PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 1, 1967

OPERATORS AND INNER FUNCTIONS

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Let $L^2_{\mathscr{H}}$ denote the Hilbert space of weakly measurable functions on the unit circle of the complex plane with values in a separable Hilbert space $\mathscr{H}$, and whose pointwise norms are square integrable with respect to Lebesgue measure. We are concerned with invariant subspaces of $L^2_{\mathscr{H}}$, by which we mean closed subspaces invariant under the right shift operator, and will be especially interested in those invariant subspaces which arise from a bounded operator on $\mathscr{H}$, using a construction due to Rota and Lowdenslager. We begin by relating the determinant of the "Rota inner function" of an operator to the characteristic polynomial of the operator, along with a similar interpretation of the minimal polynomial, when $\mathscr{H}$ is finite dimensional. We then consider some general questions about intersections and unions of invariant subspaces, and use the results to establish a factorization theorem for finite dimensional inner functions (the set of all $\mathscr{U}^*\mathscr{V}$, where $\mathscr{U}$, $\mathscr{V}$ are inner, is the same as the set of all $\mathscr{U}\mathscr{V}^*$). We show this theorem false if $\mathscr{H}$ is infinite dimensional, by exhibiting invariant subspaces $\mathscr{M}$, $\mathscr{N}$ (which are also Rota subspaces) such that $\mathscr{M} \cap \mathscr{N} = (0)$—a result of independent interest.

Rota subspaces seem to exhibit all of the pleasant and all of the pathological properties of invariant subspaces in general, and enable one to use properties of operators to provide counterexamples for general questions about invariant subspaces. It is also to be expected that invariant subspaces and their corresponding inner functions, like the analogous theory of characteristic matrix functions [8], can be used to study operators. Our results go in both directions, though they are of more interest, we believe, when using operators to study invariant subspaces.

A word about notation. $H^2_{\mathscr{H}}$ will denote the subspace of $L^2_{\mathscr{H}}$ consisting of all functions with weak analytic extensions to the disk. The inner product of $\mathscr{H}$ will be denoted by (x, y) and the norm of $\mathscr{H}$ by $|x| = (x, x)^{\frac{1}{2}}$. For the inner product in $L^2_{\mathscr{H}}$ we will write

$$[F, G] = \int (F(e^{ix}), G(e^{ix}))d\sigma(x) ,$$

where $d\sigma = (1/2\pi)dx$ is normalized Lebesgue measure on the circle. The norm of $L^2_{\mathscr{H}}$ will be denoted by $\|F\| = [F, F]^{\frac{1}{2}}$. T will always be a bounded operator on $\mathscr{H}$ whose uniform norm $\|T\|$ is less than one. We emphasize that when discussing subspaces of $L^2_{\mathscr{H}}$ the term invariant means invariant under the right shift operator. For the definitions and basic properties of $L^2_{\mathscr{H}}$ and $H^2_{\mathscr{H}}$, consult Helson's book [2].

1. Rota subspaces. The first section will be devoted to the proof of Theorem 10 (which gives the relationship between the characteristic polynomial of an operator and the determinant of the corresponding inner function), and to related results. We begin with a sketch of Rota's construction.

Given $e \in \mathscr{H}$ and a fixed operator T on $\mathscr{H}$ such that $\| T \| < 1$, define $F_e \in L^2_{\mathscr{H}}$ by the formula

$$F_e(e^{ix}) = \sum_{n=0}^{\infty} (T^n e) e^{nix} = (I - e^{ix} T)^{-1} e \, .$$

The correspondence $e \leftrightarrow F_e$ defines a bounded one-to-one linear mapping onto the closed subspace $\mathscr{K} = \{F_e \colon e \in \mathscr{H}\}$ of $H^2_{\mathscr{H}}$. Even though Rota considered only the subspace $\mathscr{K}$, we are going to follow Lowdenslager and pass to the orthogonal complement $\mathscr{M}$ of $\mathscr{K}$ in $H^2_{\mathscr{H}}$. $\mathscr{M}$ is invariant under the right shift operator and we shall refer to $\mathscr{M}$ as the Rota subspace of T. T has a proper invariant subspace if and only if $\mathscr{M}$ is not maximal as an invariant subspace—a fact with which we are not directly concerned, but which serves as an important motivation for the construction. For details, see [2, pp. 103–107].

DEFINITION. Let $\mathscr{L}^\infty_{\mathscr{H}}$ be the set of weakly measurable functions A defined on the circle whose values are a.e. bounded operators on $\mathscr{H}$, and such that $\operatorname{ess\,sup} \| A(e^{ix}) \| < \infty$. If $A \in \mathscr{L}^\infty_{\mathscr{H}}$, we say $A \in \mathscr{H}^\infty_{\mathscr{H}}$ if $AH^2_{\mathscr{H}} \subseteq H^2_{\mathscr{H}}$.

It is easy to see that $A \in \mathscr{H}^\infty_{\mathscr{H}}$ if and only if $A \in \mathscr{L}^\infty_{\mathscr{H}}$ and $Ae \in H^2_{\mathscr{H}}$ for all $e \in \mathscr{H}$, which is in turn equivalent to $(Ae, f) \in H^2$ (or H^∞) for all $e, f \in \mathscr{H}$.

DEFINITION. An inner function is an element $\mathscr{U}$ of $\mathscr{H}^\infty_{\mathscr{H}}$ such that $\mathscr{U}(e^{ix})$ is a.e. a unitary operator on $\mathscr{H}$.

Note that if $\mathscr{H}$ is one dimensional, the inner functions are just those of Beurling.

Lax's theorem says that every invariant subspace of $H^2_{\mathscr{H}}$ is of the form $\mathscr{U} H^2_{\mathscr{H}}$, where $\mathscr{U}$ is a.e. a partial isometry of $\mathscr{H}$. When $\mathscr{U}$ is inner, i.e., unitary, we say that the subspace $\mathscr{U} H^2_{\mathscr{H}} = \mathscr{M}$ is of full range—which is equivalent to the existence of an at most countable collection of functions $F_1, F_2, \cdots$ in $\mathscr{M}$ such that $\{F_j(e^{ix})\}$ spans $\mathscr{H}$ a.e. We say that an arbitrary subspace of $L^2_{\mathscr{H}}$ is of full range if this last condition holds. For details consult Helson [2, pp. 57–68].

DEFINITION. We shall call any inner function $\mathscr{U}_T$ such that $\mathscr{U}_T H^2_{\mathscr{H}}$ is the Rota subspace of T, a Rota inner function of T. (See Corollary

4.) $\mathscr{U}_T$ is unique up to a constant unitary factor on the right. (See [2, p. 64].)

PROPOSITION 1. If $A \in \mathscr{L}^\infty_{\mathscr{H}}$, A is invertible a.e. and $A^{-1} \in \mathscr{L}^\infty_{\mathscr{H}}$, then $AH^2_{\mathscr{H}}$ is a closed invariant subspace of $L^2_{\mathscr{H}}$ of full range.

Proof. Clearly $e^{ix}AH^2_{\mathscr{H}} \subseteqq AH^2_{\mathscr{H}}$. Since A is invertible, the functions $\{A(e^{ix})e_n\}$, where $\{e_n\}$ is a basis for $\mathscr{H}$, span $\mathscr{H}$ a.e., and so $AH^2_{\mathscr{H}}$ is of full range. To see that $AH^2_{\mathscr{H}}$ is closed, let $\{AF_n\}$ be a Cauchy sequence in the metric of $L^2_{\mathscr{H}}$, and observe that since $A^{-1} \in \mathscr{L}^\infty_{\mathscr{H}}$,

$$\| AF_n \| \geqq \frac{1}{M} \| F_n \| , \quad \text{where } M = \operatorname{ess\,sup} \| A^{-1}(e^{ix}) \| .$$

Thus $\{F_n\}$ is convergent, say to F, and $AF_n \to AF$.

PROPOSITION 2. If $A \in \mathscr{H}^\infty_{\mathscr{H}}$ and $A^{-1} \in \mathscr{H}^\infty_{\mathscr{H}}$, then $AH^2_{\mathscr{H}} = H^2_{\mathscr{H}}$.

Proof. $A^{-1}H^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}} \Rightarrow AA^{-1}H^2_{\mathscr{H}} \subseteqq AH^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}}$.

THEOREM 3. *The Rota subspace $\mathscr{M}$ of T is $(e^{ix} - T^*)H^2_{\mathscr{H}}$.*

Proof. $\mathscr{M}$ consists of all $F \in H^2_{\mathscr{H}}$ such that

$$[F, F_e] = \int (F, (I - e^{ix}T)^{-1}e)d\sigma = 0$$

for all $e \in \mathscr{H}$. If we knew that for all $F \in \mathscr{M}$, $(I - e^{-ix}T^*)^{-1}F \in H^2_{\mathscr{H}}$, we could conclude that $(I - e^{-ix}T^*)^{-1}F \in e^{ix}H^2_{\mathscr{H}}$, from which the theorem follows immediately. Suppose $F \in \mathscr{M}$ and let $G = (I - e^{-ix}T^*)^{-1}F$ have Fourier series $\sum_{-\infty}^{\infty} \varphi_k e^{kix}$. Then $F = \sum_{-\infty}^{\infty} (\varphi_k - T^*\varphi_{k+1})e^{kix}$, and since $F \in H^2_{\mathscr{H}}$, we must have $\varphi_k = T^*\varphi_{k+1}$ for $k = -1, -2, \cdots$. Since $F \in \mathscr{M}$

$$[(I - e^{-ix}T^*)^{-1}F, e] = [G, e] = (\varphi_0, e) = 0$$

for all $e \in \mathscr{H}$. Therefore $\varphi_0 = 0$, and $G \in H^2_{\mathscr{H}}$ as we needed to show.

We get a result of Helson's as our first corollary.

COROLLARY 4. *A Rota subspace is always of full range.*

Proof. Since $\| T \| < 1$, $(e^{ix} - T^*)$ is invertible a.e., and therefore the functions $\{(e^{ix} - T^*)e_n\}$, where $\{e_n\}$ is a basis for $\mathscr{H}$, span $\mathscr{H}$ a.e.

COROLLARY 5. *If T is a normal operator, then a Rota inner function of T is*

$$\mathscr{U}_T(e^{ix}) = (e^{ix} - T^*)(I - e^{ix}T)^{-1} .$$

Proof. It is easy to check that $\mathscr{U}_T$ as defined by this formula is unitary. By Proposition 2, $(I - e^{ix}T)^{-1}H^2_{\mathscr{H}} = H^2_{\mathscr{H}}$, and the corollary then follows from the theorem.

The next two lemmas are due to Helson for A an inner function.

LEMMA 6. *Let $\mathscr{H}$ be finite dimensional, $A \in \mathscr{H}^\infty_{\mathscr{H}}$, $A^{-1} \in \mathscr{L}^\infty_{\mathscr{H}}$. Then* $\det A \in H^\infty$, $(1/\det A) \in L^\infty$ *and*

$$(\det A)H^2_{\mathscr{H}} \subseteqq AH^2_{\mathscr{H}} .$$

Proof. $A \in \mathscr{H}^\infty_{\mathscr{H}}$ is equivalent to $(Ae, f) \in H^\infty$ for all $e, f \in \mathscr{H}$, from which it follows that $\det A \in H^\infty$. Similarly, $\det A^{-1} = 1/\det A \in L^\infty$. Thus $(\det A)H^2_{\mathscr{H}}$ is a closed invariant subspace of $H^2_{\mathscr{H}}$ by Proposition 1. Let ${}^tA$ be the matrix of cofactors of A transposed. Then $A^{-1} = (1/\det A)^tA$, and $(\det A)A^{-1} \in \mathscr{H}^\infty_{\mathscr{H}}$. Thus $A^{-1}(\det A)H^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}}$, or

$$(\det A)H^2_{\mathscr{H}} \subseteqq AH^2_{\mathscr{H}} .$$

LEMMA 7. *Let $\mathscr{H}$ be N dimensional, $A \in \mathscr{H}^\infty_{\mathscr{H}}$, $A^{-1} \in \mathscr{L}^\infty_{\mathscr{H}}$ and let p be any scalar inner function such that $pH^2_{\mathscr{H}} \subseteqq AH^2_{\mathscr{H}}$. Then*

$$p^N H^2_{\mathscr{H}} \subseteqq (\det A)H^2_{\mathscr{H}} .$$

Proof. $A^{-1}pH^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}}$ implies that

$$\det(A^{-1}p)H^2_{\mathscr{H}} = (\det A^{-1})p^N H^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}} .$$

COROLLARY 8. *If $AH^2_{\mathscr{H}} = H^2_{\mathscr{H}}$, then* $(\det A)H^2_{\mathscr{H}} = H^2_{\mathscr{H}}$.

Proof.

$$1^N H^2_{\mathscr{H}} \subseteqq (\det A)H^2_{\mathscr{H}} \subseteqq H^2_{\mathscr{H}} .$$

THEOREM 9. *$\mathscr{H}$ finite dimensional. Let $A, B \in \mathscr{H}^\infty_{\mathscr{H}}$ and A^{-1}, $B^{-1} \in \mathscr{L}^\infty_{\mathscr{H}}$ and suppose that $AH^2_{\mathscr{H}} = BH^2_{\mathscr{H}}$. Then* $(\det A)/(\det B)$ *is an invertible element in H^∞ (and is therefore an outer function in the sense of Beurling).*

Proof. Recall that an outer function f is defined by the properties that $f \in H^2$ and $fH^2 \cap H^2$ is dense in H^2. $AH^2_{\mathscr{H}} = BH^2_{\mathscr{H}}$ means

$B^{-1}AH^2_{\mathscr{H}} = H^2_{\mathscr{H}}$ and by the corollary $(\det B^{-1})(\det A)H^2_{\mathscr{H}} = H^2_{\mathscr{H}}$.

The next theorem is our goal.

THEOREM 10. *Let $\mathscr{H}$ be N dimensional and let the characteristic polynomial of T^* be $\prod_{j=1}^N(z - \lambda_j) = \prod_{j=1}^N(e^{ix} - \lambda_j)$. Then if $\mathscr{U}_T$ is any Rota inner function of T, we have, for some α such that $|\alpha| = 1$,*

$$\det \mathscr{U}_T = \alpha \prod_{j=1}^N \left(\frac{e^{ix} - \lambda_j}{1 - \bar{\lambda}_j e^{ix}}\right).$$

Proof. Since

$$\mathscr{U}_T H^2_{\mathscr{H}} = (e^{ix} - T^*)H^2_{\mathscr{H}} ,$$

it follows that $\det(e^{ix} - T^*)/\det \mathscr{U}_T$ is outer. $\mathrm{Det}(e^{ix} - T^*)$ is by definition the characteristic polynomial of T^*. Since an outer function can have no zeros or poles inside the disk, it follows that $\det \mathscr{U}_T$ must have the same zeros to the same multiplicities as $\det(e^{ix} - T^*)$. The only inner function with these properties is a scalar of modulus one times $\prod_{j=1}^N (e^{ix} - \lambda_j)(1 - \bar{\lambda}_j e^{ix})^{-1}$.

We turn next to an interpretation of the minimal polynomial of T.

Let $\mathscr{U}$ be an inner function and suppose $\mathscr{U}H^2_{\mathscr{H}} \supseteq qH^2_{\mathscr{H}}$ where q is a scalar inner function. It follows from Zorn's lemma and Beurling's theorem that $\mathscr{U}H^2_{\mathscr{H}}$ contains a maximal subspace of the form $qH^2_{\mathscr{H}}$. The q associated with this subspace will have the property that it has the smallest set of zeros and the smallest singular measure (see [6, p. 67]) of any inner function p such that $\mathscr{U}H^2_{\mathscr{H}} \supseteq pH^2_{\mathscr{H}}$. In the spirit of this last property we call this q *the minimal inner function of $\mathscr{U}$ or of the subspace $\mathscr{U}H^2_{\mathscr{H}}$*. Helson has called this q the *characteristic* inner function of $\mathscr{U}$, but we feel that Theorems 10 and 12 justify our terminology.

LEMMA 11 (*Helson*). *The Rota subspace $\mathscr{M}$ of T consists of all $F \in H^2_{\mathscr{H}}$, $F = \sum_0^\infty \varphi_k e^{kix}$, such that*

$$\sum_0^\infty (T^*)^k \varphi_k = 0 .$$

Proof. $\mathscr{M}$ is all $F = \sum \varphi_k e^{kix}$ such that $[F, \sum (T^k e)e^{kix}] = 0$ for all $e \in \mathscr{H}$.

$$[F, \sum (T^k e)e^{kix}] = \sum (\varphi_k, T^k e) = \sum ((T^*)^k \varphi_k, e)$$

and since this expression is 0 for all e, the lemma follows.

THEOREM 12. *Let $\mathscr{H}$ be finite dimensional and let the minimal polynomial of T^* be $\prod_{j=1}^{r}(e^{ix}-\lambda_j)$. Then the minimal inner function of the Rota subspace $\mathscr{M}$ of T is (up to a scalar factor of modulus* 1)

$$q(e^{ix}) = \prod_{j=1}^{r} \frac{e^{ix}-\lambda_j}{1-\bar{\lambda}_j e^{ix}} .$$

Proof. Note that $qH^2_{\mathscr{H}} = \prod_{j=1}^{r}(e^{ix}-\lambda_j)H^2_{\mathscr{H}}$ since $\prod_{j=1}^{r}(1-\lambda_j e^{ix})^{-1}$ is outer (or, if you will, by Proposition 2). To prove that $\mathscr{M} \supseteqq qH^2_{\mathscr{H}}$ we must show that for all $\sum_0^\infty \varphi_k e^{kix}$ such that $\sum |\varphi_k|^2 < \infty$, $\prod_{j=1}^{r}(e^{ix}-\lambda_j)\sum \varphi_k e^{kix} \in \mathscr{M}$. In particular we must show that for all $\varphi \in \mathscr{H}$

$$[\prod (e^{ix}-\lambda_j)]\varphi \in \mathscr{M} .$$

Since $\mathscr{M}$ is invariant, this last condition is also sufficient. By the lemma, if we set $\prod_{j=1}^{r}(e^{ix}-\lambda_j) = \sum_{j=1}^{r} a_j e^{jix}$, then the above is equivalent to the requirement that

$$\sum_{j=1}^{r} a_j (T^*)^j \varphi = 0 \quad \text{for all } \varphi \in \mathscr{M} .$$

Since $\sum a_j z^j$ is the minimal polynomial of T^*, this condition is indeed satisfied. Thus $\mathscr{M} \supseteqq qH^2_{\mathscr{H}}$ for this particular q and we may also conclude that the minimal inner function is a finite Blaschke product. By what we have shown, if $\mathscr{M} \supseteqq pH^2_{\mathscr{H}}$ for p a finite Blaschke product, say $p(e^{ix}) = \prod_{j=1}^{k}(e^{ix}-\beta_j)(1-\bar{\beta}_j e^{ix})^{-1}$, then $\prod_{j=1}^{k}(T^*-\beta_j) = 0$. Clearly the largest such subspace is $qH^2_{\mathscr{H}}$.

2. Potopov subspaces. As a corollary to Theorem 3, we saw that if T is a normal operator, then its Rota inner function is just $\mathscr{U}_T(e^{ix}) = (e^{ix}-T^*)(I-e^{ix}T)^{-1}$. While we cannot find a formula for $\mathscr{U}_T$ in general, we can, using a theorem of Potopov, define a new correspondence between operators and inner functions having all of the essential properties of the old correspondence, and the advantage that the Potopov inner functions are given explicitly by a formula.

For our purposes, Potopov's theorem (actually a special case of it) can be stated as follows.

THEOREM 13. *Let T be a bounded operator with $\|T\| < 1$. Then*

$$\mathscr{V}_T(e^{ix}) = (I-T^*T)^{-\frac{1}{2}}(e^{ix}-T^*)(I-e^{ix}T)^{-1}(I-TT^*)^{\frac{1}{2}}$$

is an inner function.

For a proof, see [9, p. 145]. We comment that even though

Potopov only claims his result when $\mathscr{H}$ is finite dimensional, his proof that $I - \mathscr{V}_T^* \mathscr{V}_T = 0$ is valid when $\mathscr{H}$ is infinite dimensional, and, using his techniques one can easily show that $I - \mathscr{V}_T \mathscr{V}_T^* = 0$ as well.

COROLLARY 14. *If T is normal, then* $\mathscr{U}_T H^2_{\mathscr{H}} = \mathscr{V}_T H^2_{\mathscr{H}}$.

DEFINITION. We call $\mathscr{V}_T$ the Potopov inner function of T and $\mathscr{V}_T H^2_{\mathscr{H}}$ the Potopov subspace of T.

THEOREM 15. *If $\mathscr{H}$ is finite dimensional, then up to a constant factor of modulus* 1, $\det \mathscr{V}_T = \det \mathscr{U}_T$, *and the minimal inner function of $\mathscr{U}_T$ is the same as the minimal inner function of $\mathscr{V}_T$.*

Proof.

$$\det \mathscr{V}_T = \det(I - T^*T)^{-\frac{1}{2}} \det(e^{ix} - T^*) \det(I - e^{ix}T)^{-1} \det(I - TT^*)^{\frac{1}{2}} .$$

The first and last factors are constants. The second is the characteristic polynomial $\prod_{j=1}^{N} (e^{ix} - \lambda_j)$ of T^*. The third factor is just

$$\det[e^{ix}(e^{-ix} - T)]^{-1} = e^{-Nix}\left[\prod_{j=1}^{N} (e^{-ix} - \bar{\lambda}_j)\right]^{-1} = \prod_{j=1}^{N} (1 - e^{ix}\bar{\lambda}_j)^{-1} .$$

The second assertion follows from the fact that since $\mathscr{V}_T H^2_{\mathscr{H}} = (I - T^*T)^{-\frac{1}{2}} \mathscr{U}_T H^2_{\mathscr{H}}$, $\mathscr{V}_T H^2_{\mathscr{H}} \supseteq qH^2_{\mathscr{H}}$ if and only if $\mathscr{U}_T H^2_{\mathscr{H}} \supseteq qH^2_{\mathscr{H}}$.

Rota's construction can be imitated by defining the correspondence $e \leftrightarrow G_e$, where $e \in \mathscr{H}$, $G_e \in H^2_{\mathscr{H}}$, and $G_e(e^{ix}) = (I - T^*T)^{-\frac{1}{2}} \sum_{n=0}^{\infty} (T^n e) e^{nix}$. The complement in $H^2_{\mathscr{H}}$ of the set of all such G_e is then the Potopov subspace of T, and T has a proper invariant subspace if and only if $\mathscr{V}_T H^2_{\mathscr{H}}$ is not maximal as an invariant subspace of $H^2_{\mathscr{H}}$.

3. Intersections and unions of invariant subspaces. We turn now to some general questions about invariant subspaces of $L^2_{\mathscr{H}}$. Not all closed invariant subspaces $\mathscr{M}$ of $L^2_{\mathscr{H}}$ are of the form $\mathscr{U} H^2_{\mathscr{H}}$, where $\mathscr{U}$ is a.e. a partial isometry of $\mathscr{H}$, and in order for Lax's theorem to hold we need the additional hypothesis (always satisfied for subspaces of $H^2_{\mathscr{H}}$) that $\mathscr{M}$ contains no nonzero element F such that $e^{-nix}F \in \mathscr{M}$ for $n = 0, 1, 2, \cdots$.

DEFINITION. A closed invariant subspace of $L^2_{\mathscr{H}}$ not infinitely divisible by e^{ix} (in the above sense) will be called simply invariant. $\mathscr{M}$ is simply invariant if and only if $\mathscr{M} = \mathscr{U} H^2_{\mathscr{H}}$, where $\mathscr{U}$ is a.e. a partial isometry of $\mathscr{H}$. (See [2, p. 64].)

DEFINITION. If $\mathscr{M}, \mathscr{N}$ are simply invariant subspaces of $L^2_{\mathscr{H}}$, we define $\mathscr{M} \cup \mathscr{N}$ to be the smallest simply invariant subspace of $L^2_{\mathscr{H}}$ containing $\mathscr{M}$ and $\mathscr{N}$, *if such a subspace exists.*

It is clear that if $\mathscr{M}$ and $\mathscr{N}$ are both contained in *any* simply invariant subspace of $L^2_{\mathscr{H}}$, they will be contained in a smallest such. However, it is conceivable that all subspaces of $L^2_{\mathscr{H}}$ which contain $\mathscr{M}$ and $\mathscr{N}$ and are invariant under the shift operator are also infinitely divisible by e^{ix}, and therefore *not* of the form $\mathscr{U}H^2_{\mathscr{H}}$.

DEFINITION. If $\mathscr{U}, \mathscr{V}$ are unitary a.e., and $\mathscr{U}H^2_{\mathscr{H}} \cup \mathscr{V}H^2_{\mathscr{H}}$ is defined, we define $\mathscr{U} \cup \mathscr{V}$ to be any $\mathscr{W}$ such that $\mathscr{W}H^2_{\mathscr{H}} = \mathscr{U}H^2_{\mathscr{H}} \cup \mathscr{V}H^2_{\mathscr{H}}$. Clearly $\mathscr{W}$ is also unitary a.e. since $\mathscr{W}H^2_{\mathscr{H}}$ must be of full range if $\mathscr{U}H^2_{\mathscr{H}}$ and $\mathscr{V}H^2_{\mathscr{H}}$ are.

DEFINITION. If $\mathscr{U}, \mathscr{V}$ are unitary a.e., and $\mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{V}H^2_{\mathscr{H}}$ is of full range, we define $\mathscr{U} \cap \mathscr{V}$ to be any $\mathscr{W}$ such that $\mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{V}H^2_{\mathscr{H}} = \mathscr{W}H^2_{\mathscr{H}}$. The intersection of simply invariant subspaces is always simply invariant, but we wish to exclude the case where the intersection does not have full range.

DEFINITION. If $\mathscr{M}, \mathscr{N}$ are invariant subspaces of $L^2_{\mathscr{H}}$, let $\mathscr{M} + \mathscr{N}$ be the smallest invariant subspace of $L^2_{\mathscr{H}}$ containing $\mathscr{M}$ and $\mathscr{N}$.

DEFINITION. Let
$$\begin{aligned} {}^*N_{\mathscr{H}} &= \{\mathscr{U}^*\mathscr{V} : \mathscr{U}, \mathscr{V} \text{ inner}\} \\ N^*_{\mathscr{H}} &= \{\mathscr{U}\mathscr{V}^* : \mathscr{U}, \mathscr{V} \text{ inner}\} \\ N &= N^*_{\mathscr{H}} = {}^*N_{\mathscr{H}} \text{ if } \mathscr{H} \text{ is one dimensional.} \end{aligned}$$

LEMMA 16. *If $\mathscr{U}, \mathscr{V}$ are unitary a.e., then $\mathscr{U}H^2_{\mathscr{H}} \subseteqq \mathscr{V}H^2_{\mathscr{H}}$ if and only if $\mathscr{V}^*\mathscr{U}$ is inner.*

LEMMA 17. (i) $\mathscr{U} \cap \mathscr{V}$ *exists* $\Leftrightarrow \mathscr{U}^*\mathscr{V} \in N^*_{\mathscr{H}}$.
(ii) $\mathscr{U} \cup \mathscr{V}$ *exists* $\Leftrightarrow \mathscr{U}^*\mathscr{V} \in {}^*N_{\mathscr{H}}$.

Proof. (i) Let $\mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{V}H^2_{\mathscr{H}} = \mathscr{W}H^2_{\mathscr{H}}$, where $\mathscr{W}$ is unitary a.e.; i.e., assume $\mathscr{U} \cap \mathscr{V}$ exists. Then $\mathscr{U}H^2_{\mathscr{H}} \supseteqq \mathscr{W}H^2_{\mathscr{H}}$, $\mathscr{V}H^2_{\mathscr{H}} \supseteqq \mathscr{W}H^2_{\mathscr{H}}$ and therefore $\mathscr{U}^*\mathscr{W}$ and $\mathscr{V}^*\mathscr{W}$ are inner. Thus $\mathscr{U}^*\mathscr{V} = \mathscr{U}^*\mathscr{W}(\mathscr{V}^*\mathscr{W})^*$ and $\mathscr{U}^*\mathscr{V} \in N^*_{\mathscr{H}}$. Conversely, if $\mathscr{U}^*\mathscr{V} = \mathscr{A}\mathscr{B}^*$, where $\mathscr{A}, \mathscr{B}$ are inner, then $\mathscr{U}\mathscr{A} = \mathscr{V}\mathscr{B}$, and
$$\mathscr{U}\mathscr{A}H^2_{\mathscr{H}} = \mathscr{V}\mathscr{B}H^2_{\mathscr{H}} \subseteqq \mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{V}H^2_{\mathscr{H}} .$$

(ii) If $\mathscr{U}H^2_{\mathscr{H}} \cup \mathscr{V}H^2_{\mathscr{H}} = \mathscr{W}H^2_{\mathscr{H}}$, then $\mathscr{U}H^2_{\mathscr{H}} \subseteqq \mathscr{W}H^2_{\mathscr{H}}$ and $\mathscr{V}H^2_{\mathscr{H}} \subseteqq \mathscr{W}H^2_{\mathscr{H}}$, and therefore $\mathscr{W}^*\mathscr{U}$ and $\mathscr{W}^*\mathscr{V}$ are inner. Thus $\mathscr{U}^*\mathscr{V} = (\mathscr{W}^*\mathscr{U})^*\mathscr{W}^*\mathscr{V} \in {}^*N_{\mathscr{H}}$. Conversely, if $\mathscr{U}^*\mathscr{V} = \mathscr{C}^*\mathscr{D}$, where $\mathscr{C}, \mathscr{D}$ are inner, then $\mathscr{U}\mathscr{C}^* = \mathscr{V}\mathscr{D}^* \supseteqq \mathscr{U}H^2_{\mathscr{H}}$ and $\supseteqq \mathscr{V}H^2_{\mathscr{H}}$ and therefore $\mathscr{U} \cup \mathscr{V}$ exists.

We note as a consequence of the lemma (what is clear anyway) that if $\mathscr{U}, \mathscr{V}$ are inner, then $\mathscr{U} \cup \mathscr{V}$ exists. If $\mathscr{H}$ is finite dimensional, it follows from Lemma 6 that $\mathscr{U} \cap \mathscr{V}$ exists when $\mathscr{U}$ and $\mathscr{V}$ are inner, since

$$(\det \mathscr{U})(\det \mathscr{V})H^2_{\mathscr{H}} \subseteqq \mathscr{U} H^2_{\mathscr{H}} \quad \text{and} \quad \subseteqq \mathscr{V} H^2_{\mathscr{H}} .$$

THEOREM 18. *If $\mathscr{H}$ is finite dimensional, then $^*N_{\mathscr{H}} = N^*_{\mathscr{H}}$.*

Proof. Let $\mathscr{A} \in {}^*N_{\mathscr{H}}$, say $\mathscr{A} = \mathscr{U}^*\mathscr{V}$. Then since by the above comment, $\mathscr{U} \cap \mathscr{V}$ exists, we have $\mathscr{U}^*\mathscr{V} \in N^*_{\mathscr{H}}$, which proves that $^*N_{\mathscr{H}} \subseteqq N^*_{\mathscr{H}}$.

The other inclusion follows from the existence of a parallel theory of subspaces of $L^2_{\mathscr{H}}$ which are invariant under the left shift operator (multiplication by e^{-ix}) and not infinitely divisible by e^{-ix}. The prototype of such subspaces is, of course,

$$K^2_{\mathscr{H}} = \{F \in L^2_{\mathscr{H}} : F = \sum \varphi_k e^{kix}, \quad \text{where} \quad \varphi_k = 0 \text{ for } k > 0\} .$$

One can show that if $\mathscr{M}$ has the above properties, it is of the form $\mathscr{U} K^2_{\mathscr{H}}$, where $\mathscr{U}$ is almost everywhere a partial isometry, and that $\mathscr{U} K^2_{\mathscr{H}} \subseteqq K^2_{\mathscr{H}}$ if and only if $(\mathscr{U} e, f)$ is the conjugate of an H^∞ function for all $e, f \in \mathscr{H}$. Analogs of all of our theorems hold for "conjugate inner" functions. In particular, if $\mathscr{U}, \mathscr{V}$ are conjugate inner, then $\mathscr{U}^*\mathscr{V} = \mathscr{C}\mathscr{D}^*$, where $\mathscr{C}, \mathscr{D}$ are conjugate inner. But this just says that $N^*_{\mathscr{H}} \subseteqq {}^*N_{\mathscr{H}}$.

If $\mathscr{H}$ is one dimensional, one can say much more. If p, q are scalar functions of modulus 1 a.e., we will write $p \mid q$ (p divides q) if $q\bar{p} \in H^2$; i.e., if $q\bar{p}$ is an inner function. Recall that $N = \{\bar{p}q : p, q$ scalar inner functions$\}$.

THEOREM 19. *Let p, q be of modulus 1 a.e., normalized so that if, say $p(e^{ix}) = e^{nix} r(e^{ix})$, where $r(0) \neq 0$, then $r(0) > 0$. Then*

(i) $pH^2 \cap qH^2 = 0 \Leftrightarrow pH^2 + qH^2 = L^2 \Leftrightarrow p\bar{q} \notin N.$

(ii) *If $p\bar{q} \in N$, then $(p \cup q)(p \cap q) = pq$ a.e.*

Proof. Suppose $p \cup q$ exists. Then $(p \cup q) \mid p$ and $p \cup q \mid q$ and therefore $p \mid [pq/(p \cup q)]$ and $q \mid [pq/(p \cup q)]$ and therefore $p \cap q$ exists, and since $(p \cap q)H^2$ is the largest subspace contained in pH^2 and qH^2, we have that $(p \cap q) \mid [pq/(p \cup q)]$.

Suppose now that $p \cap q$ exists. Then $[pq/(p \cap q)] \mid p$ and $[pq/(p \cap q)] \mid q$ and therefore $p \cup q$ exists, and since $(p \cup q)H^2$ is the smallest subspace containing pH^2 and qH^2, we must have that $[pq/(p \cap q)] \mid p \cup q$.

We have shown that $p \cup q$ exists $\Leftrightarrow$ $p \cap q$ exists, and that when they exist both $pq/(p \cup q)(p \cap q)$ and $(p \cup q)(p \cap q)/pq$ belong to H^2. Thus

$pq = \alpha(p \cup q)(p \cap q)$ and by our normalization, $\alpha = 1$.

Clearly if $p \cap q$ does not exist; i.e., if $pH^2 \cap qH^2$ is not of full range, then $pH^2 \cap qH^2 = (0)$. If $pH^2 \cup qH^2$ does not exist, then $pH^2 + qH^2 = \mathscr{M}$ must be invariant under multiplication by e^{-ix} (since an invariant subspace of L^2 which is not "doubly invariant" is of the form qH^2, where $|q| = 1$ a.e.—see [5]). Wiener's theorem (see [11]) then says that $\exists$ a Baire set E on the circle such that $\mathscr{M} = \{f\colon f(e^{ix}) = 0$ a.e. on $E\}$. Since $p, q \in \mathscr{M}$ and $|p| = |q| = 1$ a.e., E must be of measure 0, and therefore $pH^2 + qH^2 = L^2$.

It remains to prove that $p/q \in N$ if and only if $pH^2 \cap qH^2 \neq (0)$ (which we have just shown is equivalent to $pH^2 + qH^2 \neq L^2$). If $p\bar{q} = p_1\bar{q}_1$, where $p_1, q_1 \in H^2$, then $pq_1 = p_1q \in pH^2 \cap qH^2 \neq (0)$. If $pH^2 \cap qH^2 \neq (0)$, say $pH^2 \cap qH^2 = rH^2$, where $|r| = 1$ a.e., then $\bar{p}r, \bar{q}r \in H^2$ and therefore $p\bar{q} = \overline{(\bar{p}r)}(\bar{q}r) \in N$.

There seems to be little possibility of generalizing this theorem to the case of finite dimensional inner functions. In order for the above proof to work one would need to assume that $\mathscr{U}, \mathscr{V}, \mathscr{U} \cup \mathscr{V}, \mathscr{U} \cap \mathscr{V}$ all commute. But for inner functions $\mathscr{U}, \mathscr{V}$ to commute is not a property of the subspaces $\mathscr{U}H^2_{\mathscr{H}}, \mathscr{V}H^2_{\mathscr{H}}$, but of a particular choice of inner functions. Thus $\mathscr{U}H^2_{\mathscr{H}}$ and $\mathscr{U}\mathscr{C}H^2_{\mathscr{H}}$, where $\mathscr{C}$ is a constant unitary operator both represent the same subspace, but may not commute with the same inner functions. We can, however, use the terminology suggested by the above theorem to give a kind of explicit representation for the minimal scalar inner function of a finite dimensional inner function $\mathscr{U}$.

DEFINITION. If $f, g \in H^2$ and $f = pf_1$, $g = qg_1$, where p, q are inner and f_1, g_1 are outer (see [6, p. 67]), then we define $f \cup g$ and $f \cap g$ to be $p \cup q$ and $p \cap q$.

THEOREM 20. *Let $\mathscr{H}$ be finite dimensional and let $\mathscr{M} = \mathscr{U}H^2_{\mathscr{H}}$ where $\mathscr{U}$ is inner. Let $\mathscr{U} = (u_{ij})$ and let the matrix of*

$$(\det \mathscr{U})\mathscr{U}^* = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots\cdots & a_{nn} \end{pmatrix} = \begin{matrix} \text{matrix of cofactors} \\ \text{of } \mathscr{U} \text{ transposed.} \end{matrix}$$

Then the minimal inner function of $\mathscr{U}$ is equal to

$$q = \frac{\det \mathscr{U}}{(a_{11} \cup a_{12} \cup \cdots \cup a_{1n}) \cap \cdots \cap (a_{n1} \cup \cdots \cup a_{nn})}\,.$$

Proof. Let $\dim \mathscr{H} = 2$ for simplicity. Suppose $\mathscr{M} \supseteqq tH^2_{\mathscr{H}}$, for t an inner function. Then given any $f, g \in H^2$, there exist $h_1, h_2 \in H^2$ such that

$$u_{11}h_1 + u_{12}h_2 = tf$$
$$u_{21}h_1 + u_{22}h_2 = tg .$$

Therefore

$$h_1 = \frac{t(u_{22}g - u_{12}f)}{\det \mathscr{U}}$$

and

$$h_2 = \frac{t(u_{11}g - u_{21}f)}{\det \mathscr{U}}$$

must be analytic for all f, g. Thus $\det \mathscr{U} \mid t(u_{22} \cup u_{12})$ and $\det \mathscr{U} \mid t(u_{11} \cup u_{21})$. Thus t is minimal if and only if

$$\det \mathscr{U} = t(u_{22} \cup u_{12}) \cap t(u_{11} \cup u_{21}) .$$

Clearly $tf \cap tg = t(f \cap g)$ for all $f, g \in H^2$ and thus

$$\det \mathscr{U} = t[(u_{22} \cup u_{12}) \cap (u_{11} \cup u_{21})]$$

as asserted.

We now return to the theorem that $^*N_{\mathscr{H}} = N^*_{\mathscr{H}}$ when $\mathscr{H}$ is finite dimensional. To prove this result false for $\mathscr{H}$ infinite dimensional, it would suffice to exhibit inner functions $\mathscr{U}, \mathscr{V}$ such that $\mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{V}H^2_{\mathscr{H}}$ is not of full range. For, by Lemma 17, $\mathscr{U} \cap \mathscr{V}$ exists if and only if $\mathscr{U}^*\mathscr{V} \in N^*_{\mathscr{H}}$, which says that $^*N_{\mathscr{H}} \nsubseteq N^*_{\mathscr{H}}$. (By using the same symmetry used in proving Theorem 18, one can also see that $N^*_{\mathscr{H}} \nsubseteq {}^*N_{\mathscr{H}}$.)

THEOREM 21. *If $\mathscr{H}$ is infinite dimensional, there exist invariant subspaces $\mathscr{M}, \mathscr{N} \subseteqq H^2_{\mathscr{H}}$ of full range such that $\mathscr{M} \cap \mathscr{N} = (0)$.*

Proof. Let T, U be bounded operators on $\mathscr{H}$ which are each one-to-one and whose ranges are disjoint. Let $\mathscr{M}, \mathscr{N}$ be the Rota subspaces of T^* and U^*. Then by Lemma 11

$$\mathscr{M} = \left\{F = \sum_0^\infty \varphi_k e^{kix} : \sum_0^\infty T^n \varphi_n = 0\right\}$$
$$\mathscr{N} = \{F = \sum \varphi_k e^{kix} : \sum U^n \varphi_n = 0\} .$$

Suppose $F \in \mathscr{M} \cap \mathscr{N}$, where $F = \sum \varphi_k e^{kix}$. Then

$$\varphi_0 = -T\left(\sum_{n=1}^{\infty} T^{n-1}\varphi_n\right) = -U\left(\sum_{n=1}^{\infty} U^{n-1}\varphi_n\right).$$

Since T and U have disjoint ranges, $\varphi_0 = 0$.
Therefore

$$T\varphi_1 = -T^2\left(\sum_{n=2}^{\infty} T^{n-2}\varphi_n\right)$$
$$U\varphi_1 = -U^2(\sum U^{n-2}\varphi_n).$$

Since T, U are one-to-one, we conclude that

$$\varphi_1 = -T\left(\sum_{n=2}^{\infty} T^{n-2}\varphi_n\right) = -U(\sum U^{n-2}\varphi_n).$$

Therefore $\varphi_1 = 0$, and by induction $\varphi_n = 0$ and $F = 0$.

In conclusion we might remark that the above theorem gives an example of another kind of pathology. Thus if $\mathscr{M} = \mathscr{U}H^2_{\mathscr{H}}$, $\mathscr{N} = \mathscr{V}^{\frown}H^2_{\mathscr{H}}$ we have $\mathscr{U}\mathscr{V}^{\frown}H^2_{\mathscr{H}} \subseteqq \mathscr{U}H^2_{\mathscr{H}}$, but $\mathscr{U}\mathscr{V}^{\frown}H^2_{\mathscr{H}} \cap \mathscr{V}^{\frown}H^2_{\mathscr{H}} = (0)$, and, of course, $\mathscr{V}^{\frown}\mathscr{U}H^2_{\mathscr{H}} \cap \mathscr{U}H^2_{\mathscr{H}} = (0)$.

The contents of this paper form parts of the author's thesis, written under the guidance of Henry Helson at the University of California, Berkeley. The author would like to express his thanks and to acknowledge his considerable indebtedness to Professor Helson.

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Received April 4, 1965. The research on this paper was done while the author was a National Science Foundation Graduate Fellow.

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