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A NOTE OF DILATIONS IN L^p

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The objects of study in this note are the Lebesgue spaces $L^p(1 < p < \infty)$ on the n-dimensional Euclidean space R^n. We consider a function f in one of the above-mentioned spaces, and derive results about the closure (in the relevant function space) of the set of linear combinations of functions of the form

$$f(a_1x_1 + b_1, \cdots, a_nx_n + b_n)$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n \in R$, and $a_1 \neq 0, \cdots, a_n \neq 0$.

1. Notation and main results. The Haar measure on R^n will be denoted by dx. It will be assumed normalized so that the Fourier inversion formula holds without any multiplicative constants outside the integrals involved.

If $x \in R^n$, and k is an integer such that $1 \leqq k \leqq n$, then x_k will denote the k-th component of x. Multiplication (and of course addition) in R^n is defined component-wise, in the usual manner.

We write $R^* = R^n \backslash \{x: x_k = 0 \text{ for some } k\}$.

Suppose that $1 < p < \infty$. Then q will always be written for the number satisfying

$$\frac{1}{p} + \frac{1}{q} = 1 .$$

For each integer k such that $1 \leqq k \leqq n$, J_k will denote the projection of R^n onto its k-th factor; *i.e.*

$$J_k(x) = x_k \text{ for all } x \in R^n .$$

If f is any function on R^n, and $a \in R^*, b \in R^n$, then f_b^a will denote the function defined by

$$f_b^a(x) = f(ax + b) \text{ for all } x \in R^n .$$

(The map $x \rightarrow ax + b$ is called a dilation of R^n.) Finally, the set S_f is defined by

$$S_f = \{f_b^a: a \in R^*, b \in R^n\} .$$

In what follows, several vector spaces will be considered. If $1 < p < \infty$, $L^p(R^n)$ will denote the usual Lebesgue space. $L^p(R^n)$ will be given the usual norm topology.

If f is an element of $L^p(R^n)$ we shall denote by $\boldsymbol{T}[f]$ the closed vector subspace of $L^p(R^n)$ generated by S_f.

Finally, if W is any open subset of R^n, we shall write $C^\infty(W)$ for the space of functions defined on W and indefinitely differentiable there. $\boldsymbol{D}(W)$ will denote the space of indefinitely differentiable functions with compact supports contained in W. The dual of the last space is the space $\boldsymbol{D}'(W)$ of distributions on W. For details of these spaces see e.g. Schwartz [8].

Schwartz [7] considers the space of continuous functions on the n-dimensional Euclidean space R^n equipped with the topology of uniform convergence on compact sets. He shows that if f is a function in this space, and if the linear combinations of functions of the form

$$f(ax_1 + b_1, \cdots, ax_n + b_n)\ , \qquad a, b_1, \cdots, b_n \in R$$

are not dense in the space, then f satisfies at least one distributional equation of the form

$$P(D)f = 0$$

where $P(D)$ is a nontrivial homogeneous linear partial differential operator with constant coefficients,

We shall prove the following result:

THEOREM 1. *If* $f \in L^p(R^n)$, *where* $1 < p < \infty$, *and* $f \neq 0$, *then* $\boldsymbol{T}[f] = L^p(R^n)$.

2. Discussion of problem. The Fourier transform $\hat{g}$ of a function g in $L^q(R^n)$ is defined as a distribution on R^n. (See, e.g., Schwartz [8]). It has the property of being locally a pseudomeasure; i.e., its restriction to a relatively compact open set W coincides with the restriction of some pseudomeasure to W (Gaudry [2] and [3]).

If W is an open set, $g \in L^q(R)$, $F \in \boldsymbol{D}'(W)$, and if F coincides on each relatively compact open subset of W with the Fourier transform of an element of $L^1(R^n)$, then we define $F\cdot\hat{g} \in \boldsymbol{D}'(W)$ by

$$F\cdot\hat{g}(\varphi) = \hat{g}(F\varphi) \text{ for all } \varphi \in \boldsymbol{D}(W)\ .$$

It can be shown that if W is an open set, $f \in L^p(R^n)$, $g \in L^q(R^n)$, and if $\hat{f}$ coincides on each relatively compact open subset of W with the Fourier transform of an element in $L^1(R^n)$, then

$$\widehat{f*g} = \hat{f}\cdot\hat{g} \text{ on } W\ .$$

If f and h are in $L^p(R^n)$, then from the Hahn-Banach theorem it follows that $h \in \boldsymbol{T}[f]$ if and only if

$$h*g(0) = 0$$

for all functions g in $L^q(R^n)$ such that

$$f^a * g = 0 \text{ for all } a \in R^* . \tag{2.1}$$

Therefore, to establish Theorem 1, it is sufficient to prove the following assertion: if $f \in L^p(R^n)$ for some p satisfying $1 < p < \infty$, and if g is such that (2.1) holds, then

$$\operatorname{supp} \hat{g} \subseteqq R^n \backslash R^* . \tag{2.2}$$

(We are bearing in mind the fact that $R^n \backslash R^*$ is p-thin, $1 < p \leqq \infty$. See Edwards [1].) The relation (2.2) will be established in § 4.

To prove (2.2), we shall show that if $x \in R^*$, then (2.1) implies the existence of a relatively compact neighbourhood W of x, and a function $k \in L^1(R^n)$ such that

$$\hat{k} \cdot \hat{g} = 0$$

and $|\hat{k}| > 0$ on $\bar{W}$. This will imply that $\hat{g} = 0$ on W. For there will exist a function $K \in L^1(R^n)$ such that

$$\hat{k}\hat{K} = 1 \text{ on } W$$

(Rudin [6], Theorem 2.6.2), and so if $\varphi \in \boldsymbol{D}(W)$, we have

$$\begin{aligned} \hat{g}(\varphi) &= \hat{g}(\hat{k}\hat{K}\varphi) \\ &= \hat{k}\hat{K} \cdot \hat{g}(\varphi) \\ &= \widehat{k * K * g}(\varphi) \\ &= 0 \end{aligned}$$

since $k * g = 0$. Section 3 is essentially devoted to constructing the required functions k.

3. Preliminary results. Consider any function $\varphi \in \boldsymbol{D}(R^*)$. Then if $x \in R^*$, it follows that $\varphi^{x^{-1}} \in \boldsymbol{D}(R^*)$. If s is any distribution on R^n, we define a function $s \bigtriangledown \varphi$ on R^* by

$$s \bigtriangledown \varphi(x) = s(\varphi^{x^{-1}}) \text{ for all } x \in R^* .$$

We then have

LEMMA 1. *If $\varphi \in \boldsymbol{D}(R^*)$ and $s \in \boldsymbol{D}'(R^n)$, then $s \bigtriangledown \varphi \in C^\infty(R^*)$.*

Proof. (cf. Hörmander [5], Theorem 1.6.1.)
First we show that $s \bigtriangledown \varphi$ is continuous.
Suppose that ${}^j x \to {}^0 x \in R^*$. Then $\varphi^{{}^j x^{-1}} \to \varphi^{{}^0 x^{-1}}$ in $\boldsymbol{D}(R^*)$. For let

$$a = \sup\{|x_k| : x \in \operatorname{supp} \varphi, 1 \leqq k \leqq n\} < \infty$$
$$b = \inf\{|x_k| : x \in \operatorname{supp} \varphi, 1 \leqq k \leqq n\} > 0$$

and let $A, B > 0$ be numbers such that

$$B/b < |{}^jx_k| < A/a , \qquad 1 \leqq k \leqq n$$

for all j. Then if $y \in \operatorname{supp} \varphi^{{}^jx^{-1}}$, we have $y/{}^jx \in \operatorname{supp} \varphi$. This implies that

$$b \leqq |y_k/{}^jx_k| \leqq a , \qquad 1 \leqq k \leqq n ,$$

and so

$$|{}^jx_k|\, b \leqq |y_k| \leqq |{}^jx_k|\, a , \qquad 1 \leqq k \leqq n .$$

It follows that $B < |y_k| < A$. Hence all the sets $\operatorname{supp} \varphi^{{}^jx^{-1}}$ are contained in a fixed compact subset of R^*. Furthermore, since

$$D_k(\varphi^{x^{-1}}) = \frac{1}{x_k}(D_k\varphi)^{x^{-1}} , \qquad 1 \leqq k \leqq n ,$$

it is easily shown that for each multi-index α,

$$\lim_j D^\alpha(\varphi^{{}^jx^{-1}}) = D^\alpha(\varphi^{{}^0x^{-1}})$$

uniformly. Thus $\varphi^{{}^jx^{-1}} \to \varphi^{{}^0x^{-1}}$ in $\boldsymbol{D}(R^*)$ and, since s is continuous, we have

$$\lim_j s \bigtriangledown \varphi({}^jx) = s \bigtriangledown \varphi({}^0x) .$$

Hence $s \bigtriangledown \varphi$ is continuous on R^*.

To complete the proof of the lemma, it is sufficient, in view of the above, to show that if $1 \leqq k \leqq n$, then

$$D_k(s \bigtriangledown \varphi) = -1/J_k \cdot s \bigtriangledown J_kD_k\varphi \text{ on } R^* . \tag{3.1}$$

The required result will then follow by induction.

Thus, let e_k be the unit vector along the k-axis and consider the quotient

$$[s \bigtriangledown \varphi(x + he_k) - s \bigtriangledown \varphi(x)]/h = s[\varphi^{(x+he_k)^{-1}} - \varphi^{x^{-1}}]/h$$

where $x \in R^*$ and $h \neq 0$. We have

$$\lim_{h\to 0} [\varphi^{(x+he_k)^{-1}} - \varphi^{x^{-1}}]/h = -1/x_k \cdot (J_kD_k\psi)^{x^{-1}} \text{ in } \boldsymbol{D}(R^*) . \tag{3.2}$$

To verify this, consider any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$. We have

$$\begin{aligned} &D^\alpha[\varphi^{(x+he_k)^{-1}} - \varphi^{x^{-1}}]/h \\ &= \Big[1\Big/\prod_{j\neq k} x_j^{\alpha_j}\Big][(1/(x_k + h)^{\alpha_k}) \cdot (D^\alpha\varphi)^{(x+he_k)^{-1}} - (1/x_k^{\alpha_k}) \cdot (D^\alpha\varphi)^{x^{-1}}]/h . \end{aligned}$$

The last expression converges pointwise to $D^\alpha[-(1/x_k)(J_kD_k\varphi)^{x^{-1}}]$.

The convergence is in fact uniform. This may be deduced from the fact that if ψ is any function in $\boldsymbol{D}(R^n)$, and h is a positive number, then

$$|[\psi(y + he_k) - \psi(y)]/h - D_k\psi(y)| < |h|\cdot\|D_k^2\psi\|_\infty$$

which follows easily via the mean-value theorem. This establishes (3.2), and (3.1) follows. Thus the proof of Lemma 1 is complete.

COROLLARY. *If W is any relatively compact open set such that $\bar{W} \subseteqq R^*$, and if $\varphi \in \boldsymbol{D}(R^*)$ and $s \in \boldsymbol{D}'(R^n)$, then there exists a function k in $L^1(R^n)$ such that*

$$s \bigtriangledown \psi = \hat{k} \text{ on } W .$$

Proof. In fact we may take for k any function of the form

$$(s \bigtriangledown \varphi\cdot\psi)^\vee$$

where $\psi \in \boldsymbol{D}(R^*)$ and $\psi = 1$ on $\bar{W}$. [Here and elsewhere, $^\vee$ denotes the inverse Fourier transform:

$$\check{h}(x) = \int_{R^n} e^{2\pi ixy}h(y)dy \text{ for all } h \in L^1(R^n)] .$$

LEMMA 2. *If $f \in L^p(R^n)$, $g \in L^q(R^n)$ and $\varphi, \psi \in \boldsymbol{D}(R^*)$, then*

$$\hat{f}\bigtriangledown\varphi\cdot\hat{g}(\psi) = \int_{R^n}(\varphi(t)/|J_1(t)|\cdots|J_n(t)|)\left\{\int_{R^n}\widehat{\psi|J_1\cdots J_n|}(x)\cdot g * f^{t^{-1}}(x)dx\right\}dt .$$

Proof. Choose a sequence $\{f_j\}$ of functions in $L^1(R^n) \cap L^p(R^n)$ such that

$$\lim_j f_j = f \text{ in } L^p(R^n) .$$

Then, if ψ is any function in $\boldsymbol{D}(R^*)$, we have

$$\lim_j \hat{f}_j \bigtriangledown \varphi\cdot\psi = \hat{f} \bigtriangledown \varphi\cdot\psi \text{ in } \boldsymbol{D}(R^*) . \tag{3.3}$$

For, if α is any multi-index, the Leibnitz formula for the differentiation of a product shows that $D^\alpha[(\hat{f}_j - \hat{f}) \bigtriangledown \varphi\cdot\psi]$ is a sum of terms of the form

$$A\cdot D^\beta[(\hat{f}_j - \hat{f}) \bigtriangledown \varphi]D^{\alpha-\beta}\psi$$

where $\beta_i \leqq \alpha_i$, $i = 1, \cdots, n$, and A is a constant depending only on α and β. Thus we are reduced to proving that if α is any multi-index, then

(3.4) $$\lim_j D^\alpha[(\hat{f}_j - \hat{f}) \nabla \varphi] = 0 \text{ uniformly on } \operatorname{supp} \psi .$$

Now, quite generally, if s is a distribution on R^n, $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index, and φ a function in $\boldsymbol{D}(R^*)$, then

$$D^\alpha(s \nabla \varphi) = (1/J_1^{\alpha_1} \cdots J_n^{\alpha_n}) \sum_\beta \{a_\beta s \nabla (J_1^{\beta_1} \cdots J_n^{\beta_n} D^\beta \varphi)\}$$

where the a_β are constants depending only on α and β, and the summation is carried out over all multi-indices β such that $\beta_1 \leqq \alpha_1, \cdots, \beta_n \leqq \alpha_n$. This is easily shown by induction, using (3.1). Since $J_1, \cdots, J_n$ are bounded away from zero on $\operatorname{supp} \psi$, it suffices, in order to establish (3.4), to show that for every multi-index α

(3.5) $$\lim_j (\hat{f}_j - \hat{f}) \nabla (J_1^{\alpha_1} \cdots J_n^{\alpha_n} D^\alpha \varphi) = 0 \text{ uniformly on } \operatorname{supp} \psi .$$

Thus, let

$$a = \sup \{ | x_k | : x \in \operatorname{supp} \psi, 1 \leqq k \leqq n \} .$$

Then, if $x \in \operatorname{supp} \psi$, we have

$$\begin{aligned} &|[(\hat{f}_j - \hat{f}) \nabla (J_1^{\alpha_1} \cdots J_n^{\alpha_n} D^\alpha \varphi)](x)| \\ &= \left| \int_{R^n} (f_j - f)(yx^{-1}) \cdot \widehat{J_1^{\alpha_1} \cdots J_n^{\alpha_n} D^\alpha \varphi}(y) dy \right| \\ &\leqq \| f_j - f \|_p \cdot a^n \cdot \| \widehat{J_1^{\alpha_1} \cdots J_n^{\alpha_n} D^\alpha \varphi} \|_q \end{aligned}$$

From this, (3.5) follows immediately, and hence (3.4) and (3.3).

Using (3.3), it is seen that

(3.6) $$\begin{aligned} \hat{f} \nabla \varphi \cdot \hat{g}(\psi) &= \lim_j \hat{g}(\hat{f}_j \nabla \varphi \cdot \psi) \\ &= \lim_j \int_{R^n} g(x) \cdot [\hat{f}_j \nabla \varphi \cdot \psi]^\vee(-x) dx . \end{aligned}$$

Now

$$\begin{aligned} &[\hat{f}_j \nabla \varphi \cdot \psi]^\vee(-x) \\ &= \int_{R^n} e^{-2\pi ixy} \hat{f}_j \nabla \varphi(y) \cdot \psi(y) dy \\ &= \int_{R^n} e^{-2\pi ixy} \left\{ \int_{R^n} \hat{f}_j(t) \varphi(ty^{-1}) dt \right\} \psi(y) dy \\ &= \int_{R^n} e^{-2\pi ixy} \psi(y) \left\{ \int_{R^n} \hat{f}_j(yt) \varphi(t) \, | J_1(y) | \cdots | J_n(y) | \, dt \right\} dy \\ &= \int_{R^n} \varphi(t) \left\{ \int_{R^n} \hat{f}_j(yt) \psi(y) \, | J_1(y) | \cdots | J_n(y) | \, e^{-2\pi ixy} dy \right\} dt \\ &= \int_{R^n} (\varphi(t) / | J_1(t) | \cdots | J_n(t) |) \cdot \widehat{\psi | J_1 \cdots J_n |} * f_j^{t^{-1}}(x) dt . \end{aligned}$$

Substituting this in (3.6), we have

$$f \bigtriangledown \varphi \cdot \hat{g}(\psi)$$

$$(3.7)\quad \begin{aligned} &= \lim_j \int_{R^n} g(x)\left\{\int_{R^n} (\varphi(t)/|J_1(t)| \cdots |J_n(t)|)\widehat{\psi|J_1 \cdots J_n|} * f_j^{t^{-1}}(x)dt\right\}dx \\ &= \lim_j \int_{R^n} (\varphi(t)/|J_1(t)| \cdots |J_n(t)|)\left\{\int_{R^n} \widehat{\psi|J_1 \cdots J_n|}(x) g * f_j^{t^{-1}}(x)dx\right\}dt . \end{aligned}$$

Now, if

$$a = \sup\{|t_k| : t \in \operatorname{supp}\varphi, 1 \leqq k \leqq n\} ,$$

then if $t \in \operatorname{supp}\varphi$, we have

$$\begin{aligned} &\left|\int_{R^n} \widehat{\psi|J_1 \cdots J_n|}(x)\cdot g * f_j^{t^{-1}}(x)dx - \int_{R^n} \widehat{\psi|J_1 \cdots J_n|}(x) g * f^{t^{-1}} dx\right| \\ &\leqq \|\widehat{\psi|J_1 \cdots J_n|}\|_1 \cdot \|g * (f_j^{t^{-1}} - f^{t^{-1}})\|_\infty \\ &\leqq \|\widehat{\psi|J_1 \cdots J_n|}\|_1 \cdot \|g\|_q \cdot \|f_j^{t^{-1}} = f^{t^{-1}}\|_p \\ &\leqq \|\widehat{\psi|J_1 \cdots J_n|}\|_1 \cdot \|g\|_q \|f_j - f\|_p \cdot a^n . \end{aligned}$$

Using this, and (3.7) we see that

$$\begin{aligned} &\hat{f} \bigtriangledown \varphi \cdot \hat{g}(\psi) \\ &= \int_{R^n} (\varphi(t)/|J_1(t)| \cdots J_n|(t)|)\left\{\int_{R^n} \widehat{\psi|J_1 \cdots J_n|}(x)\cdot g * f^{t^{-1}}(x)dx\right\}dt \end{aligned}$$

This completes the proof of Lemma 2.

COROLLARY. *If $\varphi \in \boldsymbol{D}(R^*)$, $f \in L^p(R^n)$, $g \in L^q(R^n)$, and if*

$$f^a * g = 0 \textit{ for all } a \in R^*$$

then $\hat{f} \bigtriangledown \varphi \cdot \hat{g} = 0$ on R^*.

LEMMA 3. *Suppose that $f \in L^p(R^n)$ and that $R^* \cap \operatorname{supp}\hat{f} \neq \varnothing$. Then if $g \in L^q(R^n)$ is such that*

$$f^a * g = 0 \textit{ for all } a \in R^* ,$$

We have

$$\operatorname{supp}\hat{g} \subseteqq R^n \backslash R^* .$$

Proof. First we observe that

$$\operatorname{supp}\widehat{f^a} = a \cdot \operatorname{supp}\hat{f}$$

and hence, since $R^* \cap \operatorname{supp}\hat{f} \neq \varnothing$,

$$\bigcup_{a \in R^*} \operatorname{supp} \widehat{f^a} \supseteqq R^* . \tag{3.8}$$

Now suppose that $x \in R^*$. By (3.8), $x \in \operatorname{supp} \widehat{f^b}$ (say). Choose a relatively compact neighbourhood W of x such that $\bar{W} \subseteqq R^*$. There exists a function $\varphi \in \boldsymbol{D}(W)$ such that

$$\widehat{f^b}(\varphi) \neq 0$$

i.e.

$$\widehat{f^b} \bigtriangledown \varphi^{x^{-1}}(x) \neq 0 .$$

This implies that $\widehat{f^b} \bigtriangledown \varphi^{x^{-1}}$ is bounded away from 0 on a neighbourhood of x. Since $\widehat{f^b} \bigtriangledown \varphi^{x^{-1}} \in C^\infty(R^*)$ (by Lemma 1) and (by the corollary to Lemma 2)

$$\widehat{f^b} \bigtriangledown \varphi^{x^{-1}} \cdot \hat{g} = 0 \text{ on } R^*$$

the corollary to Lemma 1 and the reasoning indicated in § 2 together entail that $x \notin \operatorname{supp} \hat{g}$. Thus

$$\operatorname{supp} \hat{g} \subseteqq R^n \backslash R^*$$

as we wished to show.

4. **Proof of Theorem 1.** We can now prove Theorem 1.

Let $f \in L^p(R^n)$, $(1 < p < \infty)$, $f \neq 0$, and suppose that $g \in L^q(R^n)$ is such that

$$f^a * g = 0 \text{ for all } a \in R^* .$$

Since $R^n \backslash R^*$ is q-thin if $1 < q < \infty$, we deduce that

$$\operatorname{supp} \hat{f} \cap R^* \neq \varnothing .$$

Then, by Lemma 3,

$$\operatorname{supp} \hat{g} \subseteqq R^n \backslash R^*$$

and so (since $R^n \backslash R^*$ is p-thin) $g = 0$.

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