PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 1, 1967

AN INTEGRAL INEQUALITY WITH APPLICATIONS TO THE DIRICHLET PROBLEM

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An existence theorem for the elliptic equation $\Delta u - qu = f$ can be based on minimization of the Dirichlet integral $D(u, u) = \int |\nabla u|^2 + q\,|u|^2 dx$. The usual assumption that $q(x) \geqq 0$ is relaxed in this paper.

Actually the paper deals directly with the general second order formally self-adjoint elliptic differential equation $\sum_{i,k} D_i(a_{ik} D_k u) + qu = f$ where $q(x)$ is positive and "not too large" in a sense which will be made precise later. The technique consists in showing that the quadratic form whose Euler-Lagrange equation is the P.D.E. above is positive for a sufficiently large class of functions.

Earlier inequalities of Beesack [1] and Benson [2] show that there are positive functions $q(x)$ for which $\int |\nabla u|^2 - q\,|u|^2 dx \geqq 0$ for functions u which vanish on the boundary of the domain. D. C. Benson suggested to the author that this inequality might lead to existence theorems for $\Delta u + qu = f$.

Let $x = (x_1, x_2, \cdots x_n) \in R^n$. Let D be an open domain in R^n which may be unbounded unless the contrary is assumed. Let $C^\infty(D)$ denote the set of all infinitely differentiable complex-valued functions and $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ of functions with compact support contained in D. Let $\|u\|_1^2 = \int_D \sum_{i=1}^n |D_i u|^2 + |u|^2 dx$ and let $C^{\infty *}(D)$ be the subset of $C^\infty(D)$ of functions with $\|u\|_1 < \infty$. Let $H_1(D)$ be the Sobolev space which is the completion of $C^{\infty *}(D)$ under $\|u\|_1$. For a function q of the special type encountered in §1, let $H_1^q(D)$ be the Sobolev space which is the completion of $C^{\infty *}(D)$ under the norm

$$\|u\|_q^2 = \int_D \sum_{i=1}^n |D_i u|^2 + q\,|u|^2 dx\ .$$

Let $\mathring{H}_1$ and $\mathring{H}_1^q$ be the completions of $C_0^\infty(D)$ with respect to $\|u\|_1$ and $\|u\|_q$. The reader who is not familiar with the Sobolev spaces can find a discussion of their calculus in Nirenberg [5].

1. An integral inequality.

THEOREM 1.1. *Let D be smooth enough to apply Gauss' Theorem. Let $a_{ik}(x)$ be hermitian positive definite, $a_{ik} \in C^1(D)$, and let $f_1, f_2, \cdots f_n$ be continuously differentiable complex valued functions of x, for all*

$x \in D$. Then

$$\int_D \sum_{i,k=1}^{n} a_{ik} D_i u D_k \bar{u} + (a_{ik} f_i \bar{f}_k + D_k(\operatorname{Re} a_{ik} f_i)) \,|\, u\,|^2 dx$$
$$\geqq \int_{\dot{D}} \sum_{i,k=1}^{n} \operatorname{Re}(a_{ik} f_i) \,|\, u\,|^2 \nu_k ds, \quad \textit{where} \quad \nu_k$$

is the k^{th} component of the normal, $u \in C^1(D)$, and the integral on the right is assumed to exist. In the case of unbounded D, we will understand $\lim_{R\to\infty} \int_{\Sigma_R} \sum \operatorname{Re}(a_{ik} f_i) \,|\, u\,|^2 \nu_k ds = 0$ for Σ_R a sphere of radius R. Equality holds if and only if $D_i u = u f_i$, for every i.

Proof. From $\sum a_{ik}(D_i u - u f_i)(D_k \bar{u} - \bar{u} \bar{f}_k) \geqq 0$, obtain

$$\sum a_{ik} D_i u D_k \bar{u} + \left[a_{ik} f_i \bar{f}_k + \frac{1}{2} D_k(a_{ik} f_i + \bar{a}_{ik} \bar{f}_i) \right] |\, u\,|^2$$
$$(1) \quad \geqq \sum a_{ik}(f_i u D_k \bar{u} + \bar{f}_k \bar{u} D_i u) + \frac{1}{2} D_k(a_{ik} f_i + \bar{a}_{ik} \bar{f}_i) \,|\, u\,|^2$$
$$= \sum a_{ik} f_i u D_k \bar{u} + \frac{1}{2} D_k(a_{ik} f_i) \,|\, u\,|^2 + \bar{a}_{ik} \bar{f}_i \bar{u} D_k u + \frac{1}{2} D_k(\bar{a}_{ik} \bar{f}_i) \,|\, u\,|^2 .$$

Where the last line was obtained by interchanging the order of summation and using the symmetry of a_{ik}.

Now obtain a new inequality from (1) by taking conjugates of both sides and interchanging the order of summation in the first two terms. Add this new inequality to (1) and obtain

$$\sum a_{ik} D_i u D_k \bar{u} + [a_{ik} f_i \bar{f}_k + D_k(\operatorname{Re} a_{ik} f_i)] \,|\, u\,|^2$$
$$\geqq \sum D_k(|\, u\,|^2 \operatorname{Re} a_{ik} f_i) .$$

Now integrate both sides and use Gauss' Theorem to obtain the desired result.

DEFINITION 1.1. We will reserve the notation $q(x)$ for a positive function of the form $q(x) = -\sum a_{ik} f_i \bar{f}_k + D_k(\operatorname{Re} a_{ik} f_i)$.

COROLLARY. *If D is any open set in R^n and $a_{ik}(x)$ are uniformly bounded in D, then $\int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - q \,|\, u\,|^2 dx \geqq 0$, for every $u \in \mathring{H}_1^q$ and equality holds if and only if $D_i u = u f_i$, for every i, a.e.*

Proof. Let us first establish the inequality for any $u \in C_0^\infty(D)$. Let K denote the support of u and Ω denote a sphere containing K. Let $\tilde{u} \in C_0^\infty(\Omega)$ such that $\tilde{u} = \begin{cases} u \text{ on } K \\ 0 \text{ on } \Omega - D \end{cases}$ and let $\tilde{u}, \tilde{f}_i, \tilde{q}$ be continuously differentiable extensions of u, f_i, q to Ω. Then

$$\int_D \sum a_{ik} D_i u D_k \bar{u} - q\,|u|^2 dx = \int_\Omega \sum \tilde{a}_{ik} D_i \tilde{u} D_k \tilde{u} - \tilde{q}\,|\tilde{u}|^2 dx$$
$$\geqq \int_{\dot{D}} \sum \operatorname{Re}(\tilde{a}_{ik}\tilde{f}_i)\,|\tilde{u}|^2 \nu_k ds = 0\ .$$

Now let $|a_{ik}(x)| \leqq M$ for every $i, k, x \in D$. For any $u \in \mathring{H}_1^q$, choose a sequence $u_m \in C_0^\infty$ such that $\| u - u_m \|_q \to 0$.

Then

$$\int_D \sum_i |D_i u_m|^2 dx \xrightarrow{m} \int_D \sum_i |D_i u|^2 dx$$

and

$$\int_D q\,|u_m|^2 dx \xrightarrow{m} \int_D q\,|u|^2 dx$$

and we have established that

$$\int_D \sum a_{ik} D_i u_m D_k \bar{u}_m - q\,|u_m|^2 dx \geqq 0\ , \quad \text{for every} \quad m.$$

We need only show that

$$\int_D a_{ik} D_i u_m D_k \bar{u}_m dx \xrightarrow{m} \int_D a_{ik} D_i u D_k \bar{u} dx$$

which follows from

$$\int_D |a_{ik}|\,|D_i u_m D_k \bar{u}_m - D_i u D_k \bar{u}|\,dx$$
$$\leqq M \int_D (|D_i u_m| \cdot |D_k(\bar{u}_m - \bar{u})| + |D_k \bar{u}| \cdot |D_i(u_m - u)|) dx$$
$$\leqq M\Big(\int_D |D_i u_m|^2 dx\Big)^{1/2} \Big(\int_D |D_k(u_m - u)|^2 dx\Big)^{1/2}$$
$$+ M\Big(\int_D |D_k u|^2 dx\Big)^{1/2} \Big(\int_D |D_i(u_m - u)|^2 dx\Big)^{1/2} \xrightarrow{m} 0\ .$$

After proving three existence theorems, we will give some examples for choices for $q(x)$.

2. Existence theorems.

THEOREM 2.1. *Let $q(x)$ be a function of the special form of definition 1.1 and let $p(x)$ be a continuously differentiable function such that $0 < p(x) \leqq (1 - \varepsilon)q(x)$, where $\varepsilon > 0$ and fixed. Let*

$$\int_D q^{-1} |f|^2 dx < \infty\ ,$$

$g \in H_1^q$ and let

$$Au = \sum_{i,k} D_i(a_{ik}D_k u) + pu \quad \textit{be a}$$

uniformly elliptic operator. That is, a_{ik} is hermitian and there exist positive constants M and λ such that $|a_{ik}(x)| \leqq M$ and

$$\lambda \sum_i |\xi_1|^2 \leqq \sum_{i,k} a_{ik}\xi_i\bar{\xi}_k \ ,$$

for any $(\xi_1, \xi_2, \cdots \xi_n)$.

Then the Dirichlet problem

$$\begin{cases} Au = f \text{ in } D \\ u = g \text{ on } \dot{D} \\ \int_D \sum_i |D_i u|^2 + q\,|u|^2 dx < \infty \end{cases}$$

has a weak solution and any two weak solutions differ only on a set of measure zero.

Proof. We must show that there is a function $u \in H_1^q$ such that $u - g \in \mathring{H}_1^q$ and $(u, A^*\varphi) = (f, \varphi)$ for every $\varphi \in C_0^\infty$. Here A^* denotes the formal adjoint of A (actually $A = A^*$ on the domain of A). Equivalently, we can set $u_0 = u - g$ and consider the problem of finding $u_0 \in \mathring{H}_1^q$ such that $(u_0, A^*\varphi) = (f, \varphi) - (g, A^*\varphi)$.

Let

$$\begin{aligned} B(u, v) &= \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{v} - pu\bar{v}\, dx \\ &= \int_D \sum_{i,k} \bar{a}_{ik} D_k u D_i \bar{v} - pu\bar{v}\, dx \\ &= -\int_D \sum_{i,k} u D_k(\bar{a}_{ik} D_i \bar{v}) + pu\bar{v}\, dx \\ &= -(u, A^*v), \quad \text{for every } v \in C_0^\infty(D) \,. \end{aligned}$$

We will show that there exist $C_1, C_2 > 0$ such that

$$|B(u, v)| \leqq C_1 \,||\, u \,||_q \,||\, v \,||_q$$

and

$$B(u, u) \geqq C_2 \,||\, u \,||_q^2, \quad \text{for every } u, v \in \mathring{H}_1^q \,.$$

For, having shown this, we can apply the Lax-Milgram Theorem which guarantees that any bounded linear functional $F(\varphi)$ on the Hilbert space $\mathring{H}_1^q$ can be represented as $F(\varphi) = \overline{B(u_0, \varphi)}$ for some $u_0 \in \mathring{H}_1^q$.

Take $F(\varphi) = -\overline{(f, \varphi)} - \overline{B(g, \varphi)}$, then

$$| F(\varphi) | \leqq \left(\int_D q^{-1} | f |^2 dx\right)^{1/2}\left(\int_D q | \varphi |^2 dx\right)^{1/2} + C_1 || \varphi ||_q || g ||_q$$
$$\leqq \text{const} || \varphi ||_q .$$

So $B(u_0, \varphi) = -(f, \varphi) - B(g, \varphi)$ which was to be shown.

To see that $B(u, u)$ is positive, consider

$$B(u, u) = \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - p | u |^2 dx$$
$$\geqq \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - q | u |^2 dx + \varepsilon \int_D q | u |^2 dx .$$

By the corollary to Theorem 1.1, both integrals are positive and, therefore,

$$B(u, u) \geqq \varepsilon \int_D q | u |^2 dx \quad \text{and}$$
$$B(u, u) \geqq \int_D \sum a_{ik} D_i u D_k \bar{u} - q | u |^2 dx .$$

Then

$$\left(1 + \frac{2}{\varepsilon}\right) B(u, u) \geqq \int_D \sum a_{ik} D_i u D_k \bar{u} + q | u |^2 dx$$
$$\geqq \int_D \lambda \sum_i | D_i u |^2 + q | u |^2 dx \geqq C || u ||_q^2$$
$$\text{with} \quad C = \min(1, \lambda) .$$

The positivity of $B(u, u)$ implies

$| B(u, v) |^2 \leqq B(u, u) \cdot B(v, v)$ so that we need only show that $B(u, u) \leqq \text{const} || u ||_q^2$ to see that $| B(u, v) | \leqq C_1 || u ||_q || v ||_q$

$$B(u, u) \leqq M \int_D \sum_{i,k} | D_j u D_k \bar{u} | + p | u |^2 dx$$
$$\leqq M \int_D \sum_{i,k} \frac{1}{2} (| D_i u |^2 + | D_k \bar{u} |^2) + p | u |^2 dx$$
$$\leqq Mn \int_D \sum_i | D_i u |^2 + p | u |^2 dx = Mn || u ||_p^2 \leqq Mn || u ||_q^2 .$$

To obtain the uniqueness result, let $Au = 0, u \in \mathring{H}_1^q$, then

$$0 = -(u, Au) = B(u, u) \geqq C_2 || u ||_q^2 \qquad \therefore \ u = 0 \quad \text{a.e.}$$

THEOREM 2.2. *Suppose that D is bounded and $\dot{D}$ is smooth enough for integration by parts, that (a_{ik}) is real symmetric positive definite, that $a_{ik} \in C^4(D)$, and that $| a_{ik}(x) | \leqq M$, for every $i, j = 1, \cdots, n$ and $x \in D$. Let $q(x) = -\sum_{i,k} (a_{ik} f_i f_k + D_k(a_{ik} f_i))$ be such that $q \in C^2(D)$*

and the system

$$\begin{cases} D_i u = u f_i \\ u = 0 \text{ on } \dot{D} \end{cases} \quad \textit{has only the trivial solution.}$$

Let $Au = \sum_{i,k} D_i(a_{ik}D_k u) + qu$.

Then the Dirichlet problem $\begin{cases} Au = 0 \text{ in } D \\ u = g \text{ on } \dot{D} \end{cases}$

has a unique solution.

Proof. We use a result of Browder [4] which says that under the assumptions above uniqueness implies existence. Thus we need only show that if u is such that $Au = 0$ and $u = 0$ on $\dot{D}$, then $u \equiv 0$. But that is immediate since

$$\begin{aligned} B(u, u) &= \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - q \mid u \mid^2 dx \\ &= -(u, Au) = 0 \,. \end{aligned}$$

By Theorem 1.1, $B(u, u) = 0$ only if $D_i u = u f_i$. By the assumption, $u \equiv 0$.

It will be seen in § 3 that many functions $q(x)$ have the required uniqueness property.

THEOREM 2.3. *Let* $q(x) = -\sum_i \mid f_i \mid^2 + D_i(\operatorname{Re} f_i)$ *so that*

$$\int_D \sum_i \mid D_i u \mid^2 - q \mid u \mid^2 dx \geqq 0 \,,$$

for every $u \in H_1^q(D)$. *Suppose that* $q \in C^1(D)$ *and* $0 < m \leqq q(x) \leqq M$ *for every* $x \in D$. *Suppose that* (a_{ik}) *is hermitian and*

$$\lambda \sum_i \mid \xi_i \mid^2 \sum_i a_{ik}(x) \xi_i \bar{\xi}_k$$

for all $x \in \bar{D}$, *all* ξ *and some fixed* $\lambda > 0$. *Suppose that* $a_{ik} \in C^2(D)$, $b_i \in C^1(D)$ *and* a_{ik}, b_i *are bounded in* D. *Let*

$$Eu = \sum_{i,k} D_i(a_{ik} D_k u) + \sum_i b_i D_i u + (p(x) - \mathscr{K})u$$

where $0 < p(x) \leqq (\lambda - \mu - \varepsilon) q(x)$ *for* $x \in \bar{D}$, μ *and* ε *are any fixed positive numbers with* $\mu + \varepsilon < \lambda$, *and*

$$\mathscr{K} \geqq \frac{1}{\mu} \max_{x \in \bar{D}} \sum_i \mid b_i \mid^2 \,.$$

Then the Dirichlet problem

$$\begin{cases} Eu = f \ in \ D \\ u = g \ on \ \dot{D} \\ \|u\|_1 < \infty \end{cases}$$

has a weak solution and any two weak solutions differ only on a set of measure zero. [Note: In the usual theorem of this sort, one requires $\mathscr{K} \geqq (1/\lambda) \max_{x \in \bar{D}} [\sum_i b_i^2 + \lambda p]$ so that $p(x) - \mathscr{K}$ is necessarily negative. For example, see Hellwig [5].]

Proof.
Let

$$\begin{aligned} B(u, v) &= \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{v} - \sum_k b_k \bar{v} D_k u - (p - \mathscr{K}) u \bar{v} \, dx \\ &= -\int_D \sum_{i,k} u D_i(a_{ik} D_k \bar{v}) - \sum_i u D_k(b_k \bar{v}) + (p - \mathscr{K}) u \bar{v} \, dx \\ &= -(u, E^* v), \quad \text{for } v \in C_0^\infty(D) \,. \end{aligned}$$

we will show

$$\begin{aligned} | B(u, v) | &\leqq C_1 \|u\|_1 \|v\|_1 \\ | B(u, u) | &\geqq C_2 \|u\|_1^2 \end{aligned}$$

and the result follow from the Lax-Milgram Theorem by the argument in the proof of Theorem 2.1.

Recall μ from the statement and use the inequality derived from $[(\mu/2)^{1/2}\alpha - (\mu/2)^{-1/2}\beta]^2 \geqq 0$ to obtain for each k,

$$| b_k \bar{u} D_k u | \leqq \frac{1}{\mu} | b_k \bar{u} |^2 + \frac{\mu}{4} | D_k u |^2 \,.$$

Sum on k,

$$\begin{aligned} \left| \sum_k b_k \bar{u} D_k u \right| &\leqq \frac{1}{\mu} | u |^2 \sum_k | b_k |^2 + \frac{\mu}{4} \sum_k | D_k u |^2 \\ &\leqq \mathscr{K} | u |^2 + \frac{\mu}{4} \sum_k | D_k u |^2 \end{aligned}$$

$$\begin{aligned} | B(u, u) | &\geqq \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - (p - \mathscr{K}) | u |^2 dx - \int_D \left| \sum_k b_k \bar{u} D_k u \right| dx \\ &\geqq \int_D \lambda \sum_k | D_k u |^2 - (\lambda - \mu - \varepsilon) q | u |^2 + \mathscr{K} | u |^2 \\ &\qquad - \mathscr{K} | u |^2 - \frac{\mu}{4} \sum_k | D_k u |^2 dx \\ &\geqq (\lambda - \mu) \int_D \sum_k | D_k u |^2 - q | u |^2 dx + \varepsilon \int_D q | u |^2 dx \,. \end{aligned}$$

Since both integrals are positive,

$$| B(u, u) | \geqq (\lambda - \mu)\int_D \sum_k | D_k u |^2 - q | u |^2 dx$$

$$| B(u, u) | \geqq \varepsilon \int_D q | u |^2 dx \ .$$

Let $\delta = 2(\lambda - \mu/\varepsilon) > 0$, then

$$| B(u, u) | \geqq \frac{\lambda - \mu}{1 + \delta} || u ||_q^2 \geqq C_2 || u ||_1^2$$

$$\begin{aligned} | B(u, v) | &\leqq \text{const.} \int_D \sum_{i,k} | D_i u | \, | D_k v | + \sum_k | v | \, | D_k u | + | u | \cdot | v | \, dx \\ &\leqq \text{const.} \Big[\sum_{i,k} \Big(\int_D | D_i u |^2 \Big)^{1/2} \Big(\int_D | D_k v |^2 \Big)^{1/2} \\ &\quad + \sum_k \Big(\int_D | v |^2 \Big)^{1/2} \Big(\int_D | D_k u |^2 \Big)^{1/2} + \Big(\int | u |^2 \Big)^{1/2} \Big(\int | v |^2 \Big)^{1/2} \Big] \\ &\leqq \text{const.} \Big[\sum_{i,k} || u ||_1 \, || v ||_1 + \sum_k || v ||_1 \, || u ||_1 + || u ||_1 \, || v ||_1 \Big] \\ &\leqq \text{const.} \, || u ||_1 \, || v ||_1 \ . \end{aligned}$$

3. Examples. Let

$$q(x) = -\sum_{i,k} a_{ik} f_i f_k + D_k(a_{ik} f_i) \ ,$$

for real f_i, a_{ik}.

3.1. Let

$$a_{ik} = \delta_{ik}, f_i = \begin{cases} 1/2x_i & i = 1, \cdots, s \\ 0 & i = s + 1, \cdots, n \end{cases} \qquad 1 \leqq s \leqq n \ .$$

Then

$$q(x) = \frac{1}{4} \sum_{i=1}^{s} \frac{1}{x_i^2}$$

and the inequality is

$$\int_D \sum_k | D_k u |^2 - \frac{1}{4} \sum_{i=1}^{s} \frac{1}{x_i^2} | u |^2 dx \geqq 0 \ .$$

Notice that this generalizes the well-known inequality

$$\int_D | u |^2 dx \leqq 4\mu^2 \int_D \sum_k | D_k u |^2 dx \ , \qquad u \in \mathring{H}_1 \ ,$$

where $\mu = \min_{1 \leqq i \leqq n} \max_{x_i \in D} | x_i |$.

In particular for $s = 1$, Theorem 2.1 solves the Dirichlet problem for $\Delta u + p(x_1)u = 0$ where $0 < p(x_1) \leqq ((1 - \varepsilon)/4x_1^2)$ and the plane $x_1 = 0$

is not in $\bar{D}$. This differential equation has an application in Generalized Axially Symmetric Potential Theory where solutions of

$$\frac{\partial}{\partial x}\left(y^{n-2}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(y^{n-2}\frac{\partial u}{\partial y}\right)=0$$

are sought (see [7]). If we let $u = y^{-1/2(n-2)}v$, we obtain

$$v_{xx}+v_{yy}-\frac{(n-2)(n-4)}{4y^2}v=0$$

and $0 < (-(n-2)(n-4)/4) \leqq 1/4$ when $2 < n < 4$.

It sometimes happens that equations of mixed type, that is equations which are elliptic in one part of the plane and hyperbolic in the complementary part can be transformed into equations which are elliptic but which have singular coefficients. The Tricomi equation $yu_{xx} + u_{yy} = 0$ is of this sort. If we let $z = 2/3y^{3/2}$ we obtain $u_{xx} + u_{zz} + (1/3z)u_z = 0$. Now let $v = z^{1/6}u$ and obtain

$$(*)\qquad v_{xx}+v_{zz}+\frac{5}{36}\frac{1}{z^2}v=0\,.$$

Since $5/36 < 1/4$, Theorem 2.1 guarantees a solution to the Dirichlet problem in any domain for which $z \neq 0$. In [3], Bergman uses (*) to study the Tricomi equation by means of his technique of integral operators. His technique is, of course, limited to two dimensions, but there are analogues of (*) in any dimension.

3.2.

$$a_{ik}=\delta_{ik},\, f_i=\begin{cases}-\dfrac{s-2}{2}\dfrac{x_i}{\sum_{i=1}^{s}x_i^2} & 1\leqq i\leqq s\\[2ex] 0 & s\leqq i\leqq n\end{cases}\qquad 1\leqq s\leqq n\,.$$

Then

$$q(x)=\frac{(s-2)^2}{4\sum_{i=1}^{s}x_i^2}\,.$$

In particular, for $s = n = 3$, $q(x) = (1/4r^2)$ where $r = (\sum_{i=1}^{3} x_i^2)^{1/2}$, and

$$\int_D |\nabla u|^2-\frac{1}{4r^2}u^2dx \geqq -\frac{1}{2}\int_{\dot{D}}\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r^2}|u|^2d\sigma\,.$$

Notice that the right hand side is positive whenever D is the exterior of a region which is starshaped with respect to the origin.

Theorem 1.1 solves the Dirichlet problem for $\Delta u + ((1/4 - \varepsilon)/r^2)u = 0$. This example shows the value of having $\varepsilon > 0$. If we take D to be the exterior of the unit circle, the function $u = r^{-1/2-\sqrt{\varepsilon}}$ solves $\Delta u + (1/4 - \varepsilon/r^2)u = 0$ with boundary values $u = 1$ and

$$\| u \|_1^q = \left(\int_D | \nabla u |^2 + \frac{1}{4r^2} u^2 dx\right)^{1/2} < \infty .$$

For $\varepsilon = 0$, the expected solution of $\Delta u + (1/4r^2)u = 0$ is $u = r^{-1/2}$, but $\| r^{-1/2} \|_1^q = \infty$. It is not even clear that the solution is unique.

3.3. Let $a_{ik} = \delta_{ik}$, $\boldsymbol{f} = (f_1, \cdots, f_n) = \alpha r^t \boldsymbol{r}$ where $\boldsymbol{r} = (x_1, x_2, \cdots x_n)$. Then

$$\int_D \sum_k | D_k u |^2 + [\alpha^2 r^{2t+2} - \alpha(n + t)r^t] | u |^2 dx > 0$$

for every $u \in \mathring{H}_1^q$.

3.4. Let $a_{ii} = x_i^2$, $a_{ik} = 0$ for $i \neq k$, $f_i = -(1/2x_i)$. Then $q(x) = (n/4)$ and Theorem 2.1 applies to $\sum_{k=1}^n D_k(x_k^2 D_k u) + \alpha u = f$ where $0 < \alpha < n/4$.

3.5. It is possible to derive from Theorem 1.1 Rayleigh's characterization of the first eigenvalue of $\sum_{i,k} D_i(a_{ik} D_k u) + \lambda q u = 0$, $u = 0$ on $\dot{D}$, where $q > 0$ and continuous on $\bar{D}$ and D is bounded. Let λ_1 be the first eigenvalue and u_1 its eigenfunction. Then $u_1 \neq 0$ in D and we may set $f_i = (D_i u_1/u_1)$. Then

$$\sum_{i,k} a_{ik} f_i f_k + D_i(a_{ik} f_k) = \sum_{i,k} \frac{D_i(a_{ik} D_k u_1)}{u_1} \lambda_1 q .$$

Let $u \in C_0^\infty(D)$ and K be the support of u. Then

$$\int_K \sum_{i,k} a_{ik} D_i u D_k u - \lambda_1 q u^2 dx \geqq 0$$

since $f_i u^2 \in C^1(k)$. Since all the functions are bounded this implies that

$$\int_D \sum_{i,k} a_{ik} D_i u D_k u - \lambda_1 q u^2 dx \geqq 0 , \qquad \text{for every } u \in C_0^\infty(D) .$$

Since this is the only conclusion of Theorem 1.1 used in the corollary, we have this same inequality valid for all $u \in \mathring{H}_1^q(D)$. That is,

$$\lambda_1 \leqq \frac{\int_D \sum_{i,k} a_{ik} D_i u D_k u dx}{\int_D q u^2 dx}$$

with equality if and only if $D_i u = f_i u = (D_i u_1/u_1)u$ if and only if $u = ku_1$.

One can employ the technique of this example to obtain inequalities whenever a suitable solution of the string equation is known.

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Received October 3, 1966. This paper is part of the author's doctoral dissertation which was directed by Assoc. Prof. Donald C. Benson.

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