PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 2, 1967

AN ELEMENTARY PROOF THAT HAAR MEASURABLE ALMOST PERIODIC FUNCTIONS ARE CONTINUOUS

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It is known that a Haar measurable complex-valued (von Neumann) almost periodic function on a locally compact T_0-topological group is continuous. For by applying the Bohr-von Neumann approximation theorem for almost periodic functions and the fact that a Haar measurable representation into the general linear group is necessarily continuous one may deduce that such a function is the uniform limit of a sequence of continuous functions. This approach, while straightforward, has the disadvantage of depending on the very deep Bohr-von Neumann approximation theorem. The latter result is commonly proven through considerable usage of representation theory. This paper presents an alternative proof that Haar measurability plus almost periodicity imply continuity. The proof is elementary in the sense that it uses only the basic definitions of almost periodic function theory and topology. It does, however, depend on the standard tools of measure theory.

Suppose G is a locally compact T_0-topological group ($=LC$ group). Let Γ denote the set of Borel subsets of G, that is, the σ-algebra generated by the closed subsets of G. Let μ be a left Haar measure defined on Γ (cf. [2], pp. 184–215) and let $\bar{\Gamma}$ be the completion of Γ, that is, $\bar{\Gamma} = \{B \cup N: B \in \Gamma, N \subset N', \text{ where } N' \in \Gamma \text{ and } \mu N' = 0\}$. We extend μ to the σ-algebra $\bar{\Gamma}$ by defining $\mu(B \cup N) = \mu(B)$ for all $B \cup N \in \bar{\Gamma}$. μ, so extended, is left-invariant and regular on $\bar{\Gamma}$. By a $\bar{\Gamma}$-measurable function on G we mean a function f from G to the complex plane C such that $f^{-1}(A) \in \bar{\Gamma}$ for all Borel sets $A \subset C$. We are concerned with $\bar{\Gamma}$-measurable, rather Γ-measurable, functions so that in the real case, for example, we can deduce that Lebesgue measurable, as well as Borel measurable, almost periodic functions are continuous.

A set $A \subset G$ is called *bounded* if $\bar{A}$ is compact. We shall let e denote the identity of G and Ω the set of all bounded open neighborhoods of e in G. It is convenient to use the following "density theorem" whose proof is an exercise in Halmos' *Measure Theory* ([1], 61.5; Halmos' "Borel" sets are different from ours but his suggested proof works equally well in our setting.).

THEOREM. *Let G be an LC group. For any $U \in \Omega$, $x \in G$ and* *any bounded* $E \in \bar{\Gamma}$ *define*

$$d(U, E, x) = \frac{\mu\{E \cap (Ux)\}}{\mu(Ux)} .$$

then $d(U, E, \ )$ *converges in mean to the characteristic function* χ_E *as* $U \to e$; *that is, for any* $\varepsilon > 0$ *there is some* $V \in \Omega$ *such that for all* $U \in \Omega$, $U \subset V$, $\int | d(U, E, x) - \chi_E(x) | \, d\mu(x) < \varepsilon$. *In particular,* $d(U, E, \ )$ *converges in measure to* χ_E *as* $U \to e$, $U \in \Omega$.

LEMMA. *Let* G *be an LC group and* f *a* $\bar{\Gamma}$*-measurable almost periodic function on* G. *For any* $x_0 \in G$ *and any* $\delta > 0$ *define*

$$T(f, \delta, x_0) = \{x \in G \colon | f(x) - f(x_0) | \geqq \delta\} .$$

Then

$$\lim_{\substack{U \to e \\ U \in \Omega}} \frac{\mu\{T(f, \delta, x_0) \cap Ux_0\}}{\mu(Ux_0)} = 0 ;$$

that is, for any $\varepsilon > 0$ *there is a* $V \in \Omega$ *such that for all* $U \in \Omega$, $U \subset V$,

$$\frac{\mu\{T(f, \delta, x_0) \cap Ux_0\}}{\mu(Ux_0)} < \varepsilon .$$

If, for example, G is the additive group of real numbers, then the lemma states that a Lebesgue measurable almost periodic function is approximately continuous.

Proof of the lemma. We first show that it suffices to prove the lemma for the case $x_0 = e$. If $x_0 \in G$, define $f_{x_0}(x) = f(xx_0)$. Then for arbitrary $x_0 \in G$ f_{x_0} satisfies the same hypotheses as f (it is $\bar{\Gamma}$-measurable because right translation of the power set of G by x_0^{-1} preserves Γ and also preserves the property of being μ-null in Γ). Also

$$T(f_{x_0}, \delta, e) = T(f, \delta, x_0)x_0^{-1}$$

so

$$\frac{\mu\{T(f_{x_0}, \delta, e) \cap U\}}{\mu(U)} = \frac{\mu\{T(f, \delta, x_0)x_0^{-1} \cap U\}}{\mu(U)} = \frac{\mu\{T(f, \delta, x_0) \cap Ux_0\}}{\mu(Ux_0)}$$

Thus if the lemma is true for any $\bar{\Gamma}$-measurable almost periodic function when $x_0 = e$, then it is also true for arbitrary $x_0 \in G$.

We now suppose $x_0 = e$. Take $\delta > 0$. If the statement of the lemma is false, then there is a real number ε, $0 < \varepsilon < 1$, such that for every $V \in \Omega$ there exists $U \subset V$, $U \in \Omega$, satisfying

$$\frac{\mu\{T(f, \delta, e) \cap U\}}{\mu(U)} > \varepsilon .$$

Take $V^* \in \Omega$ and define $T = T(f, \delta, e) \cap V^*$, so that T is a bounded member of $\bar{\Gamma}$. Using the notation of the density theorem, it is the case that for every $V \in \Omega$ there is a $U \in \Omega$, $U \subset V$, such that $d(U, T, e) > \varepsilon$ (Take $U \subset V \cap V^*$). In what follows we make frequent use of the last statement in the density theorem.

Now $\mu(T) = t > 0$ and the family $\Phi = \{V \in \Omega: d(V, T, e) > \varepsilon\}$ is a base at e. By the density theorem there exists $V_0 \in \Phi$ such that

$$\mu(\{x \in G: |\, d(V_0, T, x) - \chi_T(x)\,| \geqq \varepsilon/2\}) < t/2 .$$

Thus

$$T \not\subset \{x \in G: |\, d(V_0, T, x) - \chi_T(x)\,| \geqq \varepsilon/2\} .$$

Take

$$a_1 \in T - \{x \in G: |\, d(V_0, T, x) - \chi_T(x)\,| \geqq \varepsilon/2\} .$$

Then

$$1 \geqq \frac{\mu(T \cap V_0 a_1)}{\mu(V_0 a_1)} = \frac{\mu(Ta_1^{-1} \cap V_0)}{\mu(V_0)} > 1 - \varepsilon/2 .$$

Since $V_0 \in \Phi$ we have

$$\mu(V_0) \geqq \mu(T \cap V_0) > \mu(V_0)\varepsilon$$

and

$$\mu(V_0) \geqq \mu(Ta_1^{-1} \cap V_0) > \mu(V_0)(1 - \varepsilon/2) .$$

Consequently $\mu(T \cap Ta_1^{-1}) > 0$ so ([1], 60.5) $\mu(T \cap Ta_1) = t_1 > 0$. As Φ is a base at e, there exists $V_1 \in \Phi$ such that

$$\mu(\{x \in G: |\, d(V_1, T \cap Ta_1, x) - \chi_{T \cap Ta_1}(x)\,| \geqq \varepsilon/2\}) < t_1/2 .$$

Therefore,

$$T \cap Ta_1 \not\subset \{x \in G: |\, d(V_1, T \cap Ta_1, x) - \chi_{T \cap Ta_1}(x)\,| \geqq \varepsilon/2\} .$$

Take

$$a_2 \in T \cap Ta_1 - \{x \in G: |\, d(V_1, T \cap Ta_1, x) - \chi_{T \cap Ta_1}(x)\,| \geqq \varepsilon/2\}$$

so that $a_2 a_1^{-1} \in T$ and

$$1 \geqq \frac{\mu(T \cap Ta_1 \cap V_1 a_2)}{\mu(V_1 a_2)} = \frac{\mu(Ta_2^{-1} \cap Ta_1 a_2^{-1} \cap V_1)}{\mu(V_1)} > 1 - \varepsilon/2 ,$$

where $V_1 \in \Phi$. Thus we have the following situation:

(i) $a_1, a_2 \in G$, $a_2 a_1^{-1} \in T$.

(ii) $1 \geqq \dfrac{\mu(Ta_2^{-1} \cap Ta_1 a_2^{-1} \cap V)}{\mu(V)} > 1 - \varepsilon/2$ for some $V \in \Phi$.

We shall construct by induction a sequence $\{a_i\}_{i=1}^{\infty} \subset G$ such that $a_i a_j^{-1} \in T$ whenever $1 \leqq j < i$. Suppose we are given

(i)′ $a_1, a_2, \cdots, a_{m-1} \in G$; $a_i a_j^{-1} \in T$ whenever $1 \leqq j < i \leqq m - 1$.

(ii)′ $1 \geqq \dfrac{\mu(Ta_{m-1}^{-1} \cap Ta_1 a_{m-1}^{-1} \cap \cdots \cap Ta_{m-2} a_{m-1}^{-1} \cap V)}{\mu(V)} > 1 - \varepsilon/2$

for some $V \in \Phi$. Now as $V \in \Phi$ we have

$$\mu(V) \geqq \mu(T \cap V) > \mu(V)\varepsilon$$

and

$$\mu(V) \geqq \mu(Ta_{m-1}^{-1} \cap \cdots \cap Ta_{m-2} a_{m-1}^{-1} \cap V) \geqq \mu(V)(1 - \varepsilon/2) .$$

Thus

$$\mu(T \cap Ta_{m-1}^{-1} \cap Ta_1 a_{m-1}^{-1} \cap \cdots \cap Ta_{m-2} a_{m-1}^{-1}) > 0 ,$$

whence

$$\mu(T \cap Ta_1 \cap \cdots \cap Ta_{m-1}) = t' > 0 .$$

As Φ is a base at e there exists $V' \in \Phi$ such that

$$\mu(\{x \in G: |d(V', T \cap Ta_1 \cap \cdots \cap Ta_{m-1}, x) - \chi_{T \cap \cdots \cap Ta_{m-1}}(x)| \geqq \varepsilon/2\}) < t'/2 .$$

Take

$$a_m \in T \cap Ta_1 \cap \cdots \cap Ta_{m-1} - \{x \in G: |d(V', T \cap \cdots \cap Ta_{m-1}, x) - \chi_{T \cap Ta_1 \cap \cdots \cap Ta_{m-1}}(x)| \geqq \varepsilon/2\} .$$

Then $a_m a_i^{-1} \in T$ for all $i = 1, 2, \cdots, m - 1$ and

$$1 \geqq \frac{\mu(T \cap Ta_1 \cap \cdots \cap Ta_{m-1} \cap V'a_m)}{\mu(V'a_m)} = \frac{\mu(Ta_m^{-1} \cap Ta_1 a_m^{-1} \cap \cdots \cap Ta_{m-1} a_m^{-1} \cap V')}{\mu(V')} > 1 - \varepsilon/2 ,$$

where $V' \in \Phi$. As $a_1, \cdots, a_m$ satisfy conditions analogous to (i)′ and (ii)′, it follows that there exists a sequence $\{a_i\}_{i=1}^{\infty} \subset G$ such that $a_i a_j^{-1} \in T$ whenever $1 \leqq j < i$.

Now as f is almost periodic, the sequence $\{f(a_i x)\}_{i=1}^{\infty}$ contains a uniformly convergent sub-sequence, say $\{f(a_i' x)\}$. Then there exists

$N > 0$ such that

$$\sup_{x\in G} |f(a'_i x) - f(a'_j x)| < \delta$$

whenever $i > j \geqq N$. But

$$\begin{aligned}\sup_{x\in G} &|f(a'_i x) - f(a'_j x)| \\ &= \sup_{x\in G} |f(a'_i(a'_j)^{-1}x) - f(x)| \geqq |f(a'_i(a'_j)^{-1}) - f(e)| \geqq \delta ,\end{aligned}$$

because $a'_i(a'_j)^{-1} \in T \subset T(f, \delta, e)$. Thus our assumption of the falsity of the lemma leads to a contradiction and the lemma is proven.

THEOREM. *Let G be an LC group. If an almost periodic function on G is $\bar{\Gamma}$-measurable, then it is continuous.*

Proof. The proof is indirect. Suppose that $f: G \to C$ is $\bar{\Gamma}$-measurable, almost periodic but not continuous. By translating f, if necessary, we may suppose f is discontinuous at e. Then for some $\delta > 0$ the set $T(f, \delta, e) = \{x \in G: |f(x) - f(e)| \geqq \delta\}$ intersects every neighborhood of e. Take some $V^* \in \Omega$ and let

$$S = \{x \in G: |f(x) - f(e)| < \delta/2\} \cap V^*$$

so that S is a bounded member of $\bar{\Gamma}$. By the previous lemma

$$\lim_{\substack{U\to e\\ U\in\Omega}} \frac{\mu(S \cap U)}{\mu(U)} = 1$$

and, in particular, $\mu(S) = r > 0$. We make frequent use of the last statement in the density theorem in the sequel. Also we let Δ denote symmetric difference.

There is some $U'_1 \in \Omega$ such that

$$\mu(\{x \in G: |d(U, S, x) - \chi_S(x)| \geqq 1/100\}) < r/2$$

for all $U \in \Omega$ such that $U \subset U'_1$. Choose $U_1 \in \Omega$, $U_1 \subset U'_1$, such that

$$\frac{\mu(S \cap U_1)}{\mu(U_1)} > \frac{99}{100}. \tag{1}$$

There exists $a_1 \in S - \{x \in G: |d(U_1, S, x) - \chi_S(\chi)| \geqq 1/100\}$ and a_1 satisfies

$$\frac{\mu(S \cap U_1 a_1)}{\mu(U_1 a_1)} = \frac{\mu(S a_1^{-1} \cap U_1)}{\mu(U_1)} > \frac{99}{100}. \tag{2}$$

Now $V = \{x \in G: \mu(xU_1 \Delta U_1) < (1/100)\mu(U_1)\} \cap V^* \in \Omega$ (cf. [1], 61. A). Take $y_1 \in V^{-1} \cap T(f, \delta, e)$ so that $y_1^{-1} \in V$ and $|f(y_1) - f(e)| \geqq \delta$. Then

$$\mu(y_1^{-1}U_1 \cup U_1) \geqq \frac{99}{100}\mu(U_1)\ . \tag{3}$$

Combining (3) with (1) and (2) and using the fact that μ is left invariant gives

$$\mu(Sa_1^{-1} \cap y_1^{-1}U_1) > \frac{98}{100}\mu(U_1) = \frac{98}{100}\mu(y_1^{-1}U_1)$$
$$\mu(S \cap y_1^{-1}U_1) > \frac{98}{100}\mu(U_1) = \frac{98}{100}\mu(y_1^{-1}U_1)$$
$$\mu(y_1^{-1}S \cap y_1^{-1}U_1) > \frac{99}{100}\mu(U_1) = \frac{99}{100}\mu(y_1^{-1}U_1)\ .$$

Thus $\mu(Sa_1^{-1} \cap S \cap y_1^{-1}S) > 0$ whence $\mu(S \cap Sa_1 \cap y_1^{-1}Sa_1) = s > 0$. There exists $U_2' \in \Omega$ such that

$$\mu\left(\left\{x \in G: \left|d(U, S \cap Sa_1 \cap y_1^{-1}Sa_1, x) - \chi_{S\cap Sa_1 \cap y_1^{-1}Sa_1}(x)\right| \geqq \frac{1}{100}\right\}\right) < s/2$$

for all $U \in \Omega$, $U \subset U_2'$. Choose $U_2 \in \Omega$ such that $U_2 \subset U_2'$ and

$$\frac{\mu(S \cap U_2)}{\mu(U_2)} > \frac{99}{100}\ .$$

Then

$$S \cap Sa_1 \cap y_1^{-1}Sa_1 - \left\{x \in G: \left|d(U_2, S \cap Sa_1 \cap y_1^{-1}Sa_1, x) - \chi_{S\cap Sa_1 \cap y_1^{-1}Sa_1}(x)\right| \geqq \frac{1}{100}\right\} \neq \phi$$

and we take a_2 belonging to this set. Then a_2 satisfies

$$\frac{\mu(Sa_2^{-1} \cap Sa_1a_2^{-1} \cap U_2)}{\mu(U_2)} = \frac{\mu(S \cap Sa_1 \cap U_2a_2)}{\mu(U_2a_2)} \geqq \frac{\mu(S \cap Sa_1 \cap y_1^{-1}Sa_1 \cap U_2a_2)}{\mu(U_2a_2)} > \frac{99}{100}\ .$$

Also $y_1a_2a_1^{-1} \in S$. We thus have the following situation:

(i) $a_1, a_2 \in G$; $y_1 \in G$; $y_1a_2a_1^{-1} \in S$.

(ii) $|f(y_1) - f(e)| \geqq \delta$.

(iii) There exists $U \in \Omega$ such that

$$\frac{\mu(S \cap U)}{\mu(U)} > \frac{99}{100} \quad \text{and} \quad \frac{\mu(Sa_2^{-1} \cap Sa_1a_2^{-1} \cap U)}{\mu(U)} > \frac{99}{100}\ .$$

Suppose the following situation is true for $m \geqq 2$:

(i)$'$ $a_1, a_2, \cdots, a_m \in G$; $y_1, y_2, \cdots, y_{m-1} \in G$; and $y_{k-1}a_ka_j^{-1} \in S$ for all $1 \leqq j < k \leqq m$.

(ii)$'$ $|f(y_i) - f(e)| \geqq \delta$ for all $i = 1, \cdots, m-1$.

(iii)′ There exists $U \in \Omega$ such that

$$\frac{\mu(S \cap U)}{\mu(U)} > \frac{99}{100}$$

and

$$\frac{\mu(Sa_m^{-1} \cap Sa_1a_m^{-1} \cap \cdots \cap Sa_{m-1}a_m^{-1} \cap U)}{\mu(U)} > \frac{99}{100}.$$

We shall show how to obtain $a_{m+1}, y_m \in G$ such that $a_1, \cdots, a_{m+1}$ and $y_1, \cdots, y_m$ satisfy conditions analogous to (i)′, (ii)′, (iii)′.

Exactly as was done in the paragraph leading up to equation (3) we obtain a point $y_m \in T(f, \delta, e)$ such that

$$\mu(y_m^{-1}U \cap U) \geqq \frac{99}{100}\mu(U) .$$

Thus $|f(y_m) - f(e)| \geqq \delta$. Combining the above relation with the relations of (iii)′ and using the left invariance of μ, we get

$$\mu(S \cap y_m^{-1}U) > \left(\frac{98}{100}\right)\mu(U) = \left(\frac{98}{100}\right)\mu(y_m^{-1}U)$$

$$\mu(Sa_m^{-1} \cap Sa_1a_m^{-1} \cap \cdots \cap Sa_{m-1}a_m^{-1} \cap y_m^{-1}U) > \left(\frac{98}{100}\right)\mu(y_m^{-1}U)$$

$$\mu(y_m^{-1}S \cap y_m^{-1}U) > \left(\frac{99}{100}\right)\mu(y_m^{-1}U)$$

$$\mu(y_m^{-1}Sa_m^{-1} \cap y_m^{-1}Sa_1a_m^{-1} \cap \cdots \cap y_m^{-1}Sa_{m-1}a_m^{-1} \cap y_m^{-1}U) > \frac{99}{100}\mu(y_m^{-1}U) .$$

Thus

$$\begin{aligned}\mu(S \cap Sa_m^{-1} \cap Sa_1a_m^{-1} \cap \cdots \cap Sa_{m-1}a_m^{-1}\\ \cap y_m^{-1}S \cap y_m^{-1}Sa_m^{-1} \cap y_m^{-1}Sa_1a_m^{-1} \cap \cdots \cap y_m^{-1}Sa_{m-1}a_m^{-1}) > 0\end{aligned}$$

so, letting

$$R = S \cap Sa_1 \cap Sa_2 \cap \cdots \cap Sa_m \cap y_m^{-1}S \cap y_m^{-1}Sa_1 \cap \cdots \cap y_m^{-1}Sa_m ,$$

we have $\mu(R) = t > 0$. There exists $W \in \Omega$ such that

$$\mu\left(\left\{x \in G: |d(V, R, x) - \chi_R(x)| \geqq \frac{1}{100}\right\}\right) < t/2$$

for all $V \subset W$, $V \in \Omega$. Take $U' \in \Omega$, $U' \subset W$, such that

$$\frac{\mu(S \cap U')}{\mu(U')} > \frac{99}{100}.$$

Then there exists $a_{m+1} \in R - \{x \in G: |d(U', R, x) - \chi_R(x)| \geqq 1/100\}$. By definition of R we have $y_m a_{m+1} a_k^{-1} \in S$ for all $k = 1, 2, \cdots, m$. Also

$$\frac{\mu(Sa_{m+1}^{-1} \cap Sa_1 a_{m+1}^{-1} \cap \cdots \cap Sa_m a_{m+1}^{-1} \cap U')}{\mu(U')} = \frac{\mu(S \cap Sa_1 \cap \cdots \cap Sa_m \cap U'a_{m+1})}{\mu(U'a_{m+1})} \geqq d(U', R, a_{m+1}) > \frac{99}{100}.$$

Thus $a_1, \cdots, a_{m+1}$ and $y_1, \cdots, y_m$ satisfy conditions analogous to (i)$'$, (ii)$'$ and (iii)$'$. It follows by induction that there exist two sequences $\{a_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty} \subset G$ such that $|f(y_i) - f(e)| \geqq \delta$ for all $i = 1, 2, \cdots$ and for any $k \geqq 2$ we have $y_{k-1} a_k a_j^{-1} \in S$ for $1 \leqq j < k$.

Now as f is almost periodic, the sequence $\{f(xa_m^{-1})\}_{m=1}^{\infty}$ contains a uniformly convergent subsequence, say $\{f(xa_{m_k}^{-1})\}_{k=1}^{\infty}$. Then there exists $N > 0$ such that

$$\sup_{x \in G} |f(xa_{m_{k_1}}^{-1}) - f(xa_{m_{k_2}}^{-1})| < \delta/2$$

whenever $k_1 > k_2 \geqq N$. But

$$\begin{aligned}
&\sup_{x \in G} |f(xa_{m_{k_1}}^{-1}) - f(xa_{m_{k_2}}^{-1})| \\
&\quad = \sup_{x \in G} |f(x) - f(xa_{m_{k_1}} a_{m_{k_2}}^{-1})| \geqq |f(y_{m_{k_1}-1}) - f(y_{m_{k_1}-1} a_{m_{k_1}} a_{m_{k_2}}^{-1})| \\
&\quad = |\{f(y_{m_{k_1}-1}) - f(e)\} + \{f(e) - f(y_{m_{k_1}-1} a_{m_{k_1}} a_{m_{k_2}}^{-1})\}| \\
&\quad \geqq ||f(y_{m_{k_1}-1}) - f(e)| - |f(e) - f(y_{m_{k_1}-1} a_{m_{k_1}} a_{m_{k_2}}^{-1})|| \\
&\quad > \delta - \delta/2 = \delta/2
\end{aligned}$$

because each $y_i \in T(f, \delta, e)$ and $y_{m_{k_1}-1} a_{m_{k_1}} a_{m_{k_2}}^{-1} \in S$. This contradiction assures that f is continuous if it is $\bar{\Gamma}$-measurable and almost periodic. The proof is completed.

References

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2. Edwin Hewitt, and Kenneth A. Ross, *Abstract Harmonic Analysis*, New York, New York, 1963.

Received November 10, 1965. This work was done while the author had a NASA predoctoral training grant and is part of a doctoral thesis written under Professor Gary H. Meisters at the University of Colorado. Preparation of the paper was supported by the project Special Research in Numerical Analysis for the Army Research Office, Durham, Contract Number DA-31-124-AROD-13, at Duke University.

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