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ON MUIRHEAD'S THEOREM

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Several of the interesting analytic and geometric conditions known to be equivalent to the classical partial order $\prec$ on E^n given by Hardy, Littlewood and Pólya have also been shown to be true in the continuous case. Muirhead's inequality, from which virtually all generalizations of the arithmetic-geometric-mean inequality follow, is perhaps less tractable and does not readily suggest a continuous analogue. The purpose of this paper is to discuss two such possibilities.

The author is indebted to Professor G.-C. Rota who suggested that such a generalization should exist.

Suppose that $\vec{x}$ and $\vec{y}$ are real n-vectors and that $\vec{x}^*, \vec{y}^*$ are the vectors obtained from $\vec{x}$ and $\vec{y}$ by rearranging their components in descending order. Then we say that $\vec{y}$ *majorizes* $\vec{x}$, writing $\vec{x} \prec \vec{y}$, whenever

$$\begin{aligned} x_1^* &\leqq y_1^* \\ &\vdots \\ x_1^* + \cdots + x_k^* &\leqq y_1^* + \cdots + y_k^* \\ &\vdots \\ x_1 + \cdots + x_n &= y_1 + \cdots + y_n \end{aligned} \tag{1}$$

where the numbers x_ν^*, y_ν^* are the components of $\vec{x}^*$ and $\vec{y}^*$. A continuous version of this partial order is suggested by means of the *decreasing rearrangement* x^* of a measurable function x. If x is measurable, real valued on $[0, 1]$ and μ is Lebesgue measure, then there exists a nonincreasing function x^* on $[0, 1]$ such that

$$m(s) = \mu\{x > s\} = \mu\{x^* > s\}$$

for all s. The function x^* is made unique by requiring that it be right-continuous and, in fact, it is the inverse of $m(s)$. Moreover, x will be integrable if and only if the same is true of x^* and their integrals will be equal. Further details are given in [6] and [7].

Guided by (1), we define a partial order $\prec$ in $L^1(0, 1) = L^1$ in the following manner: If x and y are in L^1 then $x \prec y$ is to mean

$$\begin{aligned} &\int_0^s x^* \leqq \int_0^s y^* , \qquad 0 \leqq s < 1 \\ &\int_0^1 x = \int_0^1 y . \end{aligned} \tag{1'}$$

We might remark in passing that (1) and (1′) do not define partial orders in the strict sense, since one may have $x \prec y$ and $y \prec x$ without $x = y$. In the vector case this will happen if and only if $\vec{x}$ and $\vec{y}$ are rearrangements of one another. In the continuous case this occurs if and only if x and y are *equimeasurable*, that is $x^* = y^*$.

We now outline the current state of affairs in the table below. It will be convenient to use the symbol $P(\vec{y})$ for the set of all vectors which are rearrangements of $\vec{y}$ and $P(y)$ for all functions equimeasurable with y. Then all statements in each column are equivalent.

n-vectors	L^1-functions
(a) $\vec{x} \prec \vec{y}$	(a′) $x \prec y$
(b) $\vec{x} = T\vec{y}$, T a doubly stochastic matrix	(b′) $x = Ty$, T a doubly stochastic operator
(c) $\vec{x}$ belongs to the convex hull of $P(\vec{y})$ whose set of extreme points is exactly $P(\vec{y})$	(c′) x belongs to the closed convex hull of $P(y)$ whose set of extreme points is exactly $P(y)$ [8].
(d) If ϕ is a convex function of one real variable, then $\sum \phi(x_i) \leqq \sum \phi(y_i)$	(d′) If ϕ is a convex function of one real variable for which $\phi \circ x$ and $\phi \circ y$ are in L^1, $\int_0^1 \phi \circ x \leqq \int_0^1 \phi \circ y$
(e) Muirhead's inequality	(e′) To be given.

There is another equivalent assertion in the discrete case that is worth mentioning. A real function F of n real variables is said to be *Schur-convex*, or simply *S-convex*, if for each $\vec{x}$, $F(T\vec{x}) \leqq F(\vec{x})$ whenever T is doubly stochastic. These (necessarily) symmetric functions have been studied by Ostrowski [4] who gives necessary and sufficient conditions on certain partial derivatives of F in order that it be S-convex. As every function of the type listed in (d) is S-convex one sees that the latter functions form a larger class. From the point of view of economy, those of type (d) have preference.

While we cannot put Muirhead's inequality in category (d), it is an inequality arising from a certain S-convex function. Suppose that $\vec{u} = (u_1, \cdots, u_n)$ is a *positive* n-vector $(u_k > 0, k = 1, \cdots, n)$, and S_n is the symmetric group of all permutations π of $(1, 2, \cdots, n)$. Set

$$M(\vec{u}, \vec{x}) = \sum_{\pi \in S_n} u_{\pi(1)}^{x_1} \cdots u_{\pi(n)}^{x_n} . \tag{2}$$

Muirhead's inequality is then

$$M(\vec{u}; \vec{x}) \leqq M(\vec{u}; \vec{y}) \tag{3}$$

for all positive n-vectors $\vec{u}$ if and only if $\vec{x} \prec \vec{y}$. We also should single out the interesting work of R. Rado [5] in which the summation in (2) is taken over any subgroup L_n of S_n. The inequality (3) then obtains in this restricted sense if and only if $\vec{x}$ belongs to the convex hull of the vectors $\vec{y}_\pi = (y_{\pi(1)}, \cdots, y_{\pi(n)})$, $\pi \in \Gamma_n$. We note also that M is symmetric in $\vec{x}$ and each term $u_{\pi(1)}^{x_1} \cdots u_{\pi(n)}^{x_n}$ in the summation (2) is a convex function of $\vec{x}$. This convexity allows one to determine when equality can occur in (3) by an elementary argument.

In order to develop a continuous version of (3) we first proceed with some heuristic remarks. If $\vec{x} = (x_1, x_2, 0, \cdots, 0)$ and $\vec{y} = (y_1, y_2, 0, \cdots, 0)$ are n-vectors and $\vec{u} = (u_1, \cdots, u_n)$ a positive n-vector, then $\vec{x} \prec \vec{y}$ gives

$$M(\vec{u}; \vec{x}) = (n-2)! \sum_{i \neq j} u_i^{x_1} u_j^{x_2} \leqq (n-2)! \sum_{i \neq j} u_i^{y_1} u_j^{y_2} = M(\vec{u}; \vec{y}) \, .$$

Adding the terms with $i = j$, we note that the inequality persists since $x_1 + x_2 = y_1 + y_2$. Thus

$$\Big(\sum_k u_k^{x_1}\Big)\Big(\sum_k u_k^{x_2}\Big) \leqq \Big(\sum_k u_k^{y_1}\Big)\Big(\sum_k u_k^{y_2}\Big) \, . \tag{4}$$

Now let u be a positive continuous function on $[0, 1]$ and set $u_k = u(k/n)$. Dividing (4) by n^{-2} and passing to the limit we obtain

$$\int_0^1 u^{x_1} \int_0^1 u^{x_2} \leqq \int_0^1 u^{y_1} \int_0^1 u^{y_2} \, .$$

This generalizes easily (replace $(x_1, x_2, 0, \cdots, 0)$ by $(x_1, x_2, \cdots, x_m, 0, \cdots, 0)$) to

$$\prod_{k=1}^m \int_0^1 u^{x_k} \leqq \prod_{k=1}^m \int_0^1 u^{y_k} \, .$$

Now suppose that x and y are continuous functions on $[0, 1]$ and we set $x_k = x(k/m)$. Taking logarithms, dividing by m and making $m \rightarrow \infty$ we arrive at the inequality

$$\begin{aligned} M(u; x) = \int_0^1 \log \left\{\int_0^1 u(t)^{x(s)} dt\right\} ds &\leqq \int_0^1 \log \left\{\int_0^1 u(t)^{y(s)} dt\right\} ds \\ &= M(u; y) \end{aligned} \tag{5}$$

2. A proof of the inequality. The remainder of this section will be devoted to the proof of the following theorem.

THEOREM. *Let x and y be bounded, measurable functions on $[0, 1]$. If $x \prec y$ and u is a positive function such that $u^v \in L^1$ for* *all* p, $-\infty < p < \infty$, *then inequality* (5) *results. Conversely, if the inequality holds for all such* u *then* $x \prec y$. *Actually, all one requires is that* p *lie in some finite interval* $[m, M]$ *for which* $m \leqq x, y \leqq M$ *almost everywhere.*

A word or two is in order concerning the existence of the integrals. The function

$$(6) \qquad \phi(p) = \log \int_0^1 u^p = p \log \| u \|_p$$

is convex and bounded on bounded subsets of the line. Therefore, it is continuous and the composite $\phi \circ x$ is measurable. The integrability of $\phi \circ x$ and $\phi \circ y$ is then a consequence of the boundedness of x and y.

Our principal concern here is the assertion that (5) implies $x \prec y$. That $x \prec y$ implied (5) was shown to be the case, at least for bounded functions u, in a paper of Hardy, Littlewood and Pólya [1, Theorem 10]. Using more recent results we offer here a short proof of this in the slightly more general setting. First we note that the function ϕ in (6) is rearrangement-invariant in the sense that

$$\int_0^1 \phi \circ \tilde{y} = \int_0^1 \phi \circ y$$

for any bounded measurable function $\tilde{y}$ equimeasurable with y. Then, if $x \prec y$, we may choose convex combinations $\sum \lambda_i y_i$ $(i = 1, \cdots, m)$ where each y_i is equimeasurable with y (see [7], Theorem 5, and (c′) of the preceding table) such that

$$\int_0^1 | \sum \lambda_i y_i - x | < \varepsilon$$

for each $\varepsilon > 0$. Now we use the convexity of ϕ to give

$$\phi \circ (\sum \lambda_i y_i) \leqq \sum \lambda_i \phi \circ y_i \, .$$

The rearrangement-invariance of ϕ now implies

$$\int_0^1 \phi \circ (\sum \lambda_i y_i) \leqq \int_0^1 \phi \circ y \, .$$

A convex function is Lipschitz continuous on bounded sets and so

$$\int_0^1 | \phi \circ (\sum \lambda_i y_i) - \phi \circ x | \leqq K \int_0^1 | \sum \lambda_i y_i - x |$$

for some constant K. This implies (5).

That inequality (5) is also a necessary condition for $x \prec y$ does not seem to be as accessible. We shall use the following approximation lemma in conjunction with a brief remark mentioned in *Inequalities* [2].

LEMMA. *If x is bounded, measurable on $[0, 1]$, there exist sequences $\{\underline{\sigma}_n\}$ and $\{\bar{\sigma}_n\}$ of simple functions such that*

$$\underline{\sigma}_n \prec x \prec \bar{\sigma}_n \qquad n = 1, 2, \cdots$$

and

$$\int_0^1 | x - \underline{\sigma}_n | \to 0 \, , \qquad \int_0^1 | x - \bar{\sigma}_n | \to 0 \text{ as } n \to \infty \, .$$

Proof. Let x^* be the decreasing rearrangement of x. Divide $[0, 1]$ into n equal subintervals $I_1, I_2, \cdots, I_n$, and set

$$a_k = n \int_{I_k} x^* \qquad (1 \leqq k \leqq n) \, .$$

Finally, denote by χ_k the characteristic function of I_k and define

$$\underline{\sigma}_n^* = \sum_k a_k \chi_k \, .$$

Since a_k lies between the infimum m_k and the maximum M_k of x^* on I_k and, since the average value of the integral of x^* in nonincreasing, one sees that $\underline{\sigma}_n^* \prec x^*$. Furthermore,

$$\int_0^1 | x^* - \underline{\sigma}_n^* | \leqq \frac{1}{n} \sum_k (M_k - m_k) \to 0 \, , \qquad n \to \infty \, .$$

To complete the argument, we maintain the same partition and define

$$\sigma_n = \sum_k M_k \chi_k$$

with M_k still representing the maximum of x^* on I_k. This function is simply the upper Darboux sum of x^* for the given partition. Moreover, σ_n is nonincreasing and, for $0 \leqq s \leqq 1$, we have

$$\psi(s) = \int_0^s x^* \leqq \int_0^s \sigma_n = \psi_n(s) \, .$$

Choose $\alpha < x^*(1) = \lim_{t \to 1^-} x^*(t)$, define $\beta_n = \psi_n(\tau_n)$ (where $\tau_n = 1 - (1/n)$) and let $\gamma = \psi(1) = \int_0^1 x^*$. Then for each s in $[0, 1]$

$$\lambda(s) = \beta_n + M_n(s - \tau_n) \geqq \psi_n(s) \geqq \psi(s)$$
$$l(s) = \gamma + \alpha(s - 1) \geqq \psi(s) \, ,$$

both of which follow directly from the preceding definitions. The graphs of these two linear functions intersect at

$$\tau' = \tau'(\alpha) = \frac{\gamma - \alpha}{M_n - \alpha} + \frac{\tau_n M_n - \beta_n}{M_n - \alpha} \leqq 1 .$$

To see this, we note that

$$0 \leqq \int_0^{\tau_n} (\sigma_n - x^*) + \int_{\tau_n}^1 (M_n - x^*)$$

with equality occurring only in the case where $x^* = \sigma_n$. The inequality may be rewritten as

$$0 \leqq \beta_n + M_n(1 - \tau_n) - \gamma .$$

Adding α to both sides and remembering that $M_n > \alpha$ we establish the inequality. Equality obtains only if $x^* = \sigma_n$, in which case x^* is a step function and therefore x is already a simple function.

As α is still at our disposal (subject only to $\alpha < x^*(1)$) and since $\tau' \to 1$ as $\alpha \to -\infty$ we may choose α such that $\tau' = \tau'_n$ satisfies $\tau'_n = (\tau_n + 1)/2$. Set

$$\bar{\sigma}_n^*(s) = \begin{cases} \sigma_n(s) & 0 \leqq s \leqq \tau'_n \\ \alpha & \tau'_n < s \leqq 1 . \end{cases}$$

Then $\bar{\sigma}_n^*$ is nonincreasing, and for $0 \leqq s \leqq \tau'_n$ we have

$$\int_0^s x^* \leqq \int_0^s \bar{\sigma}_n^* .$$

Also,

$$\begin{aligned} \int_0^1 \bar{\sigma}_n^* &= \int_0^{\tau_n} \sigma_n + \int_{\tau_n}^{\tau'_n} M_n + \int_{\tau'_n}^1 \alpha \\ &= \beta_n + M_n(\tau'_n - \tau_n) + \alpha(1 - \tau'_n) \\ &= \gamma = \int_0^1 x^* \end{aligned}$$

as a direct calculation will bear out. If $\tau'_n < s < 1$

$$\int_0^s (\bar{\sigma}_n^* - x^*) = \int_0^{\tau'_n} (\sigma_n - x^*) + \int_{\tau'_n}^s (\alpha - x^*) .$$

The integral on the left tends to zero as $s \to 1$. On the right, the first integral is nonnegative while the second is negative and becomes more so as $s \to 1$. It must be that the integral on the left is nonnegative. That is, $x^* \prec \bar{\sigma}_n^*$.

Now $x^* - \bar{\sigma}_n^* = (x^* - \sigma_n) + (\sigma_n - \bar{\sigma}_n^*)$, so that

$$\begin{aligned}\int_0^1 |x^* - \bar{\sigma}_n^*| &\leqq \int_0^1 (\sigma_n - x^*) + \int_{\tau_n'}^1 (M_n - \alpha) \\ &\leqq \int_0^1 (\sigma_n - x^*) + (1 - \tau_n)M_n + (\beta_n - \gamma) \\ &\leqq \int_0^1 (\sigma_n - x^*) + \int_0^{\tau_n} (\sigma_n - x^*) + (1 - \tau_n)M_n - \int_{\tau_n}^1 x^* .\end{aligned}$$

As $n \to \infty$ we have both τ_n and τ_n' tending to 1. Since x^* is Riemann integrable and bounded, the left side of the inequality tends to zero as $n \to \infty$. The approximation of x is now obtained by means of a measure preserving transformation ω of $[0, 1]$ into itself such that $x = x^* \circ \omega$ (by virtue of Lemma 2, [7]). Then $\bar{\sigma}_n = \bar{\sigma}_n^* \circ \omega$ and $\underline{\sigma}_n = \underline{\sigma}_n^* \circ \omega$ are simple functions which posses the properties set forth in the statement of the lemma. (Both the partial order $\prec$ and L^1—norms are preserved under composition with measure preserving transformations.)

The remainder of our argument becomes technically simpler if we think of the functions $\underline{\sigma}_n^*$ and $\bar{\sigma}_n^*$ as step functions associated with a partition of $[0, 1]$ into $2n$ equal subintervals. We keep the same functions and just make the partition finer. This step is necessary because $\bar{\sigma}_n^*$ is not constant on the last subinterval of the original subdivision but becomes so relative to the refinement. Both functions remain step functions when we do this.

Next we point out an elementary but useful connection between the partial order $\prec$ for vectors and for certain step functions. If $(x_1, \cdots, x_m) \prec (y_1, \cdots, y_m)$ then

$$\sum_{k=1}^m x_k \chi_k \prec \sum_{k=1}^m y_k \chi_k$$

where χ_k is the characteristic function of $[(k-1)/m, k/m]$, $(k = 1, \cdots, m)$. The reader can easily fill in the details.

Assume that (5) is valid for all functions $u > 0$ for which the integrals exist. As the quantities $M(u; x)$ and $M(u; y)$ are rearrangement-invariant it will be enough to work with x^* and y^*. Moreover, we need only require that (5) hold for positive step functions as the following argument demonstrates. Select sequences $\underline{\sigma}_n^* \prec x^*$, $y^* \prec \bar{\sigma}_n^*$ $(n = 1, 2, \cdots)$ of step functions associated with subdivisions of $[0, 1]$ into $m = 2n$ equal subintervals as given by the lemma and our remark above. We should then have

$$(7) \quad M(u; \underline{\sigma}_n^*) \leqq M(u; x^*) \leqq M(u; y^*) \leqq M(u; \bar{\sigma}_n^*) \qquad n = 1, 2, \cdots$$

where

$$u = \sum_{k=1}^m \alpha_k \chi_{J_k}$$

with $\alpha_k > 0$ $(k = 1, \cdots, m)$ and $\{J_k\}$ is an arbitrary partition of $[0, 1]$ into m mutually disjoint subintervals. Denote the length of J_k by θ_k. Then (7) becomes

$$\frac{1}{m}\sum_{k=1}^{m}\log(\alpha_1^{x_k}\theta_1 + \cdots + \alpha_m^{x_k}\theta_m) \leqq \frac{1}{m}\sum_{k=1}^{m}\log(\alpha_1^{y_k}\theta_1 + \cdots + \alpha_m^{y_k}\theta_m)$$

where

$$\underline{\sigma}_n^* = \sum_{k=1}^{m} x_k\chi_k\ , \qquad \bar{\sigma}_n^* = \sum_{k=1}^{m} y_k\chi_k\ .$$

Of course $x_1 = x_2, x_3 = x_4, \cdots, x_{m-1} = x_m$ and $y_1 = y_2, \cdots, y_{m-3} = y_{m-2}$ but the last two components y_{m-1} and y_m are, in general, distinct. Set $\vec{x} = (x_1, x_2, \cdots, x_m)$ and $\vec{y} = (y_1, y_2, \cdots, y_m)$. Then

$$\prod_{k=1}^{m}\sum_{i=1}^{m}\alpha_i^{x_k}\theta_i \leqq \prod_{k=1}^{m}\sum_{i=1}^{m}\alpha_i^{y_k}\theta_i \tag{8}$$

and this inequality is true for all positive $\vec{\theta} = (\theta_1, \cdots, \theta_m)$ for which $\sum \theta_i = 1$ and all positive choices of $\vec{\alpha} = (\alpha_1, \cdots, \alpha_m)$. It is known [2, p. 51], and not very difficult to prove, that this implies $\vec{x} \prec \vec{y}$. Hence $\underline{\sigma}_n^* \prec \bar{\sigma}_n^*$, from which we derive $x^* \prec y^*$.

There is a certain bias in the derivation of (5) regarding the order of summation. Suppose that we first started with arbitrary exponents $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$ and rather special positive $\vec{u}$ of the form $\vec{u} = (u_1, u_2, 1, \cdots, 1)$. Then if $\vec{x} \prec \vec{y}$ Muirhead's inequality becomes

$$(n-2)!\sum_{i \neq j} u_1^{x_i}u_2^{x_j} \leqq (n-2)!\sum_{i \neq j} u_1^{y_i}u_2^{y_j}\ .$$

Inserting the terms $\sum (u_1u_2)^{x_i}$ the inequality holds (because by (d) of our table, $s \to (u_1u_2)^s$ is a convex function of s) and we have

$$\Big(\sum_k u_1^{x_k}\Big)\Big(\sum_k u_2^{x_k}\Big) \leqq \Big(\sum_k u_1^{y_k}\Big)\Big(\sum_k u_2^{y_k}\Big)\ . \tag{4'}$$

As before, this suggests that perhaps

$$\int_0^1 \log\left\{\int_0^1 u(t)^{x(s)}ds\right\}dt \leqq \int_0^1 \log\left\{\int_0^1 u(t)^{y(s)}ds\right\}dt \tag{5'}$$

is valid whenever $x \prec y$. This will be the case for bounded x and y except that the integrals may not be finite.

In order to prove this, assume that u satisfies the conditions of theorem together with the restriction $u(t) \geqq \delta > 0$ in $[0, 1]$. Then both integrals in (5′) exist (finite). For each fixed value of t, $s \to u(t)^s$ is a convex function so that by (d′) of the table

$$\int_0^1 u(t)^{x(s)}ds \leqq \int_0^1 u(t)^{y(s)}ds \ .$$

This inequality implies (5′). If u is not bounded away from 0, replace u by $u + \delta, \delta > 0$. Then, as $\delta \to 0$, $(u(t) + \delta)^{x(s)}$ converges downward to $u(t)^{x(s)}$ for each fixed t. The inequality now follows by an application of the monotone convergence principle.

One cannot conclude that the validity of (5′) for all admissible functions u implies $x \prec y$. It is because of this that we feel (5) is the preferable generalization of the Muirhead inequality. The example we shall give utilizes the nonnegativity of the polynomial

$$p(t) = t^6 - 2t^5 + 2t^3 - 2t + 1 = (t-1)^2(t^4 - t^2 + 1)$$

for all values of t. Equivalently,

$$2t^5 + 2t \leqq t^6 + 2t^3 + 1^1 \ .$$

Divide the unit interval into 4 equal subintervals and let χ_k denote the characteristic function of $[(k-1)/4, k/4]$ $(k = 1, 2, 3, 4)$. Set

$$x = 5\chi_1 + 5\chi_2 + \chi_3 + \chi_4$$
$$y = 6\chi_1 + 3\chi_2 + 3\chi_3 + 0 \cdot \chi_4 \ .$$

For any positive (admissible) u

$$\int_0^1 u(t)^{x(s)}ds = \frac{1}{4}[2u(t)^5 + 2u(t)]$$
$$\leqq \frac{1}{4}[u(t)^6 + 2u(t)^3 + 1] = \int_0^1 u(t)^{y(s)}ds \ .$$

Taking logarithms and integrating again we find that (5′) is indeed valid. But by inspection, it is clear that $x \prec y$ is false and so (5′) is not a sufficient condition.

3. The case of equality. It would be of interest to establish when equality can occur in (5). In view of the results in the discrete case we conjecture that equality occurs only when u is constant or else when x and y are equimeasurable. We give here a relatively simple proof for the discrete case which may lend itself to generalization.

Let equality hold in (3) and $\vec{x} \prec \vec{y}$. Write

$$\vec{u} = (u_1, \cdots, u_n) = (e^{v_1}, \cdots, e^{v_n})$$

and $\vec{x} = \sum \lambda_i \vec{y}_i$, where the $\vec{y}_i$ are distinct rearrangements of $\vec{y}$, $\sum \lambda_i =$

[1] We wish to thank G. A. Converse for bringing this to our attention.

$1, \lambda_i > 0$ $1 \leqq i \leqq n$, with $n \geqq 2$. Then equality in (3) becomes

$$M(\vec{u}; \vec{x}) = \sum_{\pi} \exp \langle \sum \lambda_i \vec{y}_i, \vec{v}_\pi \rangle = \sum_{\pi} \exp \langle \vec{y}, v_\pi \rangle = M(\vec{u}; \vec{y}) .$$

The exponential is strictly convex, hence

$$M(\vec{u}; \vec{x}) = M(u; \sum \lambda_i y_i) \leqq \sum \lambda_i M(\vec{u}; \vec{y}_i) = M(\vec{u}; \vec{y}) ,$$

since $M(\vec{u}; \vec{y})$ is unaffected by a rearrangement of $\vec{y}$. It must be that the numbers $\langle \vec{y}_i, \vec{v}_\pi \rangle$, $1 \leqq i \leqq n$, are all equal for each π. An elementary argument shows that all the components of $\vec{v}$ (hence $\vec{u}$) are the same. We could exploit this argument further if it were possible to answer the following.

Question 1. Let $x \prec y$. Then is it possible to approximate x by convex combinations $\sum \lambda_i y_i$ (y_i equimeasurable with y) such that $x \prec \sum \lambda_i y_i$? A second question which arose during the course of this investigation represents a possible generalization of (8).

Question 2. If for all positive n-vectors $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$ one has the inequality

$$\prod_{k=1}^{n} \sum_{i=1}^{n} \alpha_i^{x_k} \leqq \prod_{k=1}^{n} \sum_{i=1}^{n} \alpha_i^{y_k}$$

does it follow that $x \prec y$? The inequality is not true if we interchange i and k when $n > 3$.

REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. E. Pólya, *Some simple inequalities satisfied by convex functions*, Messenger of Math. **58** (1929), 145-152.
2. ———, *Inequalities*, Cambridge, 1952.
3. R. F. Muirhead, *Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters*, Proc. Edinburgh Math. Soc. **21** (1903), 144-157.
4. A. Ostrowski, *Sur quelques applications des fonctions convexes et concaves au sens de I. Schur*, J. Math. Pures Appl. IX Ser., **31** (1952), 253-292.
5. R. Rado, *An inequality*, J. London Math. Soc. **27** (1952), 1-6.
6. J. V. Ryff, *On the representation of doubly stochastic operators*, Pacific J. Math. **13** (1963), 1379-1386.
7. ———, *Orbits of L^1-functions under doubly stochastic transformations*, Trans. Amer. Math. Soc. **117** (1965), 92-100.
8. ———, *Extreme points of some convex subsets of* L'(0,1), Proc. Amer. Math. Soc. (To appear).

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