PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 2, 1967

THE PTOLEMAIC INEQUALITY IN HILBERT GEOMETRIES

DAVID C. KAY

Let M be a metric space, and if x and y are points in M, let xy denote the metric. The space M and its metric are called *ptolemaic* if for each quadruple of points x_i $(i = 1, 2, 3, 4)$ the *ptolemaic inequality*

$$x_1x_2 \cdot x_3x_4 + x_1x_3 \cdot x_2x_4 \geqq x_1x_4 \cdot x_2x_3$$

holds. If the inequality holds only in some neighborhood of each point the space and its metric are said to be *locally ptolemaic*. Euclidean space is known to be ptolemaic and therefore, locally ptolemaic. We are interested here in certain non-euclidean spaces which may possibly be locally ptolemaic. The author has proved in his thesis (Michigan State University Doctoral Dissertation, 1963) that a Riemannian geometry is locally ptolemaic if and only if it has nonpositive curvature, and that a Finsler space which is locally ptolemaic is Riemannian. The main result established here extends the theorem regarding Finsler spaces to include Hilbert geometries as well: A Hilbert geometry is locally ptolemaic if and only if it is hyperbolic.

The ptolemaic inequality is related to problems of curvature in metric geometry. Assuming this condition enables one to prove that a curve is a geodesic if and only if its metric curvature is zero at each of its points (see [3]). Blumenthal has investigated a number of properties peculiar to ptolemaic metric spaces in [2]. It is then significant to determine what metric spaces are ptolemaic. A question which remains unsettled is whether a non-Riemannian G-space (Busemann [4, p. 37]) can be locally ptolemaic. The result obtained here concerning Hilbert geometries, together with several in the author's thesis lends support to the conjecture that such a space does not exist.

Hilbert geometry is a generalization of the well-known Klein model for hyperbolic geometry. Consider an arbitrary bounded convex body C with nonempty interior D in euclidean space. If x and y be any two points in D, a distance function may be defined as

$$h(x, y) = k \mid \log R(xy, ab) \mid$$

where k is a positive constant, a and b are the points of intersection of C with the affine line L_{xy} determined by x and y, and $R(xy, ab)$ is the usual (euclidean) cross-ratio. It can be proved that $h(x, y)$ is a metric, and moreover if C is either strictly convex or has the property that all linear triples which it contains belong to the same hyperplane, linearity under h coincides with linearity under the euclidean metric. Hence, the geodesics for h are the portions of affine lines contained in D. The function h is then called a *Hilbert metric* for D and with this metric, D is called a *Hilbert geometry*.

Let p be any point in D. With the euclidean metric understood, let C' be the reflection of C in p. Each ray r from p will cut C and C' in unique points x_r and x'_r respectively. Define U_p as the set of all points u_r such that $u_r \in r$ and

$$\frac{2}{\overline{pu_r}} = \frac{1}{\overline{px_r}} + \frac{1}{\overline{px'_r}}, \tag{1}$$

where $\overline{xy}$ denotes the euclidean metric. It is clear that U_p is symmetric about p. With our former assumptions on C we can convince ourselves that U_p is strictly convex. Suppose rays r, s, and t with origin p meet C in x, y, and z, and C' in x', y', and z'. If further, u, v, and w are the intersections of those rays with U_p we may use (1) to deduce

$$R(pu, xx') = R(pv, yy') = R(pw, zz') = -1 .$$

Assume that v_1 is a point on the segment joining u and w, S_{uw}, and that s passes through v_1. Finally, let $y_1 = s \times S_{xz}$ and $y'_1 = s \times S_{x'z'}$. Now, $q = L_{xz} \times L_{x'z'}$ is the center of a projectivity which maps x into z, u into w, and x' into z'. It also maps x, u, and x' into y_1, v_1, and y'_1, so

$$R(pv_1, y_1y'_1) = -1$$

which is equivalent to

$$\frac{2}{\overline{pv_1}} = \frac{1}{\overline{py_1}} + \frac{1}{\overline{py'_1}} .$$

Since C and C' are convex we have $\overline{py_1} \leqq \overline{py}$ and $\overline{py'_1} \leqq \overline{py'}$, with strict inequality in at least one case. Hence

$$1/\overline{py_1} + 1/\overline{py'_1} > 1/\overline{py} + 1/\overline{py'} = 2/\overline{pv} .$$

Therefore, $\overline{pv_1} < \overline{pv}$ proving that U_p is strictly convex.

As proved in [5], it follows that U_p is the unit sphere of a Minkowski metric $m_p(x, y)$ defined on D having the same geodesics as h. Further,

$$\lim_{x,y\to p} \frac{h(x, y)}{m_p(x, y)} = 1 .$$

In this sense a Hilbert metric is "locally" Minkowskian. The metric $m_p(x, y)$ is called the *associated local Minkowski metric of* $h(x, y)$ *at* p.

1. Perpendicularity in Hilbert geometries. Suppose D is metrized as a straight space by the metric $d(x, y)$. We shall say that a geodesic L_1 is perpendicular to another geodesic L_2 at $p \in L_1 \cap L_2$ whenever $x \in L_1$ and $y \in L_2$ implies

$$d(x, p) \leqq d(x, y) .$$

This condition will be denoted $L_1 \perp_d L_2$. The problem of existence has been solved in [4, pp. 119–122]: Given a geodesic L_2 and a point q not on it there exists a unique geodesic L_1 such that $q \in L_1$ and $L_1 \perp_d L_2$ at some point p, and if p is any point on L_2 and H is any planar section of D containing L_2 (H is then a two-dimensional straight space) then precisely one geodesic L_1 exists in H such that $L_1 \perp_d L_2$ at p. Since the Hilbert metric and the metrics $m_p(x, y)$ for $p \in D$ metrize D as a straight space this statement applies to those metrics.

Perpendicularity is called *symmetric* if for any two geodesics L_1 and L_2, $L_1 \perp_d L_2$ implies $L_2 \perp_d L_1$. The following theorem was proved in [6]:

THEOREM 1. *Perpendicularity is symmetric in a Hilbert geometry if and only if it is hyperbolic.*

Of significance to us is

THEOREM 2. *At any point in* D, *perpendicularity under the local Minkowski metric coincides with perpendicularity under the original Hilbert metric.*

Proof. Let $L_1 \perp_h L_2$ at $p \in D$ and suppose $x \in L_1$, $y \in L_2$, $x \neq p$, and $y \neq p$. With p as origin, define the points $x_\lambda = \lambda x$ and $y_\lambda = \lambda y$ for each real λ, $0 < \lambda < 1$, where λx denotes the usual scalar multiplication in euclidean space. Under the local Minkowski metric $m(x, y)$ at p, the triangles determined by the triples (p, x, y) and $(p, x_\lambda, y_\lambda)$ are similar and thus $m(x, p)/m(x_\lambda, p) = m(x, y)/m(x_\lambda, y_\lambda)$. Since $h(x_\lambda, p) \leqq h(x_\lambda, y_\lambda)$ for each λ we have

$$\frac{h(x_\lambda, p)}{m(x_\lambda, p)} \cdot m(x, p) \leqq \frac{h(x_\lambda, y_\lambda)}{m(x_\lambda, y_\lambda)} \cdot m(x, y) .$$

But $\lim_{\lambda\to 0} [h(x_\lambda, p)/m(x_\lambda, p)] = \lim_{\lambda\to 0} [h(x_\lambda, y_\lambda)/m(x_\lambda, y_\lambda)] = 1$ so it follows that $m(x, p) \leqq m(x, y)$ and therefore $L_1 \perp_m L_2$ at p.

Conversely, suppose $L_1 \perp_m L_2$ at p. Consider L_1' the geodesic which passes through p, lies in the plane determined by L_1 and L_2, such that $L_1' \perp_h L_2$ at p. Then, from the preceding case, $L_1' \perp_m L_2$ at p. Since the geodesic perpendicular to L_2 at p is unique in this plane, $L_1' = L_1$. Therefore $L_1 \perp_h L_2$ at p.

If a Hilbert geometry has the property that the local Minkowski metric at each point is euclidean (that is, its unit sphere is an ellipsoid) we shall say it is locally euclidean. We have an immediate corollary, making use of Theorem 1:

COROLLARY 1. *A locally euclidean Hilbert geometry is hyperbolic.*

2. Ptolemaic metrics.

In [9] Schoenberg has proved that a ptolemaic normed linear space is an inner product space. We may state this in the more pertinent form:

THEOREM 3. *A Minkowski metric is euclidean if and only if it is ptolemaic.*

This enables us to prove

THEOREM 4. *A locally ptolemaic Hilbert geometry is hyperbolic.*

Proof. It will be shown that the given Hilbert metric $h(x, y) = xy$ is locally euclidean. The rest follows immediately from Corollary 1. Let p be any point in D and suppose $m(x, y) = \overline{xy}$ is the associated local Minkowski metric at p. In view of Theorem 3 it suffices to show that $\overline{xy}$ is ptolemaic. Let x, y, z, and w be four points in D and let N be the neighborhood about p in which the Hilbert metric is ptolemaic. As before, with p as origin define $x_\lambda = \lambda x$, $y_\lambda = \lambda y$, $z_\lambda = \lambda z$, $w_\lambda = \lambda w$ for λ a positive real number. For all sufficiently small λ the points x_λ, y_λ, z_λ, and w_λ lie in N and we therefore have

$$x_\lambda y_\lambda \cdot z_\lambda w_\lambda + x_\lambda z_\lambda \cdot y_\lambda w_\lambda \geqq x_\lambda w_\lambda \cdot y_\lambda z_\lambda$$

or,

$$\begin{aligned}(1)\qquad &\frac{x_\lambda y_\lambda \cdot z_\lambda w_\lambda}{\overline{x_\lambda y_\lambda} \cdot \overline{z_\lambda w_\lambda}} \cdot \frac{\overline{x_\lambda w_\lambda} \cdot \overline{y_\lambda z_\lambda}}{x_\lambda w_\lambda \cdot y_\lambda z_\lambda} \cdot \frac{\overline{x_\lambda y_\lambda} \cdot \overline{z_\lambda w_\lambda}}{\overline{x_\lambda w_\lambda} \cdot \overline{y_\lambda z_\lambda}} \\ &+ \frac{x_\lambda z_\lambda \cdot y_\lambda w_\lambda}{\overline{y_\lambda z_\lambda} \cdot \overline{y_\lambda w_\lambda}} \cdot \frac{\overline{x_\lambda w_\lambda} \cdot \overline{y_\lambda z_\lambda}}{x_\lambda w_\lambda \cdot y_\lambda z_\lambda} \cdot \frac{\overline{x_\lambda z_\lambda} \cdot \overline{y_\lambda w_\lambda}}{\overline{x_\lambda w_\lambda} \cdot \overline{y_\lambda z_\lambda}} \geqq 1 .\end{aligned}$$

Since $u \longrightarrow \lambda u = u_\lambda$ is a similarity mapping we have

$$(3)\qquad \begin{aligned} &\frac{x_\lambda y_\lambda \cdot z_\lambda w_\lambda}{\overline{x_\lambda y_\lambda}\cdot\overline{z_\lambda w_\lambda}}\cdot\frac{\overline{x_\lambda w_\lambda}\cdot\overline{y_\lambda z_\lambda}}{x_\lambda w_\lambda\cdot y_\lambda z_\lambda}\cdot\frac{\overline{xy}\cdot\overline{zw}}{\overline{xw}\cdot\overline{yz}} \\ &+\frac{x_\lambda z_\lambda\cdot y_\lambda w_\lambda}{\overline{x_\lambda z_\lambda}\cdot\overline{y_\lambda w_\lambda}}\cdot\frac{\overline{x_\lambda w_\lambda}\cdot\overline{y_\lambda z_\lambda}}{x_\lambda w_\lambda\cdot y_\lambda z_\lambda}\cdot\frac{\overline{xz}\cdot\overline{yw}}{\overline{xw}\cdot\overline{yz}}\geqq 1\,. \end{aligned}$$

Taking the limit in (3) as $\lambda \to 0$ yields

$$\frac{\overline{xy}\cdot\overline{zw}}{\overline{xw}\cdot\overline{yz}}+\frac{\overline{xz}\cdot\overline{yw}}{\overline{xw}\cdot\overline{yz}}\geqq 1\,,$$

the desired inequality.

If x, y, and z are any three points in D and m is the midpoint of segment S_{yz}, the euclidean formula for the median S_{xm} is

$$(4)\qquad (xm)^2=(xy)^2/2+(xz)^2/2-(yz)^2/4\,.$$

It can be shown that for a hyperbolic metric, instead of (4), we have

$$(5)\qquad (xm)^2\leqq(xy)^2/2+(xz)^2/2-(yz)^2/4$$

with equality only when x, y, and z are linear. Inequality (5) in turn implies the ptolemaic inequality as we shall see. This fact may then be used to derive the ptolemaic inequality in hyperbolic geometry.

We shall use (5) to derive the more general inequality

$$(5')\qquad (xm)^2\leqq\lambda(xy)^2+(1-\lambda)(xz)^2-\lambda(1-\lambda)(yz)^2\,,\qquad 0\leqq\lambda\leqq 1\,,$$

for any point m on S_{yz} with $\lambda = mz/yz$. Induction will establish (5′) for λ a diadic rational $\mu/2^\nu$ where μ and ν are nonnegative integers and $0 \leqq \mu \leqq 2^\nu$. The inequality is clear for the case $\nu = 1$. Assume it has been proved for all diadic rationals of the form $\mu/2^\kappa$, $\kappa < \nu$, and let $\mu/2^\nu$ be given, excluding the cases $\mu = 0$, $\mu = 2^\nu$, and μ even, as trivial. Then, there exist points y' and z' on S_{yz} such that $(\mu+1)/2^\nu = y'z/yz$ and $(\mu-1)/2^\nu = z'z/yz$. Since μ is odd, $(\mu+1)/2^\nu$ and $(\mu-1)/2^\nu$ are diadic rationals of the form $\mu'/2^\kappa$ for $\kappa < \nu$ and the induction hypothesis implies

$$(6)\qquad \begin{aligned} (xy')^2 &\leqq \frac{\mu+1}{2^\nu}(xy)^2 \\ &\quad+\Big(1-\frac{\mu+1}{2^\nu}\Big)(xz)^2-\frac{\mu+1}{2^\nu}\Big(1-\frac{\mu+1}{2^\nu}\Big)(yz)^2 \\ (xz')^2 &\leqq \frac{\mu-1}{2^\nu}(xy)^2 \\ &\quad+\Big(1-\frac{\mu-1}{2^\nu}\Big)(xz)^2-\frac{\mu-1}{2^\nu}\Big(1-\frac{\mu-1}{2^\nu}\Big)(yz)^2\,. \end{aligned}$$

Since $\mu/2^\nu = mz/yz$, m is the midpoint of the segment $S_{y'z'}$ and (5) yields

$$(7) \qquad (xm)^2 \leqq (xy')^2/2 + (xz')^2/2 - (y'z')^2/4 .$$

Substituting $yz/2^{\nu-1}$ for $y'z'$ and making use of (6), (7) becomes

$$(xm)^2 \leqq \frac{\mu}{2^\nu}(xy)^2 + \left(1 - \frac{\mu}{2^\nu}\right)(xz)^2 - \frac{\mu}{2^\nu}\left(1 - \frac{\mu}{2^\nu}\right)(yz)^2$$

and induction carries. Then (5′) is true for arbitrary real λ, $0 \leqq \lambda \leqq 1$.

THEOREM 5. *The ptolemaic inequality holds in any metric space D in which the inequality* (5) *holds, provided the metric is complete and convex.*

Proof. The completeness and convexity of the metric guarantees the existence of segments. Moreover, (5) implies the *uniqueness* of the segment joining each pair of points. Let x, y, z, and w be any four points in D. We must prove that

$$(8) \qquad xy \cdot zw + xz \cdot yw \geqq xw \cdot yz .$$

If the four points are not distinct, (8) follows immediately; hence we set aside this case, as well as the case when the four points lie on a segment. In a euclidean plane, the euclidean metric denoted $\overline{pq}$, let x', y', and z' be the vertices of a triangle such that $\overline{x'y'} = xy$, $\overline{x'z'} = xz$, and $\overline{y'z'} = yz$, and let w' be a point such that $\overline{y'w'} = yw$, $\overline{w'z'} = wz$, and the segment $S_{x'w'}$ has at least one point m' in common with the line $L_{y'z'}$. Point m' is determined uniquely since the linearity of x', y', z', and w' would imply that x, y, z, and w lie on a metric segment (see [4] p. 29).

Case 1. m' lies on $S_{y'z'}$. Locate m in D such that m lies on the metric segment joining y and z, and $mz = \overline{m'z'}$. Put $\lambda = mz/yz = \overline{m'z'}/\overline{y'z'}$. Applying (5′),

$$\begin{aligned}(xm)^2 &\leqq \lambda(xy)^2 + (1-\lambda)(xz)^2 - \lambda(1-\lambda)(yz)^2 \\ &= \lambda(\overline{x'y'})^2 + (1-\lambda)(\overline{x'z'})^2 - \lambda(1-\lambda)(\overline{y'z'})^2 \\ &= (\overline{m'x'})^2 .\end{aligned}$$

Therefore $xm \leqq \overline{m'x'}$ and similarly, $mw \leqq \overline{m'w'}$. Thus,

$$xw \leqq xm + mw \leqq \overline{x'm'} + \overline{m'w'} = \overline{x'w'} .$$

Apply the ptolemaic inequality for the euclidean metric

$$\overline{x'y'} \cdot \overline{z'w'} + \overline{x'z'} \cdot \overline{y'w'} \geqq \overline{x'w'} \cdot \overline{y'z'}$$

and we have

$$xy \cdot zw + xz \cdot yw \geqq \overline{x'w'} \cdot yz \geqq xw \cdot yz \ .$$

Case 2. m' is exterior to $S_{y'z'}$. We may assume that $\overline{x'w'} < xw$, for otherwise we could derive (8) just as we did above. Since x', y', z', and w' are not linear, at most one of the points x', w' lies on $L_{y'z'}$. We may suppose that x' does not lie on $L_{y'z'}$, for a reversing of the roles of x and w leaves (8) unchanged. Similarly, it may be assumed that y' is between m' and z'. Let K be the semicircle with center at y' and radius $\overline{x'y'}$ which passes through x' and whose base lies on $L_{y'z'}$. Further, let u' be the endpoint of K lying on the same side of y' as z', and v' the point where the ray from w' through y' meets K. Since the sum of $\angle x'y'u'$ and $\angle w'y'x'$ is greater than or equal to π, v' lies on, or in the interior of, $\angle x'y'u'$ and therefore belongs to K', the sub-arc of K whose endpoints are x' and u'. Now, every point t' on K' has the property that $\overline{t'z'} \leqq \overline{x'z'}$. Consider the continuous function $f(t') = \overline{t'w'}$, $t' \in K'$. $f(x') = \overline{x'w'} < xw$ and $f(v') = \overline{v'w'} = \overline{w'y'} + \overline{y'v'} = wy + yx \geqq xw$, so there exists a point t' on K' such that $\overline{t}'\overline{w}' = xw$. Applying the ptolemaic inequality to t', y', z', and w',

$$\overline{t'y'} \cdot \overline{z'w'} + \overline{t'z'} \cdot \overline{y'w'} > \overline{t'w'} \cdot \overline{y'z'} \ ,$$

which gives us

$$\begin{aligned} xy \cdot zw + xz \cdot yw &\geqq xy \cdot zw + \overline{t'z'} \cdot yw \\ &= \overline{t'y'} \cdot \overline{z'w'} + \overline{t'z'} \cdot \overline{y'w'} \geqq \overline{t'w'} \cdot \overline{y'z'} = xw \cdot yz \ . \end{aligned}$$

Remark. This proof applies to G-spaces. Unfortunately, even in G-spaces, it is not known whether (5) characterizes the ptolemaic inequality. It is interesting to note in this connection that Blumenthal has investigated a property he calls the *euclidean four-point property* which is merely our (5′) (or its equivalent (5) in complete convex metric spaces) with equality prevailing (see [1]).

Theorem 5 then provides an easy proof of

THEOREM 6. *Hyperbolic geometry is ptolemaic.*

Proof. If xy is a hyperbolic metric, we observe from the cosine inequality (see [4, p. 268]) that if x, y, and z are the vertices of a triangle and m is the midpoint of S_{yz}, with A one of the angles at m, then

$$(xy)^2 \geqq (xm)^2 + (my)^2 - 2(xm)(my) \cos A$$
$$(xz)^2 \geqq (xm)^2 + (mz)^2 + 2(xm)(mz) \cos A \ ,$$

and, since $my = mz = yz/2$, we have

$$(xy)^2 + (xz)^2 \geqq 2(xm)^2 + (yz)^2/2 ,$$

which is (5). By Theorem 5 the metric is ptolemaic.

Theorems 4 and 6 combine to give

COROLLARY 2. *A Hilbert geometry is locally ptolemaic if and only if it is hyperbolic.*

3. **Related inequalities.** In a G-space the ptolemaic inequality appears to be related to the "curvature" of the space. It can be easily verified that spherical geometry (a space having positive curvature) is *not* ptolemaic. This illustrates the theorem in the author's thesis that a Riemannian space is locally ptolemaic if and only if its curvature is nonpositive, which leads to the definition: A metric space has *nonpositive curvature* if and only if the ptolemaic inequality (8) holds locally. Other concepts of space curvature have been proposed. In [4, p. 237] Busemann defines nonpositive curvature as follows: If for each point there exists a neighborhood such that given any triple of points (x, y, z) in that neighborhood, with m_y the midpoint between x and y, and m_z the midpoint between x and z, the inequality

$$(9) \qquad 2m_y m_z \leqq yz$$

holds, the space is said to have nonpositive curvature. In his thesis the author proposes this definition (making use of inequality (5) above): If for each point there exists a neighborhood such that if (x, y, z) be any triple of points in that neighborhood with m the midpoint between y and z then

$$(10) \qquad (mx)^2 \leqq (xy)^2/2 + (xz)^2/2 - (yz)^2/4$$

holds, the space is said to have nonpositive curvature. F. P. Pederson [8] has investigated still a different concept of nonpositive curvature and relates it to (9).

Relatively little is known concerning the various implications which may exist among these concepts of curvature. The seemingly stronger (10) is shown to imply (9) in [4, pp. 268–269]. Our Theorem 5 shows that (10) implies (8) (locally). In Riemannian spaces they are each equivalent to nonpositive Riemannian curvature, but the situation is completely unsettled in Finsler geometry. In view of our Corollary 2 and the theorem of P. Kelly and E. Strauss [7] that a

Hilbert geometry has nonpositive curvature in the sense of (9) if and only if it is hyperbolic, we may conclude (trivially) that the conditions (8), (9), and (10) are equivalent in Hilbert geometries. It would be of interest to determine further implications—if such exist—among (8), (9), and (10) outside of Hilbert and Riemannian geometry.

REFERENCES

1. L. M. Blumenthal, *Remarks concerning the Euclidean four-point property*, Ergebnisse eines mathematischen Kolloquiums (Wien) **7** (1936), 7-10.
2. ———, *Remarks on a Weak Four-Point Property*, Revista de Ciencias, **45** (1943), 183-193.
3. ———, *Theory and Applications of Distance Geometry*, Oxford University Press, London 1953.
4. H. Busemann, *The Geometry of Geodesics*, Academic Press, New York, 1957.
5. H. Busemann, and P. J. Kelly, *Projective Geometry and Projective Metrics*, Academic Press, New York, 1953.
6. P. J. Kelly, and L. J. Paige, *Symmetric perpendicularity in Hilbert geometries*, Pacific J. Math. **2** (1952), 319-322.
7. P. J. Kelly, and E. Straus, *Curvature in Hilbert geometries*, Pacific. J. Math. **8** (1958), 119-125.
8. F. P. Pederson, *On spaces with negative curvature*, Mat. Tidsskrift B, (1952), 66-89.
9. I. J. Schoenberg, *A Remark on M. M. Day's Characterization of Inner-Product Spaces and a Conjecture of L. M. Blumenthal*, Proc. Amer. Math. Soc. **3** (1952), 961-964.

Received April 27, 1965. Supported by a National Science Foundation Grant while the author was a Research Participant at the University of Oklahoma, summer 1964.

THE UNIVERSITY OF OKLAHOMA