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ON MAPS WITH IDENTICAL FIXED POINT SETS

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If two maps on a space X, which admits a fixed point index, have identical sets of fixed points and agree on an open subset of X which contains the fixed point set, then the maps have the same Lefschetz number. If the subset is closed, the conclusion is no longer true in general. However, a theorem of Leray implies that some kinds of maps on cartesian products of convexoid spaces which agree on a certain closed subset of their common fixed point set do have the same Lefschetz number, even though the maps may not be homotopic and may not agree on any open set containing the fixed point set. The purpose of this note is to prove a very general form of Leray's theorem for maps on ANR's.

LEMMA. *Let X be a compact ANR, let A be a closed subset of X, and let $f, g\colon X \to X$ be maps such that $f(a) = g(a)$ for all $a \in A$. Given $\gamma > 0$, there exists an open subset V of X containing A and a map $H\colon V \times I \to X$ such that $H(x, 0) = f(x), H(x, 1) = g(x)$ and*

$$d(f(x), H(x, t)) < \gamma$$

for all $x \in V, t \in I$, where d denotes the metric of X.

Proof. Imbed X in the Hilbert cube I^∞, then there is a retraction $r\colon U \to X$ defined on some open subset U of I^∞ containing X. Let d be the metric of I^∞, then there exists $\eta > 0$ such that

$$X \subseteqq \bar{N}(X, \eta) \subseteqq U ,$$

where $\bar{N}(X, \eta) = \{e \in I^\infty \mid \inf_{x \in X} d(e, x) \leqq \eta\}$. One can find $\delta > 0$ such that $e_1, e_2 \in \bar{N}(X, \eta)$ and $d(e_1, e_2) < \delta$ implies $d(r(e_1), r(e_2)) < \gamma$. Furthermore, there exists $\zeta > 0$ such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \zeta$, then $d(f(x_1), f(x_2))$ and $d(g(x_1), g(x_2))$ are both less than the smaller of $\delta/2$ and η. Let $V = \{x \in X \mid \inf_{a \in A} d(x, a) < \zeta\}$ and define $H\colon V \times I \to X$ by $H(x, t) = r((1 - t)f(x) + tg(x))$ for $x \in V, t \in I$.

For maps $f, g\colon X \to Y$ let $C(f, g) = \{x \in X \mid f(x) = g(x)\}$. Let i denote the fixed point index for the category of compact ANR's [1]. For $U \subseteqq X$, let ∂U be the boundary of U and $\bar{U}$ the closure of U.

THEOREM. *Let $\mathfrak{F} = (X, p, B)$ be a Hurewicz fibre space where X and B are compact* ANR's. *Given a map $f\colon X \to X$ and an open subset U of X such that $f(x) \neq x$ for all $x \in \partial U$, if $g\colon X \to X$ is a map such that $pg = pf$ and $C(p, pf) \cap \bar{U} \subseteqq C(f, g)$, then*

$$i(f, U) = i(g, U) .$$

In particular, if $C(p, pf) \subseteqq C(f, g)$, *then* $L(f) = L(g)$, *where* L *denotes the Lefschetz number.*

Proof. We recall that since B is an ANR, it is ULC, that is, there exists an open subset W of $B \times B$ containing the diagonal and a map $\theta: W \to B^I$ such that for $b_1, b_2 \in W$,

$$\theta(b_1, b_2)(0) = b_1, \theta(b_1, b_2)(1) = b_2 ,$$

and $\theta(b, b)(t) = b$ for all $t \in I$. Let d' be the metric of B. There exists $\varepsilon > 0$ such that $d'(b_1, b_2) < \varepsilon$ implies $(b_1, b_2) \in W$. Furthermore, there exists $\gamma > 0$ such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \gamma$, then

$$d'(p(x_1), p(x_2)) < \varepsilon .$$

Applying the lemma for $A = C(p, fp) \cap \bar{U}$ and this γ, we have an open subset V of X containing A and a homotopy $H: V \times I \to X$. Let 0 be open in X such that $A \subseteqq \bar{O} \subseteqq O \subseteqq V$. Set $Q = U \cap O$ and consider

$$H: \bar{Q} \times I \to X .$$

There is a regular lifting function λ for $\mathfrak{F}$ [3]. Define $G_1: \bar{Q} \times I \to X$ by

$$G_1(x, t) = \lambda[f(x), \theta'_t(x)](1)$$

where $\theta'_t: \bar{Q} \to X^I$ is given by

$$\theta'_t(x)(s) = \theta(pH(x, s), pf(x))(t) .$$

For the map $\bar{\lambda}: X^I \to X^I$ defined by $\bar{\lambda}(\alpha) = \lambda[\alpha(0), p\alpha]$, there is a homotopy $K: X^I \times I \to X^I$ such that

$$K(\alpha, 0) = \alpha, K(\alpha, 1) = \bar{\lambda}(\alpha), pK(\alpha, s)(t) = p\alpha(t)$$

for all $\alpha \in X^I, t \in I$, and if α is a constant path then $K(\alpha, t) = \alpha$ for all $t \in I$ [2, Proposition 1]. Define $G_2: \bar{Q} \times I \to X$ by

$$G_2(x, t) = K(H'(x), t)(1)$$

where $H': \bar{Q} \to X^I$ is induced by H. Finally, consider $G: \bar{Q} \times I \to X$ where

$$G(x, t) = \begin{cases} G_1(x, 1 - 2t) & \text{if } 0 \leqq t \leqq 1/2 \\ G_2(x, 2 - 2t) & \text{if } 1/2 \leqq t \leqq 1 \end{cases}$$

then $G(x, 0) = f(x), G(x, 1) = g(x)$ and $pG(x, t) = pf(x)$ for all $t \in I$. If $x \in \partial Q$ and $x \notin A$, then $p(x) \neq pf(x)$ so $x \neq G(x, t)$ for all $t \in I$. If

$$x \in \partial Q \cap A ,$$

then $f(x) = g(x)$ and by construction $H(x, t) = f(x)$ for all $t \in I$ so by the properties of θ and K, $G(x, t) = f(x)$ for all $t \in I$. Since $\partial O \cap A = \phi$ and $\partial Q \subseteqq \partial U \cup \partial O$, then $x \in \partial U$ so $f(x) \neq x$ by hypothesis which implies $G(x, t) \neq x$. Hence, by the homotopy axiom of the fixed point index [1], $i(f, Q) = i(g, Q)$. If $x \in \partial U$, then $f(x) \neq x$ and $g(x) \neq x$ while if $x \in U$ and $f(x) = x = g(x)$ then $x \in A \cap U \subset Q$. Therefore f and g have no fixed points on $\bar{U} - \bar{Q}$ and by the additivity axiom,

$$i(f, U) = i(g, U) .$$

The last sentence of the theorem follows by taking $U = X$ and using the normalization axiom: $i(f, X) = L(f)$.

As a special case, we obtain the result which Leray proved for convexoid spaces [4, Theorem 26].

COROLLARY. *Let X and Y be compact* ANR's. *Maps*

$$f: X \times Y \to X, g: X \times Y \to Y$$

induce $f \times g: X \times Y \to X \times Y$ defined by

$$(f \times g)(x, y) = (f(x, y), g(x, y)) .$$

Suppose U is an open subset of $X \times Y$ such that $f \times g$ has no fixed points on ∂U. If $h: X \times Y \to Y$ is a map with the property that $(x, y) \in \bar{U}$ and $f(x, y) = x$ implies $h(x, y) = g(x, y)$, then

$$i(f \times h, U) = i(f \times g, U) .$$

Proof. The maps $f \times g$ and $f \times h$ satisfy the hypotheses of the theorem with respect to the trivial fibration of $X \times Y$ over X.

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