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ON OPERATORS WHOSE FREDHOLM SET IS THE COMPLEX PLANE

M. A. KAASHOEK AND D. C. LAY

Let T be a closed linear operator with domain and range in a complex Banach space X. The Fredholm set $\Phi(T)$ of T is the set of complex numbers λ such that $\lambda - T$ is a Fredholm operator. If the space X is of finite dimension then, obviously, the domain of T is closed and $\Phi(T)$ is the whole complex plane C. In this paper it is shown that the converse is also true. When T is defined on all of X this is a well-known result due to Gohberg and Kreĭn.

Examples of nontrivial closed operators with $\Phi(T) = C$ are the operators whose resolvent operator is compact. A characterization of the class of closed linear operators with a nonempty resolvent set and a Fredholm set equal to the complex plane will be given.

Throughout the present paper X and Y will denote complex Banach spaces. Let T be an arbitrary closed linear operator with domain $\mathscr{D}(T)$ in X and range $\mathscr{R}(T)$ in Y. The *nullity* $n(T)$ of T is the dimension of the null space $\mathscr{N}(T)$ of T. The *defect* $d(T)$ of T is the dimension of the quotient space $Y/\mathscr{R}(T)$. No distinction is made between infinite dimensions, so that $n(T)$ and $d(T)$ may be nonnegative integers or $+\infty$. We say that T is *Fredholm* if $n(T)$ and $d(T)$ are both finite. Note that $d(T) < \infty$ implies $\mathscr{R}(T)$ is closed (cf. [5], Lemma 332).

In 1957 Gohberg and Kreĭn [3] showed that if A is a bounded linear operator on X with $\Phi(A) = \boldsymbol{C}$, then the dimension of X (denoted by $\dim X$) is finite. The following theorem extends this result.

THEOREM 1. *Let T and S be bounded linear operators from X into Y. Suppose that S is a homeomorphism, and that $T + \lambda S$ is Fredholm for each $\lambda \in \boldsymbol{C}$. Then*

$$\dim X \leqq \dim Y < \infty .$$

Proof. Since S is a homeomorphism, $\mathscr{R}(S)$ is closed and $n(S) = 0$. By a well-known stability theorem (cf. [5], Theorem 1), this implies the existence of a positive constant ρ such that for $0 < |\mu| < \rho$

$$d(S) = d(S) - n(S) = d(S + \mu T) - n(S + \mu T) .$$

The right-hand side is finite because $S + \mu T$ is Fredholm for $\mu \neq 0$. Hence $d(S) < \infty$, and so S has a bounded left inverse, say R. Then $n(R) \leqq d(S) < \infty$ and $d(R) = 0$, so R is Fredholm. Define $A = RT$.

Then A is a bounded linear operator on X and

$$\lambda - A = \lambda RS - RT = R(\lambda S - T) .$$

For each complex value of λ, $\lambda - A$ is the product of two bounded Fredholm operators and hence is Fredholm. But $\Phi(A) = \boldsymbol{C}$ implies that $\dim X < \infty$ by the result of Gohberg and Kreĭn ([3], Theorem 3.2). Then $\dim Y = \dim X + d(S) < \infty$, concluding the proof.

COROLLARY. *Let T be a closed linear operator with domain $\mathscr{D}(T)$ and range in X. Then $\dim X < \infty$ if and only if $\mathscr{D}(T)$ is closed and $\Phi(T) = \boldsymbol{C}$.*

In [1] Caradus has proved that if T is a closed linear operator with domain and range in X such that $\dim X/\mathscr{D}(T) < \infty$, $\Phi(T) = \boldsymbol{C}$ and such that the resolvent set of T is neither empty nor the whole complex plane, then $\dim X < \infty$. The following lemma shows that Caradus' result is contained in the Corollary.

LEMMA. *Let T be a closed linear operator with domain in X and range in Y. Suppose there exists a closed subspace M of X such that $X = \mathscr{D}(T) \oplus M$. Then $\mathscr{D}(T)$ is closed.*

Proof. Let Y_1 be the Banach space $Y \times M$, with the norm

$$\| (y, m) \| = \| y \| + \| m \| .$$

Define the linear operator J from X into Y_1 by setting

$$J(x + m) = (Tx, m)$$

for each $x \in \mathscr{D}(T)$ and $m \in M$. It is easily verified that J is a well-defined closed linear operator. Since the domain of J is the Banach space X, the closed graph theorem implies that J is bounded. Hence

$$(\| Tx \| + \| m \|) \leqq \| J \| \cdot \| x + m \|$$

for each $x \in \mathscr{D}(T)$ and $m \in M$. In particular,

$$\| Tx \| \leqq \| J \| \cdot \| x \|$$

for each $x \in \mathscr{D}(T)$. Thus T is both closed and bounded, implying that $\mathscr{D}(T)$ is closed.

We have learned recently that similar statements for the range of a closed linear operator are proved by S. Goldberg in [4]. That this can be done follows easily from the observation that the range of a closed linear operator is always the domain of some other closed linear operator, and conversely (cf. [6], Chapter IV).

The Corollary states that the closed linear operators T with closed domain and $\Phi(T) = \boldsymbol{C}$ are trivial. Examples of nontrivial closed operators whose Fredholm set is the complex plane are the operators with compact resolvent (cf. [7], § 2). The following theorem shows that each closed operator T with a nonempty resolvent set $\rho(T)$ and with $\Phi(T) = \boldsymbol{C}$ is characterized by the fact that for each $\mu \in \rho(T)$ the resolvent $(\mu - T)^{-1}$ is a Riesz operator. For the definition of Riesz operators and one of their characterizations we refer to Dieudonné ([2], XI. 4, problem 5).

THEOREM 2. *Let T be a closed linear operator with domain and range in X. If $\Phi(T) = \boldsymbol{C}$, then $(\mu - T)^{-1}$ is a Riesz operator for all $\mu \in \rho(T)$. Conversely, if $(\mu - T)^{-1}$ is a Riesz operator for some $\mu \in \rho(T)$, then $\Phi(T) = \boldsymbol{C}$.*

Proof. We may assume that $\dim X = \infty$ and that $\rho(T)$ is not empty. Take μ in $\rho(T)$ and let $A = (\mu - T)^{-1}$. Then for $\lambda \neq \mu$,

$$(\lambda - T)(\mu - T)^{-1} = (\mu - \lambda)(\zeta - A) ,$$

where $\zeta = (\mu - \lambda)^{-1}$. This implies that $\Phi(T) = \boldsymbol{C}$ if and only if $\Phi(A) = \boldsymbol{C}\backslash\{0\}$. Hence it is enough to show that A is a Riesz operator if and only if $\Phi(A) = \boldsymbol{C}\backslash\{0\}$. In order to do this, let $\mathscr{K}$ be the ideal of all compact linear operators in the Banach algebra $\mathscr{L}(X)$ of all bounded linear operators on X, and let π denote the canonical homomorphism from $\mathscr{L}(X)$ onto the quotient algebra $\mathscr{L}(X)/\mathscr{K}$. Then it follows from Atkinson's characterization of the class of all Fredholm operators in $\mathscr{L}(X)$ that $\zeta - A$ is Fredholm if and only if $\zeta - \pi(A)$ has an inverse in $\mathscr{L}(X)/\mathscr{K}$. So $\Phi(A) = \boldsymbol{C}\backslash\{0\}$ if and only if the spectrum of $\pi(A)$ in $\mathscr{L}(X)/\mathscr{K}$ is $\{0\}$, i.e., the spectral radius $r(\pi(A))$ of $\pi(A)$ is zero. But

$$\begin{aligned} r(\pi(A)) &= \lim_{n\to\infty} \| [\pi(A)]^n \|^{1/n} \\ &= \lim_{n\to\infty} \| \pi(A^n) \|^{1/n} = \lim_{n\to\infty} [d(A^n, \mathscr{K})]^{1/n} , \end{aligned}$$

where $d(A^n, \mathscr{K})$ is the infimum of $\| A^n - K \|$ for $K \in \mathscr{K}$. Thus $\Phi(A) = \boldsymbol{C}\backslash\{0\}$ if and only if

$$\lim_{n\to\infty} [d(A^n, \mathscr{K})]^{1/n} = 0 ,$$

which is equivalent to the statement that A is a Riesz operator (cf. [2], XI. 4, problem 5).

When T is a self-adjoint closed linear operator in a Hilbert space Theorem 2 can be strengthened. This is because $(\mu - T)^{-1}$ is normal for $\mu \in \rho(T)$, and a normal operator is Riesz if and only if it is compact.

Hence, in this special case, $\Phi(T) = \boldsymbol{C}$ if and only if $(\mu - T)^{-1}$ is compact for each μ in $\rho(T)$.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES