generate a commutative associative subalgebra; we write $b = a^{-1}$. In a special Jordan algebra the notion of Jordan inverse is equivalent to inverse in the associative sense.

Given a set $\mathfrak{X}$ and a subset $\mathfrak{Y}$ we denote by $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ the *free*

*Jordan algebra generated by* $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$. If $\mathfrak{Y} \to \mathfrak{Y}^{-1}$ is a bijection of $\mathfrak{Y}$ onto a set $\mathfrak{Y}^{-1}$ disjoint from $\mathfrak{X}$ we may set $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) = \mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})/\mathfrak{K}$ where $\mathfrak{K}$ is the ideal in the free Jordan algebra $\mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})$ generated by all $y \cdot y^{-1} - 1$, $y^2 \cdot y^{-1} - y$ for $y \in \mathfrak{Y}$. Similarly we have the *free special Jordan algebra* $\mathfrak{J}_s(\mathfrak{X}/\mathfrak{Y})$ *generated by* $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$; this may be regarded as the subalgebra of $\mathfrak{F}(\mathfrak{X}/\mathfrak{Y})^+$ generated by $\mathfrak{X} \cup \mathfrak{Y}^{-1}$, where $\mathfrak{F}(\mathfrak{X}/\mathfrak{Y})$ is the *free associative algebra generated by* $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$.

If $L_a$ denotes left-multiplication by an element $a$ of a Jordan algebra $\mathfrak{A}$ we have the following operator identities

$$(2)\qquad \begin{gathered}[L_a, L_{b\cdot c}] + [L_b, L_{c\cdot a}] + [L_c, L_{a\cdot b}] = 0\\ L_aL_bL_c + L_cL_bL_a + L_{b\cdot(a\cdot c)} = L_{a\cdot b}L_c + L_{b\cdot c}L_a + L_{c\cdot a}L_b\ .\end{gathered}$$

If we set

$$(3)\qquad U_{a,b} = L_aL_b + L_bL_a - L_{a\cdot b}$$

then we have $U_a = U_{a,a}$, $L_a = U_{a,1} = U_{1,a}$. It is well known [3, p. 243] that if $\mathfrak{X}$ is a set of generators (containing 1) for a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ then the operators $U_{x,y}$ for $x, y \in \mathfrak{X}$ generate the same algebra of linear transformations as the $L_b$ for $b \in \mathfrak{B}$. In particular, it is not hard to see that if $\mathfrak{B}$ is generated by $x, y, x^{-1}, y^{-1}$ then the $U_{x^i,y^j}$ for $i, j \in Z$ generate the same algebra $\mathfrak{E}$ of linear transformations as do the $L_b$ for $b \in \mathfrak{B}$.

**2. The presentation $\pi$.** The above remarks show that the homomorphism $\pi: \mathfrak{F} \to \mathfrak{E}$ in the Theorem is surjective. We next show that the ideal $\mathfrak{R}$ generated by the elements (1) is contained in the kernel of $\pi$, i.e. $\pi(f) = 0$ for $f$ of the form (i), (ii), (iii) in (1). Part (i) is trivial since $U_{x^0,y^0} = I$. Parts (ii) and (iii) follow from the first part of (ii) by symmetry in $x$ and $y$ and symmetry in the operator relations (a consequence of the symmetry in (2); more precisely, this "symmetry" corresponds to the canonical involution in the universal multiplication envelope). The first part of (ii) follows from the following lemma by taking $a = x^i$, $b = x^{j-i}$, $c = y^k$ and noting [5, p. 19] that $[L_{x^n}, L_{x^m}] = 0$ for all $n, m \in Z$.

LEMMA 1. *If elements $a, b, c$ of a Jordan algebra satisfy*

$$[L_a, L_b] = [L_{a^2}, L_b] = 0$$

*then*

$$2L_aU_{a\cdot b,c} = U_aU_{b,c} + U_{a^2\cdot b,c}\ .$$

*Proof.* By (2), (3) and our hypotheses we have

$$\begin{aligned}
2L_aU_{a\cdot b,c} &= 2L_a\{L_{a\cdot b}L_c + L_cL_{a\cdot b} - L_{c\cdot(a\cdot b)}\} + 2[L_a, L_b]\{L_{a\cdot c} - L_cL_a\} \\
&= \{2L_aL_{a\cdot b}\}L_c + 2L_a\{L_cL_{a\cdot b} - L_{c\cdot(a\cdot b)} + L_bL_{a\cdot c} - L_bL_cL_a\} \\
&\quad + L_b\{2L_aL_cL_a - 2L_aL_{a\cdot c}\} \\
&= \{2L_aL_bL_a + L_{a^2\cdot b} - L_bL_{a^2}\}L_c + 2L_a\{L_aL_cL_b - L_aL_{b\cdot c}\} \\
&\quad + L_b\{L_cL_{a^2} - L_{c\cdot a^2}\} \\
&= \{2L_a^2L_b + L_{a^2\cdot b} - L_{a^2}L_b\}L_c + 2L_a^2\{L_cL_b - L_{b\cdot c}\} \\
&\quad + \{L_cL_{a^2\cdot b} + L_{a^2}L_{b\cdot c} - L_{a^2}L_cL_b - L_{c\cdot(a^2\cdot b)}\} \\
&= \{2L_a^2 - L_{a^2}\}\{L_bL_c + L_cL_b - L_{b\cdot c}\} \\
&\quad + \{L_{a^2\cdot b}L_c + L_cL_{a^2\cdot b} - L_{c\cdot(a^2\cdot b)}\} \\
&= U_aU_{b,c} + U_{a^2\cdot b,c}\ .
\end{aligned}$$

Thus $\pi$ induces a homomorphism $\sigma$ of $\mathfrak{A} = \mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{E}$.

LEMMA 2. *If $e_{i,j} \in \mathfrak{A} = \mathfrak{F}/\mathfrak{R}$ is the image of $f_{i,j} \in \mathfrak{F}$ and we set $a_i = e_{i,0}$, $b_i = 2a_i^2 - a_{2i}$, $c_i = e_{0,i}$, $d_i = 2c_i^2 - c_{2i}$ then we have the following identities*:

$$\begin{aligned}
&(\mathrm{i}) && a_0 = b_0 = c_0 = d_0 = e_{0,0} = 1 \\
&(\mathrm{ii}) && 2a_ie_{j,k} = b_ie_{j-i,k} + e_{i+j,k},\ 2c_ie_{k,j} = d_ie_{k,j-i} + e_{k,i+j} \\
&(\mathrm{iii}) && 2e_{j,k}a_i = e_{j-i,k}b_i + e_{i+j,k},\ 2e_{k,j}c_i = e_{k,j-i}d_i + e_{k,i+j} \\
&(\mathrm{iv}) && 2a_ia_j = b_ia_{j-i} + a_{i+j},\ 2c_ic_j = d_ic_{j-i} + c_{i+j} \\
(4)\quad &(\mathrm{v}) && 2a_ja_i = a_{j-i}b_i + a_{i+j},\ 2c_jc_i = c_{j-i}d_i + c_{i+j} \\
&(\mathrm{vi}) && a_i = a_{-i}b_i = b_ia_{-i},\ c_i = c_{-i}d_i = d_ic_{-i} \\
&(\mathrm{vii}) && b_ib_{-i} = b_{-i}b_i = 1,\ d_id_{-i} = d_{-i}d_i = 1 \\
&(\mathrm{viii}) && [a_i, a_j] = [a_i, b_j] = 0,\ [c_i, c_j] = [c_i, d_j] = 0 \\
&(\mathrm{ix}) && b_ib_j = b_{i+j},\ d_id_j = d_{i+j}\ .
\end{aligned}$$

*Proof.* (i)–(vi) follow immediately from the relations (1). (vii) follows from

$$b_ib_{-i} = b_i\{2a_{-i}^2 - a_{-2i}\} = 2a_ia_{-i} - b_ia_{-2i}$$

(by vi) $= a_0$ (by iv) $= 1$ (by i). For (viii) it suffices to show $[a_i,a_j] = 0$, and this only for $i, j \geqq 0$ since $b_i = 2a_i^2 - a_{2i}$ and $a_{-i} = b_i^{-1}a_i$ by (vi), and finally only for $i = 1$, $j = 2$ since (iv) shows by induction that the $a_i$ for $i \geqq 0$ are generated by $a_1, b_1$ (hence $a_1, a_2$). But (iv), (v) show $2[a_1, a_2] = [b_1, a_1] = -[a_2, a_1] = [a_1, a_2]$, so $[a_1, a_2] = 0$ as desired. For (ix) it suffices to show $b_i = b_1^i$, and this only for $i \geqq 0$ by (vii); this follows by induction from (i) and

$$\begin{aligned}
b_{i+1} &= 2a_{i+1}^2 - a_{2i+2} \\
&= 2a_{i+1}\{2a_i a_1 - a_{i-1}b_1\} - a_{2i+2} \quad \text{(by v)} \\
&= 2\{a_1 b_i + a_{2i+1}\}a_1 - \{a_2 b_{i-1} + a_{2i}\}b_1 - \{2a_{2i+1}a_1 - a_{2i}b_1\} \quad \text{(by v)} \\
&= 2a_1 b_i a_1 - a_2 b_{i-1} b_1 \\
&= \{2a_1^2 - a_2\}b_i \quad \text{(by viii and induction)} \\
&= b_1 b_i \,.
\end{aligned}$$

3. **The idea of the proof.** We have surjective homomorphisms $\sigma: \mathfrak{A} \to \mathfrak{E}$ and $\nu: \mathfrak{E} \to \mathfrak{E}_s$, and a linear mapping $\tau: \mathfrak{E}_s \to \mathfrak{F}_s$ by $L \to L(z)$. The theorem will be proven if we show $\mu = \tau \circ \nu \circ \sigma$ is injective, for then $\sigma$ and $\nu$ will be isomorphisms. This will be the case if we find a spanning set in $\mathfrak{A}$ whose image under $\mu$ is independent in $\mathfrak{F}_s$. A hint is provided by Cohn's Theorem [1, p. 307] which says that $\mu(\mathfrak{A})$ is precisely the set of all elements of the free associative algebra $\mathfrak{F}(x, y, z/x, y)$ which are linear in $z$ and symmetric under the reversal involution $*$. A basis for this set consists of the distinct

$$f_s(p, q) = \tfrac{1}{2}\{pzq^* + qzp^*\} = f_s(q, p)$$

for monomials $p, q \in \mathfrak{F}(x, y/x, y)$. The idea of the proof [3, p. 249] is to construct pre-images

$$f(p, q) = f(q, p)$$

in $\mathfrak{A}$ satisfying

$$\mu(f(p, q)) = f_s(p, q) \,. \tag{5}$$

By definition the images in $\mathfrak{F}_s$ will be independent, and the only question is whether these elements span $\mathfrak{A}$. Since $\mathfrak{A}$ is generated by 1 and the elements $b_k, d_k, e_{k,l}, a_k, c_k$ it suffices to show the set of $f(p, q)$ contains 1

$$f(1, 1) = 1 \tag{6}$$

and is invariant under left multiplication by the generators

$$\begin{aligned}
&\text{(i)} && b_k f(p, q) = f(x^k p, x^k q) \\
&\text{(ii)} && d_k f(p, q) = f(y^k p, y^k q) \\
&\text{(iii)} && e_{k,l} f(p, q) = \tfrac{1}{2}\{f(x^k p, y^l q) + f(y^l p, x^k q)\} \quad (k, l \neq 0) \\
&\text{(iv)} && a_k f(p, q) = \tfrac{1}{2}\{f(x^k p, q) + f(p, x^k q)\} \\
&\text{(v)} && c_k f(p, q) = \tfrac{1}{2}\{f(y^k p, q) + f(p, y^k q)\} \,.
\end{aligned} \tag{7}$$

To this end we define $f(p, q)$ by induction as follows. First we inductively define sets $\mathfrak{X}_n, \mathfrak{Y}_n$ $(n \geqq 0)$ of monomials in $\mathfrak{F}(x, y/x, y)$ by

$$\mathfrak{X}_0 = \mathfrak{Y}_0 = \{1\}$$
$$\mathfrak{X}_{n+1} = \{x^k p \mid k \neq 0, p \in \mathfrak{Y}_n\} \qquad \mathfrak{Y}_{n+1} = \{y^k p \mid k \neq 0, p \in \mathfrak{X}_n\} .$$

Next we define sets of pairs of monomials by

$$\mathfrak{X}_{n,m} = \mathfrak{X}_n \times \mathfrak{X}_m \cup \mathfrak{X}_m \times \mathfrak{X}_n = \mathfrak{X}_{m,n}$$
$$\mathfrak{Y}_{n,m} = \mathfrak{Y}_n \times \mathfrak{Y}_m \cup \mathfrak{Y}_m \times \mathfrak{Y}_n = \mathfrak{Y}_{m,n}$$
$$\mathfrak{Z}_{n,m} = \mathfrak{X}_n \times \mathfrak{Y}_m \cup \mathfrak{Y}_m \times \mathfrak{X}_n .$$

Finally, $f$ is defined recursively on the sets $\mathfrak{X}_{n,m}, \mathfrak{Y}_{n,m}, \mathfrak{Z}_{n,m}$ by

(D.0) On $\mathfrak{X}_{0,0} = \mathfrak{Y}_{0,0} = \mathfrak{Z}_{0,0} = \{(1, 1)\}$:

$$f(1, 1) = 1 .$$

(D.1) On $\mathfrak{X}_{n+1,m+1}$: for $i, j \neq 0, i \geqq j, (r, s) \in \mathfrak{Y}_{n,m}$

$$f(x^i r, x^j s) = f(x^j s, x^i r) = b_j f(x^{i-j} r, s) .$$

(D.2) On $\mathfrak{Y}_{n+1,m+1}$: for $i, j \neq 0, i \geqq j, (r, s) \in \mathfrak{X}_{n,m}$

$$f(y^i r, y^j s) = f(y^j s, y^i r) = d_j f(y^{i-j} r, s) .$$

(D.3) On $\mathfrak{Z}_{n+1,m+1}$: for $i, j \neq 0, r \in \mathfrak{Y}_n, s \in \mathfrak{X}_m$

$$f(x^i r, y^j s) = f(y^j s, x^i r) = 2e_{i,j} f(r, s) - f(y^j r, x^i s)$$

which is defined by induction unless $n = m = 0, r = s = 1$, and on $\mathfrak{Z}_{1,1}$:

$$f(x^i, y^j) = f(y^j, x^i) = e_{i,j} .$$

(D.4) On $\mathfrak{X}_{n+1,0} = \mathfrak{X}_{0,n+1} = \mathfrak{Z}_{n+1,0}$: for $i \neq 0, r \in \mathfrak{Y}_n$

$$f(x^i r, 1) = f(1, x^i r) = 2a_i f(r, 1) - f(r, x^i)$$

which is defined by induction if $n \neq 0$, and on $\mathfrak{X}_{1,0} = \mathfrak{X}_{0,1} = \mathfrak{Z}_{1,0}$:

$$f(x^i, 1) = f(1, x^i) = a_i .$$

(D.5) Similarly, on $\mathfrak{Y}_{n+1,0} = \mathfrak{Y}_{0,n+1} = \mathfrak{Z}_{0,n+1}$:

$$f(y^i r, 1) = f(1, y^i r) = 2c_i f(r, 1) - f(r, y^i)$$

and on $\mathfrak{Y}_{1,0} = \mathfrak{Y}_{0,1} = \mathfrak{Z}_{0,1}$:

$$f(y^i, 1) = f(1, y^i) = c_i .$$

It is easy to verify that $f(p, q) = f(q, p)$ is a well-defined element of $\mathfrak{A}$ for all monomials $p, q$ in $\mathfrak{F}(x, y/x, y)$.

**4. The main lemma.** The previous considerations have reduced the proof of the theorem to the following.

LEMMA 3. *The elements* $f(p, q) = f(q, p) \in \mathfrak{A}$ *defined by* (D.0)—(D. 5) *satisfy* (5), (6), (7).

*Proof.* (5) can be verified at each step of the inductive definition, and (6) is just (D.0). We will prove (7.i)-(7.v) for $(p, q)$ in $\mathfrak{X}_{n,m}$, $\mathfrak{Y}_{n,m}$, $\mathfrak{Z}_{n,m}$ by induction on the *weight* $n + m$; the case $n + m = 0$ follows immediately from the definitions (D.1)-(D.5), and we assume the result proven for all weights less than $n + m$. We claim that if $i, j \neq 0$ (but $k, l = 0$ are allowed) then

$$
(8)\quad
\begin{aligned}
&(\mathrm{i}) && (r, s) \in \mathfrak{Y}_{n-1,m-1} \Rightarrow b_k f(x^i r, x^j s) = f(x^{k+i} r, x^{k+j} s)\\
&(\mathrm{ii}) && (r, s) \in \mathfrak{Y}_{n-1} \times \mathfrak{X}_{m-1} \Rightarrow b_k f(x^i r, y^j s) = f(x^{k+i} r, x^k y^j s)\\
&(\mathrm{iii}) && r \in \mathfrak{Y}_{n-1} \Rightarrow b_k f(x^i r, 1) = f(x^{k+i} r, x^k)\\
&(\mathrm{iv}) && (r, s) \in \mathfrak{Y}_{n-1,m-1} \Rightarrow 2e_{k,l} f(x^i r, x^j s)\\
& && \qquad = f(x^{k+i} r, y^l x^j s) + f(y^l x^i r, x^{k+j} s)\\
&(\mathrm{v}) && (r, s) \in \mathfrak{Y}_{n-1} \times \mathfrak{X}_{m-1} \Rightarrow 2a_k f(x^i r, y^j s)\\
& && \qquad = f(x^{k+i} r, y^j s) + f(x^i r, x^k y^j s)\\
&(\mathrm{vi}) && r \in \mathfrak{Y}_{n-1} \Rightarrow 2a_k f(x^i r, 1) = f(x^{k+i} r, 1) + f(x^i r, x^k)\ .
\end{aligned}
$$

These suffice to establish the various cases of (7) according to the following table:

| | 7.i | 7.iv | 7.iii | 7.v | 7.ii |
|---|---|---|---|---|---|
| $\mathfrak{X}_{n,m}$ | 8.i | 8.iv | 8.iv | 8.iv | def |
| $\mathfrak{X}_{n,0} = \mathfrak{Z}_{n,0}$ | 8.iii | 8.vi | def | def | def |
| $\mathfrak{Z}_{n,m}$ | 8.ii | 8.v | def | 8.v* | 8.ii* |
| $\mathfrak{Y}_{n,0} = \mathfrak{Z}_{0,n}$ | def | def | def | 8.vi* | 8.iii* |
| $\mathfrak{Y}_{n,m}$ | def | 8.iv* | 8.iv* | 8.iv* | 8.i* |

Here the columns indicate the particular cases of (7) and the rows the particular possibilities for $(p, q)$, with $n, m > 0$; "def" means the result follows directly from the definitions, and * denotes the dual formula obtained by everywhere interchanging $x$ and $y$. The proof of (8.i)-(8.vi) will be broken into corresponding Cases I-VI.

*Case* I. (a) If $k + i, k + j \neq 0$, say $i \geqq j$, then

$$
\begin{aligned}
b_k f(x^i r, x^j s) &= b_k b_j f(x^{i-j} r, s) && (\mathrm{D}.1)\\
&= b_{k+j} f(x^{i-j} r, s) && (4.\mathrm{ix})\\
&= f(x^{k+i} r, x^{k+j} s)\ . && (\mathrm{D}.1)
\end{aligned}
$$

(b) If, say, $k + j = 0$ then

$$\begin{aligned} b_k f(x^i r, x^j s) &= b_k b_j f(x^{i-j} r, s) && \text{(induction 7.i)} \\ &= f(x^{i+k} r, x^{j+k} s) \,. && \text{(4.vii)} \end{aligned}$$

*Case* II. (a) If $0 \neq i + k \geqq k$ the result follows from (D.1).
(b) If $0 = i + k$, $m = n = 0$, $r = s = 1$ we have

$$\begin{aligned} b_k f(x^i, y^j) &= b_k e_{-k,j} && \text{(D.3)} \\ &= 2a_k e_{0,j} - e_{k,j} && \text{(4.ii)} \\ &= f(1, x^k y^j) && \text{(D.4)} \\ &= f(x^{k+i}, x^k y^j) \,. \end{aligned}$$

(c) If $0 = i + k$ but $r \neq 1$ or $s \neq 1$ then

$$\begin{aligned} b_k f(x^i r, y^j s) &= b_k \{2e_{i,j} f(r, s) - f(y^j r, x^i s)\} && \text{(D.3)} \\ &= 2\{2a_k e_{0,j} - e_{k,j}\} f(r, s) - f(x^k y^j r, s) \end{aligned}$$

by 4.ii and induction 7.i—which is applicable since by our assumptions on $r$ and $s$ $(y^j r, x^i s)$ has weight less than $n + m$)

$$\begin{aligned} &= 4a_k c_j f(r, s) - \{f(x^k r, y^j s) + f(y^j r, x^k s)\} - f(x^k y^j r, s) \\ & && \text{(induction 7.iii)} \\ &= 4a_k c_j f(r, s) - f(x^k r, y^j s) - 2a_k f(y^j r, s) && \text{(induction 7.iv)} \\ &= 2a_k f(r, y^j s) - f(x^k r, y^j s) && \text{(induction 7.v)} \\ &= f(r, x^k y^j s) && \text{(induction 7.iv)} \\ &= f(x^{k+i} r, x^k y^j s) \,. \end{aligned}$$

(d) If $0 \neq i + k < k$ then

$$\begin{aligned} b_k f(x^i r, y^j s) &= b_k \{2a_i f(r, y^j s) - f(r, x^i y^j s)\} && \text{(induction 7.iv)} \\ &= b_k b_i \{2a_{-i} f(r, y^j s) - f(x^{-i} r, y^j s)\} && \text{(4.vi, Case IIb, c above)} \\ &= b_{k+i} f(r, x^{-i} y^j s) && \text{(4.ix, induction 7.iv)} \\ &= f(x^{k+i} r, x^k y^j s) \,. && \text{(D.1)} \end{aligned}$$

*Case* III. The proof is obtained from that of Case II by setting $j = 0$, $s = 1$; the second line of the proof of (c) is justified by Case II rather than by induction 7.i.

*Case* IV. We allow $k$ or $l$ to be zero, and we induct on $|i| + |j|$; the result follows from the induction hypothesis if $i$ or $j$ is zero.

(a) If $i, j$ have the same sign, say $|i| \geqq |j| > 0$, then $|i - j| < |i|$ so

$$
\begin{aligned}
2e_{k,l}f(x^ir, x^js) &= 2e_{k,l}b_jf(x^{i-j}r, s) \\
&\qquad \text{(induction 7.i)} \\
&= 2\{2e_{k+j,l}a_j - e_{k+2j,l}\}f(x^{i-j}r, s) \\
&\qquad \text{(4.iii)} \\
&= 2e_{k+j,l}\{f(x^ir, s) + f(x^{i-j}r, x^js)\} \\
&\quad - \{f(x^{i+j+k}r, y^ls) + f(y^lx^{i-j}r, x^{k+2j}s)\} \\
&\qquad \text{(induction 7.iii–iv)} \\
&= \{2e_{k+j,l}f(x^{i-j}r, x^js) - f(y^lx^{i-j}r, x^{k+2j}s)\} \\
&\quad + \{2e_{k+j,l}f(x^ir, s) - f(x^{i+j+k}r, y^ls)\} \\
&= f(x^{i+k}r, y^lx^js) + f(y^lx^ir, x^{k+j}s) . \\
&\qquad \text{(induction, } |i-j| + |j| < |i| + |j|)
\end{aligned}
$$

(b) If $i, j$ have opposite signs, say $|i| \geqq |j| > 0$, then $|i+j| < |i|$ so

$$
\begin{aligned}
2e_{k,l}f(x^ir, x^js) &= 2e_{k,l}\{2a_jf(x^ir, s) - f(x^{i+j}r, s)\} \\
&\qquad \text{(induction 7.iv)} \\
&= 2\{e_{k-j,l}b_j + e_{k+j,l}\}f(x^ir, s) - 2e_{k,l}f(x^{i+j}r, s) \\
&\qquad \text{(4.iii)} \\
&= 2e_{k-j,l}f(x^{i+j}r, x^js) - f(y^lx^{i+j}r, x^ks) - f(x^{i+j+k}r, y^ls) \\
&\quad + f(x^{i+j+k}r, y^ls) + f(y^lx^ir, x^{k+j}s) \\
&\qquad \text{(induction 7.i, 7.iii)} \\
&= f(x^{k+i}r, y^lx^js) + f(y^lx^ir, x^{k+j}s) . \\
&\qquad (|i+j| + |j| < |i| + |j|, \text{ induction})
\end{aligned}
$$

*Case* V. (a) If $m = n = 1$, $r = s = 1$ we have

$$
\begin{aligned}
2a_kf(x^i, y^j) &= 2a_ke_{i,j} = e_{i+k,j} + b_ke_{i-k,j} && \text{(D.3, 4.ii)} \\
&= f(x^{k+i}, y^j) + b_kf(x^{i-k}, y^j) && \text{(D.3)} \\
&= f(x^{k+i}, y^j) + f(x^i, x^ky^j) . && \text{(Case II–III above)}
\end{aligned}
$$

(b) If $r \neq 1$ or $s \neq 1$ then

$$
\begin{aligned}
2a_kf(x^ir, y^js) &= 2a_k\{2e_{i,j}f(r, s) - f(y^jr, x^is)\} \qquad \text{(D.3)} \\
&= 2\{e_{i+k,j} + b_ke_{i-k,j}\}f(r, s) \\
&\quad - \{f(x^ky^jr, x^is) + f(y^jr, x^{i+k}s)\}
\end{aligned}
$$

(by 4.ii and induction 7.iv—which is applicable since by our assumptions on $r$ and $s$ $(y^jr, x^is)$ has weight less than $n + m$)

$$
\begin{aligned}
&= \{2e_{i+k,j}f(r, s) - f(y^ir, x^{i+k}s)\} \\
&\quad + b_k\{2e_{i-k,j}f(r, s) - f(y^jr, x^{i-k}s)\}
\end{aligned}
$$

(induction 7.i applicable to $(y^j r, x^{i-k}s)$—or use Case II above)

$$= f(x^{i+k}r, y^j s) + b_k f(x^{i-k}r, y^j s) \qquad \text{(induction 7.iii)}$$
$$= f(x^{k+i}r, y^j s) + f(x^i r, x^k y^j s) \, . \qquad \text{(Case 2 above)}$$

*Case* VI. This follows from Case V by setting $j = 0$, $s = 1$ throughout the proof; the second line in the proof of (b) is justified by Case V rather than by induction 7.iv.

This completes the proof of (8), the Lemma, and all the Theorems.

**5. Remarks and conjectures.** We will now indicate how the above proof can be modified to prove Macdonald's original theorem without inverses; in a similar manner we obtain a one-inverse form of the theorem.

We require that all indices $i, j, k, l$ etc. be nonnegative; this modifies the free algebra $\mathfrak{F}$ of the theorem, so we add to the relations (1) the further elements of $\mathfrak{R}$

$$(1.\text{iv}) \quad \begin{array}{l} f_{i,0}f_{j,k} + f_{j,0}f_{i,k} - (2f_{j,0}^2 - f_{2j,0})f_{i-j,0}f_{0,k} - f_{i+j,k} \\ f_{0,i}f_{k,j} + f_{0,j}f_{k,i} - (2f_{0,j}^2 - f_{0,2j})f_{0,i-j}f_{k,0} - f_{k,i+j} \end{array}$$

for $i \geqq j$ corresponding to the relations

$$(4.\text{x}) \quad \begin{array}{l} a_i e_{j,k} + a_j e_{i,k} = b_j a_{i-j} c_k + e_{i+j,k} \\ c_i e_{k,j} + c_j e_{k,i} = d_j c_{i-j} a_k + e_{k,i+j} \end{array}$$

in the algebra $\mathfrak{A}$. It suffices to establish the first relation in (4.x), and this follows by putting $a = x^j$, $b = x^{i-j}$, $c = y^k$ in the following addition to Lemma 1: if $a, b, c$ are elements of a Jordan algebra satisfying

$$[L_a, L_b] = [L_a, L_{a\cdot b}] = 0$$

then

$$L_a U_{a\cdot b, c} + L_{a\cdot b} U_{a,c} = U_a L_b L_c + U_{a^2\cdot b, c} \, .$$

The only other thing to be changed is the proof of (8). Cases Ib, IIb–c–d, and IVb are unnecessary, but the proof of Case V works only for $i \geqq k$; for $i < k$ we must use the relations (4.x).

It would be nice if the inverse-less and one-inverse theorems could be obtained directly from the two-inverse form, which leads to a general

*Conjecture.* If $\mathfrak{X}_0 \subset \mathfrak{X}$, $\mathfrak{Y}_0 \subset \mathfrak{Y}$ then the canonical homomorphism $\mathfrak{J}(\mathfrak{X}_0/\mathfrak{Y}_0) \rightarrow \mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ is injective.

If we represent $\mathfrak{J}(\mathfrak{X}_0/\mathfrak{Y}_0)$ by $\mathfrak{J}(\mathfrak{X}_0 \cup \mathfrak{Y}_0^{-1})/\mathfrak{K}_0$ and $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ by $\mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})/\mathfrak{K}$ as in the first section of the paper then the conjecture amounts to

$$\mathfrak{J}(\mathfrak{X}_0 \cup \mathfrak{Y}_0^{-1}) \cap \mathfrak{K} = \mathfrak{K}_0 .$$

It is also sufficient to consider only the case $\mathfrak{X} = \mathfrak{X}_0$.

More generally, we have a

*Conjecture.* $\mathfrak{J}(\mathfrak{X})$ can be imbedded in a universal Jordan division algebra $\mathfrak{D}(\mathfrak{X})$ such that the canonical homomorphisms $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) \rightarrow \mathfrak{D}(\mathfrak{X})$ are all injective.

It is easy to see that this implies the first conjecture by considering the commutative diagram

$$\begin{array}{ccc} \mathfrak{J}(\mathfrak{X}/\mathfrak{Y}_0) & \longrightarrow & \mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) \\ & \searrow \quad \swarrow & \\ & \mathfrak{D}(\mathfrak{X}) & \end{array}$$

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