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ON COMPACT UNITHETIC SEMIGROUPS

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A topological semigroup is a Hausdorff space S together with a continuous, associative multiplication. If each element of S has unique roots in S of each positive integral order, then S is said to be uniquely divisible. The closure of the set of positive rational powers of an element x in a compact uniquely divisible semigroup S is a commutative clan (compact connected semigroup with identity) called the unithetic semigroup generated by x.

The purpose of this paper is to discuss the structure of compact unithetic semigroups. It is established that if the cartesian product of two semigroups is unithetic, then both factors are unithetic, and at least one factor is a group.

A partial converse is presented. If S is a compact first countable unithetic semigroup, and G is a finite dimensional compact unithetlc group, then $G \times S$ is a unithetic semigroup. These results are used to give the precise of a unithetic semigroup with zero whose maximal group containing the identity is finite dimensional. A complete converse to the first result is not known. In particular, the question as to whether one or both of the conditions that S be first countable and G be finite dimensional can be omitted is open.

Throughout this paper R denotes the set of all positive rational numbers, and N denotes the set of all positive integers.

A semigroup S is said to be *uniquely divisible* if each element of S has a unique root of each positive integral order. If S is uniquely divisible, $x \in S$, and $n \in N$, then $x^{1/n}$ denotes the unique nth. root of x in S. If $r \in R$, $r = m/n$, $m, n \in N$, define $x^r = (x^{1/n})^m$. It is not difficult to show that x^r is unique and independent of the choice of m and n. Define $[x] = \{x^r : r \in R\}^*$ (closure in S). If $S = [y]$ for some $y \in S$, then S is said to be *unithetic* and y a *unithetic generator* of S.

EXAMPLE. Let $I = [0, 1]$ be the unit interval under usual real multiplication. Then I is unithetic and is generated by any $x \in I$ such that $0 < x < 1$. A semigroup which is iseomorphic (topologically isomorphic) to I is called a *U-semigroup*.

Note that although I is unithetic, $I \times I$ is not unithetic. However, $I \times I$ is uniquely divisible. Indeed, the cartesian product of two uniquely divisible semigroups is uniquely divisible. One might ask "Under what conditions is the cartesian product of two unithetic semigroups unithetic?" This question is partially answered in §4.

The study of unithetic semigroups is partially motivated by the fact that each clan determines an irreducible uniquely divisible clan. This is established in the following theorem sequence:

LEMMA 1.1 *Let A be a commutative divisible clan. For each $n \in N$, let $S_n = A$. For $m > n$ in N, define $f_{mn} : S_m \to S_n$ by $f_{mn}(z) = z^{\phi(m,n)}$, where $\phi(m, n) = m!/n!$. Let $S = \underleftarrow{\lim} (S_n, f_{mn}, N)$. Then S is a commutative uniquely divisible clan. Moreover, if A is uniquely divisible, then S is iseomorphic to A.*

The proof of Lemma 1.1 is straight forward and will not be presented here. Although it is stated for clans, it holds for compact semigroups.

The definition of an *irreducible clan* (semigroup) is given in [5]. Note that any irreducible clan is commutative [6].

LEMMA 1.2. *An irreducible clan A is divisible.*

Proof. Let $n \in N$. Define $g_n : A \to A$ by $g^n(z) = z^n$, $z \in A$. Then, since A is a commutative semigroup, g_n is a continuous homomorphism. Thus $g_n(A)$ is a subclan of A which contains the identity of A and meets the kernel (minimal ideal) of A. Since A is irreducible, $g_n(A) = A$. Thus A is divisible.

LEMMA 1.3. *Let S be a clan and T an irreducible subclan (which contains the identity and meets the kernel of S). If S is iseomorphic to T, then $T = S$.*

Proof. Let $g : S \to T$ be an iseomorphism. Suppose $T \neq S$. Then, since g is one-to-one, $g(T)$ is a proper subset of T. But $g(T)$ is a subclan of T. This contradicts the fact that T is irreducible. Hence $T = S$.

THEOREM 1.4. *Each clan S determines an irreducible uniquely divisible clan S_0.*

Proof. The clan S contains an irreducible subclan A which is commutative [6]. By Lemma 1.2, A is divisible. Let $A_n = A$ for each $n \in N$ and apply Lemma 1.1 to obtain $S_0 = \underleftarrow{\lim} A_n$. Then S_0 is a commutative uniquely divisible clan.

Now S_0 contains an irreducible subclan T (which contains the identity and meets the kernel af S_0) [6]. Let π_n, $n \in N$, denote the projection of S_0 onto A_n. If $\pi_n(T) \neq A_n$ for some $n \in N$, then $\pi_n(T)$ is a proper subclan of A_n which contains the identity of

A_n and meets the kernel of A_n. This contradicts the fact that A_n is irreducible. Hence $\pi_n(T) = A_n$ for each $n \in N$. Now T is iseomorphic to $\varprojlim \pi_n(T) = \varprojlim A_n = S_0$. Thus, S_0 and T are iseomorphic. By Lemma 1.3, $S_0 = T$, and hence S_0 is irreducible.

COROLLARY 1.5. *Let S be a clan containing exactly two idempotents. Then S determines a compact unithetic semigroup S_0.*

2. Unithetic groups.

THEOREM 2.1. *Let G be a compact uniquely divisible abelian group. These are equivalent:*
(i) *G is monothetic;*
(ii) *G is unithetic;*
(iii) *G is separable.*

Proof. Since G is divisible and compact, G is connected [3, p. 385].
(i) *implies* (ii). Suppose G is monothetic. Then there exists $g \in G$ such that $G = \{g, g^2, \cdots\}^*$. Thus $G = \{g, g^2, \cdots\}^* \subset [g] \subset G$, and hence $G = [g]$.
(ii) *implies* (iii). Suppose G is unithetic. Then $G = [g]$ for some $g \in G$. Hence $\{g^r : r \in R\}$ is dense in G, and thus G is separable.
(iii) *implies* (i). Suppose G is separable. Then, since G is connected, G is monothetic [2].

Notation. Let Σ denote the a-adic solenoid with $a = (2, 3, 4, \cdots)$ [3, p. 114].

Let $\Sigma^{\wedge} = \prod_{\alpha \varepsilon \wedge} \Sigma_\alpha$, where $\Sigma_\alpha = \Sigma$ for each $\alpha \varepsilon \wedge$.

THEOREM 2.2 *Let G be a nondegenerate compact uniquely divisible abelian group. Then $G = \Sigma^{\wedge}$ for some $\wedge$.*

Proof. Since G is uniquely divisible, it is both divisible and torsion-free. The result now follows from [3, p. 406].

THEOREM 2.3. *Let G be a compact group. Then G is unithetic if and only if $G = \Sigma^{\wedge}$ and* card $\wedge \leqq c$.

Proof. Suppose G is unithetic. Then G is a uniquely divisible abelian group. Hence, by Theorem 2.2, $G = \Sigma^{\wedge}$ for some $\wedge$. By Theorem 2.1, G is separable. Thus card $\wedge \leqq c$ [10].

Suppose $G = \Sigma^{\wedge}$ and card $\wedge \leqq c$. Then G is separable [10]. Hence, by Theorem 2.1, G is unithetic.

COROLLARY 2.4. *Any element of Σ which is not the identity is a unithetic generator of Σ.*

LEMMA 2.5. *Let $\{r_n\}$ be a sequence in R which converges to 0. Then there exist $g, g_0 \in \Sigma$, $g_0 \neq 1$, such that $\{g^{r_n}\}$ clusters to g_0.*

Proof. Let N denote the set of all positive integers and C the circle group (the boundary of the complex unit disk under multiplication). For each $n \in N$, let $C_n = C$. Define $f_{mn}: C_m \to C_n$ for $m > n$, by $f_{mn}(z) = z^{\phi(m,n)}$, where $\phi(m, n) = m!/n!$. Then $\Sigma = \varprojlim (C_n, f_{mn}, N)$.

Let L denote the closed left half of C, i.e., $L = \{e^{i\theta} : \cos\theta \leqq 0\}$. We will construct, by induction, an element $g = (g_1, g_2, \cdots)$ in Σ such that $\{r_n\}$ has a subsequence $\{r_{n_j}\}$ such that each first coordinate of $g^{r_{n_j}}$ lies in L. This will be sufficient to insure that $\{g^{r_{n_j}}\}$ does not converge to the identity $(1, 1, 1, \cdots)$ of Σ. Thus $\{g^{r_{n_j}}\}$ (hence $\{g^{r_n}\}$) will have a subsequence converging to some $g_0 \neq 1$ in Σ.

Note that if $r = a/b$, where a and b are relatively prime positive integers, and $(x_1, x_2, \cdots)$ is an element of Σ, then $(x_1, x_2, \cdots)^r = (x_b^{rb!}, \cdots)$.

For each r_n in $\{r_n\}$ let m_n be the least positive integer such that $M_n = m_n!\, r_n$ is an integer. We may assume that $\{r_n\}$ is such that $m_1 > 2$ and that $m_{n-1} < m_n$ for each n, since $\{r_n\}$ will have a subsequence satisfying these conditions.

Let $r_n = a_n/b_n$, where a_n and b_n are relatively prime positive integers for each n.

Let $g_1 = g_2 = \cdots = g_{m_1} = 1$. We want to define $g = (g_1, g_2, \cdots)$ so that:

(i) $g_n^n = g_{n-1}$ for each n, i.e., $g \in \Sigma$, and

(ii) $g_{m_n}^{M_n} \in L$ for each $n \geqq 2$.

Note that having defined g_{n-1} we can always find $z \in C$ such that $z^n = g_{n-1}$ by selecting z to be one of the nth. roots of g_{n-1}. However, the manner in which z is chosen when $n = m_n - 1$ will be more specific, so that (ii) can be satisfied.

Having defined g_{m_n}, define $g_{m_n+1}, \cdots, g_{m_{n+1}-1}$ just to satisfy condition (i).

We now define g_{m_n} by induction. Suppose that $g_{m_n-1} = e^{i\theta}$, where $0 \leqq \theta < 2\pi$. Let $\beta = M_n/m_n$. Then $\beta \in R\backslash N$, by the way in which m_n is defined.

Suppose $2^p\beta \in N$ for some $p \in N$. Let $p_0 = \min. \{p \in N : 2^p\beta \in N\}$. Then $2^{p_0}\beta$ is odd and hence $\cos(2^{p_0}\beta\pi) + \sin(2^{p_0}\beta\pi) = -1 \leqq 0$.

Suppose $2^p\beta \notin N$ for all $p \in N$. Then, using the dyadic expansion of β, one can show that there exists $p_0 \in N$ such that

$$\cos(2^{p_0}\beta\pi) + \sin(2^{p_0}\beta\pi) \leqq 0 .$$

Thus, in either case, we have

$$\cos(\theta\beta + 2^{p_0}\beta\pi) = \cos(\theta\beta)\cos(2^{p_0}\beta\pi) - \sin(\theta\beta)\sin(2^{p_0}\beta\pi)$$
$$\leqq \cos(2^{p_0}\beta\pi) + \sin(2^{p_0}\beta\pi) \leqq 0 .$$

Let $g_{m_n} = e^{i[(\theta/m_n)+(2^{p_0}\pi)/m_n]}$. Then

$$g_{m_n}^{m_n} = e^{i(\theta+2^{p_0}\pi)} = e^{i\theta} = g_{m_{n-1}} .$$

Moreover, $g_{m_n}^{M_n} = e^{i(\theta\beta+2^{p_0}\beta\pi)}$, which is in L. Thus the required conditions are satisfied.

Let $g = (g_1, g_2, \cdots)$. Then $g^{r_n} = (g_{b_n}^{r_n b_n'}, \cdots) = (g_{m_n}^{M_n}, \cdots)$ and $g_{m_n}^{M_n} \in L$ for each n. This completes the proof of the lemma.

3. **Preliminary results.** Throughout this section $S = [x]$ denotes a compact unithetic semigroup which is not a group. The results in Theorem 3.1 can be obtained as a consequence of [4, p. 275] or by using the techniques and results in [7], [8], and [9].

For a net $\{x_\alpha\}$ in S, we use $x_\alpha \xrightarrow{e} x$ to denote the fact that $\{x_\alpha\}$ converges to x, and $x_\alpha \xrightarrow{f} x$ to denote the fact that $\{x_\alpha\}$ clusters to x.

THEOREM 3.1. *The semigroup S is a commutative clan containing exactly two idempotents such that:*

(i) *The kernel $K(x)$ of S is a unithetic group generated by ex, where e is the identity of $K(x)$.*

(ii) *The maximal subgroup $H(x)$ containing the identity 1 of S is a unithetic group.*

(iii) *There is a continuous one-to-one homomorphism σ from the additive nonnegative real numbers $\bar{R}$ into S such that $S = H(x)(\sigma(\bar{R})^*)$. The kernel of $\sigma(\bar{R})^*$ is $\sigma(\bar{R})^*\backslash\sigma(\bar{R})$ and is contained in $K(x)$. If $\sigma(\alpha) = \sigma(\beta)g$, for $\alpha, \beta \in \bar{R}$ and $g \in H(x)$, then $\alpha = \beta$ and $g = 1$. Moreover, if $\alpha \in \bar{R}\backslash\{0\}$, then $\sigma(\bar{R})^* = [\sigma(\alpha)]$.*

LEMMA 3.2. *Let $\{r_\alpha\}$ be a net in R.*

(i) *If $r_\alpha \xrightarrow{e} 0$ and $x^{r_\alpha} \xrightarrow{f} z$, then $z \in H(x)$.*

(ii) *If $x^{r_\alpha} \xrightarrow{e} z$ and $z \in H(x)$, then $r_\alpha \xrightarrow{f} 0$.*

Proof. For $s \in R$, let $H(x, s) = \{x^r : r \in R, r < s\}^*$. Then $H(x) = \bigcap \{H(x, s) : s \in R\}$ (See [8]).

(i) Let $s \in R$. Then there exists α_0 such that $r_\alpha < s$ for all $\alpha > \alpha_0$. Thus $x^{r_\alpha} \in H(x, s)$ for all $\alpha > \alpha_0$. Since $H(x, s)$ is closed and $x^{r_\alpha} \xrightarrow{f} z$, $z \in H(x, s)$. Hence $z \in H(x)$.

(ii) Suppose there exists $r \in R$ such that $r_\alpha > r$ eventually. Then $\{x^{r_\alpha - r}\}$ clusters to some $w \in S$. Hence $x^{r_\alpha} \xrightarrow{f} wx^r$. Thus $wx^r = z$. Therefore, $w^{1/r}x = z^{1/r}$, and $z^{1/r} \in H(x)$. Since $S\backslash H(x)$ is an ideal, $x \in H(x)$.

This implies that S is a group, which contradicts the assumption that S is not a group. Hence $r_\alpha \xrightarrow{f} 0$.

THEOREM 3.3. *The function* $\Psi : R \rightarrow S$ *defined by* $\Psi(r) = x^r$, $r \in R$, *is continuous if and only if* $H(x) = \{1\}$.

Proof. If $H(x) = \{1\}$, then the fact that Ψ is continuous follows from Lemma 3.2 (i).

Suppose $H(x) \neq \{1\}$. Let $p \in H(x)$, $p \neq 1$. Let $\{s_\alpha\}$ be a net in R such that $x^{s_\alpha} \xrightarrow{e} p$. Then, by Lemma 3.2 (ii), $s_\alpha \xrightarrow{f} 0$. Let $s \in R$. Then there exists a subnet $\{s_{\alpha_\beta}\}$ of $\{s_\alpha\}$ such that $s_{\alpha_\beta} \xrightarrow{e} 0$ and $s_{\alpha_\beta} < s$ for all β. Thus $s - s_{\alpha_\beta} \xrightarrow{e} s$. If Ψ were continuous, we would have $x^{s - s_{\alpha_\beta}} \xrightarrow{e} x^s$, and thus $x^s = px^s$ or $x = p^{1/s}x$. This would imply that $p = 1$. Since $p \neq 1$, Ψ is not continuous.

THEOREM 3.4. *The quotient semigroup* $S/K(x)$ *is a compact unithetic semigroup which is iseomorphic to* $(H(x) \times I)/(H(x) \times \{0\})$, *where* $I = [0, 1]$ *is a U-semigroup. A generator of* $S/K(x)$ *is* $\phi(x)$, *where* $\phi : S \rightarrow S/K(x)$ *is the natural map.*

Proof. It is not difficult to show that $S/K(x)$ is a uniquely divisible commutative clan with zero $z = \phi(K(x))$.

Let $\phi(y) \in S/K(x)$, $y \in S$. Then there exists a net $\{r_\alpha\}$ in R such that $x^{r_\alpha} \xrightarrow{e} y$. Since ϕ is continuous, $\phi(x^{r_\alpha}) \xrightarrow{e} \phi(y)$. Since ϕ is a homomorphism $\phi(x^{r_\alpha}) = \phi(x)^{r_\alpha}$ for each α. Thus $\phi(y) \in [\phi(x)]$, and hence $S/K(x) = [\phi(x)]$.

Let σ be the map of Theorem 3.1. Then $S = H(x)(\sigma(\bar{R})^*)$, and hence $S/K(x) = \phi(H(x))\phi(\sigma(\bar{R})^*)$. Since the kernel of $\sigma(\bar{R})^*$ is contained in $K(x)$, z is a zero for $\phi(\sigma(\bar{R})^*)$.

Define $f : \phi(H(x)) \times \phi(\sigma(\bar{R})^*) \rightarrow S/K(x)$ by $f((a, b)) = ab$. Then f is a continuous homomorphism onto $S/K(x)$. Define a relation Q on $\phi(H(x)) \times \phi(\sigma(\bar{R})^*$ by $Q = \{(a, b), (c, d) : f((a, b)) = f((c, d))\}$. Then $(\phi(H(x)) \times \phi(\sigma(\bar{R})))/Q$ is iseomorphic to $S/K(x)$.

It will be established that

$$(\phi(H(x)) \times \phi(\sigma(\bar{R})^*)/Q = (\phi(H(x)) \times \phi(\sigma(\bar{R})^*))/(\phi(H(x)) \times \{z\}) .$$

This is done by showing that f is one-to-one on $\phi(H(x)) \times (\phi(\sigma(\bar{R})^*)\backslash\{z\})$ ($\backslash$ denotes complement).

Suppose $f((a', b')) = f((c', d'))$, where $a', c' \in \phi(H(x))$ and

$$b', d' \in \phi(\sigma(\bar{R})^*)\backslash\{z\} .$$

Then $a'b' = c'd'$. There exist $a, c \in H(x)$ and $b, d \in \sigma(\bar{R})^*$ such that $a' =$

$\phi(a)$, $b' = \phi(b)$, $c' = \phi(c)$, and $d' = \phi(d)$. Since $b' \neq z \neq d'$, b' and d are not in the kernel of $\sigma(\bar{R})^*$. Thus, by Theorem 3.1, $b, d \in \sigma(\bar{R})$. Now $\phi(ab) = \phi(a)\phi(b) = a'b' = c'd' = \phi(c)\phi(d) = \phi(cd)$.

Suppose $ab \in K(x)$. Let a^{-1} denote the inverse of a in $H(x)$. Then $b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab)$ is in $K(x)$. Thus $\phi(b) = z$. This contradicts $b' \neq z$. Hence $ab \in S \backslash K(x)$. Similarly, $cd \in S \backslash K(x)$.

Since ϕ is one-to-one on $S \backslash K(x)$ and $\phi(ab) = \phi(cd)$, $ab = cd$. Let $\alpha \in \bar{R}$ and $\beta \in \bar{R}$ such that $b = \sigma(\alpha)$ and $d = \sigma(\beta)$. Then $a\sigma(\alpha) = c\sigma(\beta)$. Thus $\sigma(\alpha) = (a^{-1}c)\sigma(\beta)$ and $a^{-1}c \in H(x)$. Hence, by Theorem 3.1, $\alpha = \beta$ and $a^{-1}c = 1$. Therefore, $a = c$ and $b = d$, and hence f is one-to-one $\phi(H(x)) \times (\phi(\sigma(R)^*) \backslash \{z\})$. This establishes the fact $S/K(x)$ is iseomorphic to $(\phi(H(x)) \times \phi(\sigma(\bar{R})^*)/(\phi(H(x)) \times \{z\})$.

Since ϕ is one-to-one on $S \backslash K(x)$, $\phi(H(x))$ is iseomorphic to $H(x)$.

Let $I = [0, 1]$ under usual real multiplication. Then I is a compact unithetic semigroup. Let σ_0 be the map of Theorem 3.1 corresponding to I. Since the maximal group of I containing 1 is trivial, $I = \sigma_0(\bar{R})^*$. Moreover, since the kernel of I is $\{0\}$, $(0, 1] = \sigma_0(\bar{R})$. Define $u : \phi(\sigma(\bar{R})^*) \rightarrow I$ by $u(z) = 0$ and $u(b) = \sigma_0(\alpha)$, where $b = \phi(\sigma(\alpha))$ for $\alpha \in \bar{R}$ if $b \neq z$. Then u is an iseomorphism of $\phi(\sigma(\bar{R})^*)$ onto I. This completes the proof of the theorem.

4. Structure theorems.

THEOREM 4.1. *Let each of S_1 and S_2 be compact semigroups such that $S_1 \times S_2$ is unithetic. Then each of S_1 and S_1 is unithetic. Moreover, either S_1 or S_2 is a group.*

Proof. Let $S_1 \times S_2 = [(x_1, x_2)]$. Then $S_1 = [x_1]$ and $S_2 = [x_2]$.

Suppose that neither S_1 nor S_2 is a group. Now $(1, x_2) \in S_1 \times S_2$. Hence there exists a net $\{r_\alpha\}$ in R such that $x_1^{r_\alpha} \xrightarrow{e} 1$ and $x_2^{r_\alpha} \xrightarrow{e} x_2$. Since $x_1^{r_\alpha} \xrightarrow{e} 1$, $r_\alpha \xrightarrow{f} 0$. Let $\{r_{\alpha_\beta}\}$ be a subnet such that $r_{\alpha_\beta} \xrightarrow{e} 0$. Then $x_2^{r_{\alpha_\beta}} \xrightarrow{e} x_2$. But this implies that $r_{\alpha_\beta} \xrightarrow{f} 1$. This contradiction implies that either S_1 or S_2 is a group.

THEOREM 4.2. *Let $S = [x]$ be a compact first countable unithetic semigroup, and G a compact finite dimensional unithetic group. Then $G \times S$ is a unithetic semigroup.*

Proof. If S is a group, then the theorem follows from Theorem 2.3, since $G = \Sigma^\wedge$ for some finite $^\wedge$.

Suppose that S is not a group. We prove that $\Sigma \times S$ is unithetic. The conclusion follows by induction.

Now $\Sigma \times S$ is a compact uniquely divisible semigroup with identity

$(1, 1)$ and maximal group $\Sigma \times H(x)$.

Let $\{r_n\}$ be a sequence in R such that $x^{r_n} \xrightarrow{e} 1$ and $r_n \xrightarrow{e} 0$. Then, by Lemma 2.5, there exist $g \in \Sigma$ and $g_0 \in \Sigma$, $g_0 \neq 1$, such that $g^{r_n} \xrightarrow{f} g_0$. Thus $(g, x)^{r_n} \xrightarrow{f} (g_0, 1)$. Hence $(g_0, 1) \in H((g, x))$. Therefore, $H((g, x)) \cap [\Sigma \times \{1\}]$ is a nondegenerate compact uniquely divisible subgroup of $\Sigma \times \{1\}$. Thus, by Corollary 2.4, $H((g, x)) \cap (\Sigma \times \{1\}) = \Sigma \times \{1\}$, and hence $\Sigma \times \{1\} \subset [(g, x)]$.

Let $(w, y) \in \Sigma \times S$, and $x^{s_j} \xrightarrow{e} y$, $s_j \in R$. Then $g^{s_j} \xrightarrow{f} g_0 \in \Sigma$. Hence $(g_0, y) \in [(g, x)]$. Since $\Sigma \times \{1\} \subset [(g, x)]$, $(wg_0^{-1}) \in [(g, x)]$. Hence $(w, y) = (wg_0^{-1}, 1)(g_0, y)$ is in $[(g, x)]$. Thus $\Sigma \times S \subset [(g, x)]$, and hence $\Sigma \times S = [(g, x)]$. This completes the proof of the theorem.

The following theorem is a consequence of Theorem 3.4.

THEOREM 4.3. *Let $S_1 = [x_1]$ and $S_2 = [x_2]$ be compact unithetic semigroups such that $H(x_1)$ is iseomorphic to $H(x_2)$. Then $S_1/K(x_1)$ is iseomorphic to $S_2/K(x_2)$.*

EXAMPLE. Let $I = [0, 1]$ be a U-semigroup and $\hat{}$ a set of n elements. Then, by Theorem 4.2, $\Sigma^{\hat{}} \times I$ is unithetic. Moreover, $D_n = (\Sigma^{\hat{}} \times I)/(\Sigma^{\hat{}} \times \{0\})$ is a unithetic semigroup with zero whose maximal group containing the identity has dimension n.

COROLLARY 4.4. *Let $S = [x]$ be a compact unithetic semigroup with zero such that $H(x)$ has dimension n. Then S is iseomorphic to D_n.*

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