PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 2, 1967

ERGODIC PROPERTIES OF NONNEGATIVE MATRICES-I

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This paper contains an attempt to develop for discrete semigroups of infinite order matrices with nonnegative elements a simple theory analogous to the Perron-Frobenius theory of finite matrices. It is assumed throughout that the matrix is irreducible, but some consideration is given to the periodic case. The main topics considered are

(i) nonnegative solutions to the inequalities

$$r \sum_k x_k t_{kj} \leqq x_j \qquad (r > 0)$$

(ii) nonnegative solutions to the inequalities

$$r \sum_k x_k t_{kj} \geqq x_j \qquad (r > 0)$$

(iii) the limiting behaviour of sums $P_j(n; r) = \sum_k u_k t_{kj}^{(n)} r^n$ **as** $n \to \infty$, **where** $\{u_k\}$ **is arbitrary nonnegative vector. An extensive use is made of generating function techniques.**

It is well-known that an $n \times n$ matrix with positive elements t_{ij} has an eigenvalue with very special properties: it is positive, greater in modulus than all other eigenvalues, a simple root of the characteristic equation, and associated with unique positive eigenvectors for both the original matrix and its transpose. In the present paper we shall develop some related results when the matrix is infinite (of denumerable order). Although this work was suggested by recent results for Markov chains, we shall not here make the assumption that the matrix is stochastic (i.e. that $\Sigma_j t_{ij} = 1$). Nor shall we place any restrictions on the matrix of the type that it should act as a bounded linear operator on one of the standard sequence spaces. Thus our results are not directly covered by recent extensions of the Perron-Frobenius theorem to linear operators leaving invariant a positive cone in a normed linear space; the relation of our results to these theorems will be discussed in a sequel (part II of the present paper).

It is convenient to relax the requirements that the matrix elements be strictly positive, and to suppose only that they are nonnegative and that the matrix is *irreducible*. If we also assume (as we shall throughout) that the matrix iterates ($T^n = \{t_{ij}^{(n)}\}$) are defined and finite for $n = 2, 3, \cdots$, the condition of irreducibility is equivalent to the condition that for each pair of indices i, j there exists an integer $n > 0$ (depending in general on i and j) such that $t_{ij}^{(n)} > 0$.

As in the Markov chain case the weakening from matrices with positive elements to matrices that are irreducible and have nonnegative elements, introduces the possibility that the set of indices J, labelling the matrix entries, may split into a finite number of subclasses

$$C_1, C_2, \cdots C_d, d \geqq 1 .$$

The integer d can be obtained as the H.C.F. of all integers n for which $t_{ii}^{(n)}$ is nonzero. The subclasses C_α are characterized by the property that if $i \in C_\alpha$ and $j \in C_\beta$, the only nonzero terms in the sequence $\{t_{ij}^{(n)}\}$ occur when n falls into a given residue class (modulo d) determined by α and β. Moreover, the labelling can (and will) be imposed in such a way that

$$t_{ij}^{(n)} \neq 0 \text{ only if } n = (\alpha - \beta) \bmod d .$$

These assertions are a simple consequence of irreducibility, and are quite independent of whether the matrix is stochastic. We shall refer to the subclasses C_α as the *periodic subclasses* of J, and the integer d as the *period* of the matrix.

A fundamental result for irreducible matrices may be described now as follows.

THEOREM A. *If T is an irreducible matrix with nonnegative elements, the quantities*

$$\lambda_{ij} = \lim_{n\to\infty} \{t_{ij}^{(n)}\}^{1/n}$$

(*where n tends to infinity through the residue class, modulo d, for which the terms in the sequence are not identically zero*) *exist for all $i, j \in J$, and have a common value say $\lambda_M = 1/R$.*

For $i = j$, the existence of the limit follows directly from the inequality

$$t_{ii}^{(m+n)} \geqq t_{ii}^{(m)} t_{ii}^{(n)} ,$$

and the remaining assertions are a fairly straightforward consequence of irreducibility (Kingman [8]).

It is clear that the quantity R is the common radius of convergence of the generating functions $T_{ij}(z) = \sum_{n=0}^{\infty} t_{ij}^{(n)} z^n$ (where $t_{ij}^{(0)} = \delta_j^i$ (Kronecker delta)). Since $zT_{ij}(z)$ is nothing other than the i-j element of the resolvent operator for T (with $z = 1/\lambda$—see equation (6)), it follows easily that when T is finite-dimensional, $\lambda_M = 1/R$ must be the familiar maximum eigenvalue. When the matrix is infinite-dimensional, however, this interpretation may break down. Thus the matrix may not even define a bounded operator on the particular sequence space in which we are interested. Alternatively, it may define an operator on several such spaces, but its spectrum in these spaces may vary, and in general it will not be true that the spectral radius is equal to $1/R$ (see the discussion in Vere-Jones, [18]).

The aim of the present paper is to investigate the extent to which the Perron-Frobenius theorem holds true if we regard $1/R$ as the natural analogue of the maximum eigenvalue for infinite-dimensional matrices, while making no assumptions as to whether the matrix defines a bounded operator. The structure of the paper is as follows. In the next two sections we collect some preliminary results. In § 4 and § 5 we consider nonnegative solutions to the two sets of inequalities

(1a) $$r \sum_i x_i t_{ij} \leqq x_j \qquad (j \in J)$$

(1b) $$r \sum_j t_{ij} y_j \leqq y_i \qquad (i \in J)$$

(2a) $$r \sum_i x_i t_{ij} \geqq x_j \qquad (j \in J)$$

(2b) $$r \sum_j t_{ij} y_j \geqq y_i \qquad (i \in J)$$

where $0 < r < \infty$. The importance of these inequalities in treating the finite case is evident from, say, the discussion of Wielandt ([20]); in the infinite case they have been discussed by Šidák ([12]-[14]), Pruitt ([10]) and also in the author's thesis ([16]). Although some of the results we shall obtain here are covered by these writers, the techniques we shall use are different, and lead to some lemmas that will be needed in the later sections; it also seems worthwhile to present a unified treatment.

In § 6 we turn to a problem which has no real analogue in the finite-dimensional case: to determine conditions on the sequences $\{u_i\}$, $\{v_i\}$ sufficient to ensure that the sums

$$P_j(n; r) = \sum_i u_i t_{ij}^{(n)} r^n$$

$$Q_i(n; r) = \sum_j t_{ij}^{(n)} v_j r^n$$

and

$$S(n; r) = \sum_i \sum_j u_i t_{ij} v_j r^n$$

exhibit the same limiting behaviour as the individual sequences $(t_{ij}^{(n)} r^n)$. Some results in this direction were proved by Kingman ([8]), but the theorems we shall develop here appear to be essentially new, and lead to some new applications to Markov chains (see [11], [19]).

Finally, in § 7, we apply the earlier results to discussing the existence and uniqueness of eigenvectors on the boundary of the disc $|z| = 1/R$, recovering in particular the conclusions of the Perron-Frobenius theorem for finite-dimensional matrices.

2. Definitions and preliminary results. In this section we collect together, with a brief indication of their proofs, some preliminary theorems on the properties of infinite, nonnegative matrices. A fuller discussion, and some further results, will be found in the author's earlier paper [17].

The following extension of Theorem A is also a consequence of irreducibility.

THEOREM B. *For any real value of $r > 0$, and all $i, j \in J$,*

(i) *the series $\sum_n t_{ij}^{(n)} r^n$ are either all convergent or all divergent;*

(ii) *as $n \to \infty$ through the appropriate residue class, either all or none of the sequences $\{t_{ij}^{(n)} r^n\}$ tend to zero.*

It is clear from the definition of R that in fact the series

$$\sum^n t_{ij}^{(n)} r^n$$

must be convergent for $r < R$, and divergent for $r > R$, so that the classification implied by this theorem is of nontrivial significance only when $r = R$. These remarks prompt the following definitions.

(i) The matrix T is *R-transient* or *R-recurrent* according as the series $\sum_n t_{ij}^{(n)} R^n$ are convergent or divergent;

(ii) an R-recurrent matrix is *R-null* or *R-positive* according as all or none of the sequences $\{t_{ij}^{(n)} R^n\}$ tend to zero.

The quantity R defined by Theorem A will be called the *convergence parameter* of the matrix T, and its reciprocal the *convergence norm* (see [17]).

From the definition above the R-positive case might seem to include a wide variety of possible behaviours; in fact (see Theorem D below) it exhibits the most regular behaviour of all three groups.

Note that the convergence parameter of the stochastic matrix associated with a recurrent Markov chain is equal to unity, and that the matrix is "1-positive" or "1-null" according as the Markov chain is positive recurrent or null recurrent in the usual sense. In a transient Markov chain, however, the convergence parameter may be greater than unity; Kingman has called this the case of "exponential transience".

One slight extension of the above terminology will be useful. If it is given that the series $\sum_n t_{ij}^{(n)} r^n$ are convergent, but it is not specified that $r = R$, we shall say that the matrix is *r-transient* (using a lower case r). This will allow us to condense the phrase "either $r < R$ or $r = R$ and the matrix is R-transient" into the phrase "the matrix is r-transient".

In order to make a deeper study of the behaviour described in the preceding theorems, it is necessary to appeal to renewal-type arguments. Exactly as in the probability case, the stage is set for such arguments by introducing analogues to "first-entrance" and "last-exit" probabilities. In the general case, these can be defined by repeated summation. We set $f_{ij}^{(0)} = 0, f_{ij}^{(1)} = t_{ij}$ and define recursively

$$f_{ij}^{(n+1)} = \sum_{k \neq j} t_{ik} f_{kj}^{(n)} \qquad (n = 1, 2, \cdots) ;$$

similarly, we set $l_{ij}^{(0)} = 0, l_{ij}^{(1)} = t_{ij}$ and

$$l_{ij}^{(n+1)} = \sum_{k \neq i} l_{ik}^{(n)} t_{kj} \qquad (n = 1, 2, \cdots) .$$

The exact duality between these two sets of quantities is somewhat obscured in the probability case by the asymmetrical condition $\sum_j t_{ij} = 1$.

Since it is clear that $f_{ij}^{(n)} \leqq t_{ij}^{(n)}, l_{ij}^{(n)} \leqq t_{ij}^{(n)}$, the generating functions $F_{ij}(z) = \sum_n f_{ij}^{(n)} z^n$ and $L_{ij}(z) = \sum_n l_{ij}^{(n)} z^n$ are certainly convergent for $|z| < R$, and may in fact be convergent over wider regions.

Of the many important identities connecting these and similar quantities, we note

$$F_{ii}(z) = L_{ii}(z)$$

$$T_{ii}(z) = 1/(1 - F_{ii}(z)) = 1/(1 - L_{ii}(z)) \tag{3}$$

$$T_{ij}(z) = T_{ii}(z) L_{ij}(z) = F_{ij}(z) T_{jj}(z) \qquad (i \neq j) . \tag{4}$$

The equations are given in the first place for $|z| < R$, but may be extended to any region for which either side has an analytic continuation. It should also be noted that the sequences $\{f_{ij}^{(n)}\}$ and $\{l_{ij}^{(n)}\}$ have the same periodic behaviour as the corresponding sequence $\{t_{ij}^{(n)}\}$.

Standard renewal arguments (in particular the Erdös-Feller-Pollard theorem; see Chung, [3] p. 27) applied to equation 3 now yield the following deeper characterization of behaviour at the point $z = R$.

THEOREM C. *For* $0 \leqq r \leqq R, F_{ii}(r) \leqq 1$, *with strict inequality except, perhaps, when* $r = R$. *Moreover,*

(i) *the series* $\sum_n t_{ii}^{(n)} R^n$ *converges or diverges according as*

$$F_{ii}(R) < 1 \textit{ or } F_{ii}(R) = 1 ;$$

(ii) *even if the series* $\sum t_{ii}^{(n)}R^n$ *is divergent, its terms are bounded and the sequence* $\{t_{ii}^{(nd)}R^{nd}\}$ *is convergent, with*

$$\lim_{n\to\infty} t_{ii}^{(nd)}R^{nd} = d/RF'_{ii}(R) ,$$

(*where the limit is to be taken as zero if the 'first moment'*

$$\mu_i = \sum_n nf_{ii}^{(n)}R^n$$

is infinite).

From this theorem we have the following alternative criteria for the matrix properties defined earlier:

(i) the matrix is R-transient or R-recurrent according as

$$F_{ii}(R) < 1 \text{ or } F_{ii}(R) = 1 ;$$

(ii) an R-recurrent matrix is R-positive or R-null according as $F'_{ii}(R) < \infty$, or $F'_{ii}(R) = \infty$.

It follows, of course, from the "solidarity" behaviour described by the two earlier theorems, that relationships such as $F'_{ii}(R) < \infty$ must hold either for all values of i or for none, so that, just as with the previous criteria, it is sufficient to investigate them for a single index.

We shall also need the following partial converse to the first statement of the theorem.

LEMMA 2.1. *If the series* $\sum_n f_{ii}^{(n)}r^n$ *is convergent, and* $F_{ii}(r) \leqq 1$, *then* $r \leqq R$.

Proof. Since R is a singularity of $T_{ii}(z)$ it is either a singularity of $F_{ii}(z)$ or a point at which $F_{ii}(z) = 1$. The monotonic character of $F_{ii}(x)$, taken with the assumptions of the lemma, precludes either of these circumstances from occurring at any point x in the range $0 < x < r$.

For the sake of completeness, we include a final theorem characterizing in more detail the behaviour of R-positive matrices (most of these properties will be recapitulated in the sequel).

THEOREM D. *If* T *is* R*-positive, there exist unique, positive, left and right eigenvectors for the eigenvalue* $1/R$, *say* $\{\alpha_k\}, \{\beta_k\}$ *respectively, such that* $\sum \alpha_k\beta_k < \infty$, *and as* $n \to \infty$ *through the appropriate residue class,*

$$t_{ij}^{(n)}R^n \to d\alpha_j\beta_i/\sum \alpha_k\beta_k . \tag{5}$$

The left and right eigenvectors figuring in this theorem can be shown to be proportional to the vectors $\{L_{ik}(R)\}$ (i fixed), $\{F_{kj}(R)\}$ (j fixed), respectively. The multiplicative form of the limit follows from a corresponding multiplicative property for these quantities, and can be viewed as a statement of the fact that the C-1 limit matrix is idempotent of unit rank.

3. Basic relations satisfied by the generating functions $T_{ij}(z)$. If the matrix T is finite dimensional or defines a bounded operator, the Neumann expansion for the resolvent operator of T,

$$R_\lambda(T) = (\lambda I - T)^{-1} = I/\lambda + T/\lambda^2 + T^2/\lambda^3 + \cdots \tag{6}$$

(which is valid for $|\lambda|$ greater than the spectral radius of T) shows that for small values of z we can identify the generating function $zT_{ij}(z)$ with the i-j element of $R_{1/z}(T)$. The use of these functions therefore goes beyond the desire for a convenient tool; they embody many of the most striking algebraic and analytic properties of the matrix itself.

Even if the matrix T does not define an operator, the matrix $R(z)$ with elements $T_{ij}(z)$ is certainly defined for $|z| < R$, and within this disc we might expect it to exhibit many properties of the resolvent operator. This expectation is well fulfilled, as the next two lemmas show. The first asserts that the matrix $R(z)$ satisfies the defining equations $R(z)[I - zT] = I = [I - zT]R(z)$, and the second that it satisfies the "resolvent equation"

$$z_1R(z_1) - z_2R(z_2) = (z_1 - z_2)R(z_1)R(z_2) .$$

LEMMA 3.1. *If $|z| \leqq r$, and T is r-transient, the quantities $T_{ij}(z)$ satisfy the identities*

$$T_{ij}(z) = z\sum_k T_{ik}(z)t_{kj} + \delta^i_j \qquad (i \text{ fixed, all } j \in J) \tag{7}$$

$$T_{ij}(z) = z\sum_k t_{ik}\, T_{kj}(z) + \delta^i_j \qquad (j \text{ fixed, all } i \in J) \tag{8}$$

Proof. The identities (7) can be obtained formally by multiplying by z^n and summing over the defining equations

$$t^{(0)}_{ij} = \delta^i_j$$
$$t^{(n+1)}_{ij} = \sum_k t^{(n)}_{ik}t_{kj}. \qquad (n = 0, 1, 2, \cdots) .$$

Taking $z = r$, the convergence of the series on the right of (7), and the equality of the two sides of this equation, follow from the convergence of the series defining $T_{ij}(r)$ and the fact that all terms in the sum are nonnegative. The identity can then be extended onto the disc $|z| \leqq r$ by absolute convergence. The dual equations are proved in an analogous manner.

LEMMA 3.2. *If z_1, z_2, and z all lie within the disc $|z| < R$, then*

$$(9) \qquad (z_1 - z_2)\sum_k T_{ik}(z_1)T_{kj}(z_2) = z_1 T_{ij}(z_1) - z_2 T_{ij}(z_2)\,. \qquad (all\ i \in J, j \in J)$$

$$(10) \qquad \sum_k T_{ik}(z)T_{kj}(z) = T_{ij}(z) + zT'_{ij}(z) \qquad (all\ i \in J, j \in J)\,.$$

Proof. To each value of n $(n = 0, 1, 2, \cdots)$ corresponds an equation

$$\sum_k \sum_{r=0}^{n} t_{ik}^{(r)} t_{kj}^{(n-r)} z_1^r z_2^{n-r} = t_{ij}^{(n)} \sum_{r=0}^{n} z_1^r z_2^{n-r}\,.$$

An argument similar to that in the previous lemma shows that the series obtained by summing these equations over n are absolutely convergent whenever $|z_1| < R$ and $|z_2| < R$. Equation (10) follows on writing $z_1 = z_2 = z$, and equation (9) on substituting

$$\textstyle\sum_{r=0}^{n} z_1^r z_2^{n-r} = (z_1 - z_2)^{-1}(z_1^{n+1} + z_2^{n+1})$$

in the right hand expression, and multiplying through by $(z_1 - z_2)$.

There is a difficulty about extending these identities onto the boundary of the disc $|z| \leqq R$ even when T is R-transient, because the derivative on the right side of (10) may be infinite. When it is finite, however, the equations may be extended as in the previous lemma.

In general, it is not possible to assert that the solutions to the equations

$$(11) \qquad x_j = \delta_j^i + z\sum_k x_k t_{kj} \qquad (i \text{ fixed, all } j \in J)$$

$$(12) \qquad y_i = \delta_j^i + z\sum_k t_{ik} y_k \qquad (j \text{ fixed, all } i \in J)$$

given by Lemma 3.1 are unique (although they may be so). Such an assertion would imply there were no solutions to the corresponding homogeneous equations, but this is in general false. It is not even true that equations (11) and (12) have unique nonnegative solutions when z is real and positive, for the solutions to the homogeneous equations may themselves be nonnegative.

Nevertheless, the solutions $T_{ij}(z)$ possess a number of distinguishing properties. For the present, we shall prove only the minimality property stated in the lemma below. A number of further properties follows from the discussion of the homogeneous equations in § 5; we mention in particular, that for real positive values r of z, the vectors $\{x_k = T_{ik}(r)\}$ and $\{y_k = T_{kj}(r)\}$ are the only joint solutions of (11) and (12), respectively, satisfying the condition $\sum_k x_k y_k < \infty$.

LEMMA 3.3. *If* $\{x_k\}$ *is a nonnegative solution of* (11) *with* $z = r$, *then* T *is* r*-transient, and* $x_k \geqq T_{ik}(r)$.

Proof. We show by induction that if a solution exists, then for all $j \in J$ and for $N = 0, 1, 2, \cdots, x_j \geqq \sum_{n=0}^{N} t_{ij}^{(n)} r^n$. For $N = 0$ this follows immediately from (11) and the assumption $x_j \geqq 0$. Then if the induction hypothesis holds true for N, we obtain on substituting $x_j \geqq \sum_{n=0}^{N} t_{ij}^{(n)} r^n$ in the right hand side of (11),

$$\begin{aligned} x_j &\geqq \delta_{ij} + r \sum_k \left[\sum_{n=0}^{N} t_{ik}^{(n)} r^n \right] t_{kj} \\ &= \delta_{ij} + r \sum_{n=0}^{N} t_{ij}^{(n+1)} r^n = \sum_{m=0}^{N+1} t_{ij}^{(m)} r^m . \end{aligned}$$

Thus the induction holds for all N, and letting N tend to infinity we see first that the series on the right is convergent (so that T is necessarily r-transient), and then that $x_j \geqq T_{ij}(r)$.

There is, of course, a dual result for equation (12), but here, as in the sequel, we shall omit an explicit statement of the dual result when it is sufficiently obvious.

4. **Subinvariant vectors.** For this section and the greater part of § 5 we shall restrict attention to nonnegative vectors. A nonnegative solution to the inequalities (1) will be called a *subinvariant vector*, or a *left* (or *right*) *r-subinvariant vector* when it is desired to specify the inequality more precisely.

It is a trivial consequence of Lemma 3.1 that r-subinvariant vectors exist whenever the matrix is r-transient; in this case, it is sufficient to take $x_k = T_{ik}(r)$, where i is fixed but arbitrary. Moreover, the subinvariant vectors obtained in this way are "almost invariant", in the sense that strict inequality holds for just one component (the i'th); from this it follows that no one such vector can be expressed as a finite linear combination of the others. In fact, therefore, when T is r-transient there are at least as many linearly independent left r-subinvariant vectors as there are values of i. Exactly similar remarks apply to the right equations.

In order to obtain a clearer picture of the behaviour of these solutions as r approaches the radius of convergence R, it is convenient to rewrite the identities of the previous section in terms of the functions $F_{ij}(z)$ and $L_{ij}(z)$ defined in § 3. For example, on substituting for $T_{ij}(z)$ from (3) and (4), equation (7) takes the form

$$L_{ij}(z) = z \sum_k L_{ik}(z) t_{kj} + z t_{ij}(1 - L_{ii}(z)) \qquad (i \text{ fixed}, \ j \in J) \tag{13}$$

while (8) becomes

$$(14)\qquad F_{ij}(z) = z\sum_k t_{ik}F_{kj}(z) + zt_{ij}(1 - F_{jj}(z)) \qquad (j \text{ fixed, } i \in J)\ .$$

Now put $z = r$ and let $r \uparrow R$ in these equations. Taking $i = j$ in equation (13), we see first that since $L_{ii}(R) \leqq 1$ (Theorem C), the quantities $L_{ik}(R)$ must be finite for all values of k for which $t_{ki} > 0$. Using irreducibility, it is easy to show that in fact this property extends to all values of k. Then it is evident that the quantities $x_k = L_{ik}(R)$ will form a *left R-invariant vector* (a solution to the strict equations corresponding to the inequalities (1a), with $r = R$) if, and only if, the matrix is R-recurrent.

It is now necessary to discuss whether the solutions obtained in this way are unique. At this stage it is possible to note that if T is R-recurrent, the matrix P, with elements

$$p_{ij} = Rt_{ij}F_{jk_0}(R)/F_{ik_0}(R) \qquad (k_0 \text{ fixed})\ ,$$

is stochastic, and to appeal to the corresponding theorems for Markov chains, which can often be established by probabilistic arguments. We shall prefer, however, to give a direct analytical proof, based on the lemma below, from which the Markov chain results follow as corollaries.

LEMMA 4.1. *If for some $r > 0$, the left inequalities $r\sum_k x_k t_{kj} \leqq x_j$ have a nonnegative, nonzero solution $\{x_j\}$ then for all $i, j \in J$, $x_i > 0$, the series $\sum_n l_{ij}^{(n)} r^n$ are convergent, and*

$$(15)\qquad x_j/x_i \geqq L_{ij}(r)\ .$$

A similar result holds for the corresponding right inequalities, with $F_{ji}(r)$ in place of $L_{ij}(r)$.

Proof. We consider only the left inequalities. Since the solution is nonzero, it has at least one nonzero component, say x_{i_0}, and there will be no loss of generality in supposing $x_{i_0} = 1$. Imitating the proof of Lemma 3.3, we easily prove by induction that for each N and all $j \in J$, $x_j \geqq \sum_{n=0}^N l_{i_0 j}^{(n)} r^n$. As $N \to \infty$, the series defining $L_{i_0 j}(r)$ is convergent for each j and $x_j \geqq L_{i_0 j}(r) > 0$, the strict inequality following from irreducibility. The proof is completed by noting that it is now legitimate to replace i_0 by any arbitrary value of i.

For Markov chains, the corresponding inequality was noted by Chung ([3] p. 38).

COROLLARY 1. *There are no nontrivial r-subinvariant vectors for $r > R$.*

Proof. For any value of r for which such a vector exists, we have, on taking $i = j$ in (15), that the series defining $L_{ii}(r)$ is convergent, and $L_{ii}(r) \leqq 1$. The corollary follows from Lemma 2.1.

COROLLARY 2. *If the matrix is R-recurrent, there are no R-subinvariant vectors which are not R-invariant, and the nonnegative R-invariant vectors are unique up to constant factors.*

Proof. Let $\{x_i\}$ be any nonzero, R-subinvariant vector. Then

$$1 = x_j/x_j \geqq R \sum (x_k/x_j)\, t_{kj} \geqq R \sum L_{jk}(R) t_{kj} = L_{jj}(R) = 1$$

which shows that whenever $t_{kj} > 0$ the inequality $x_k/x_j \geqq L_{jk}(R)$ of the lemma must in fact be an equality. Since similar inequalities hold with t_{kj} replaced by $t_{kj}^{(n)}$, it follows from irreducibility that in fact $x_k/x_j = L_{jk}(R)$ for all $j, k \in J$. Thus the vector $\{x_k\}$ is proportional to the vector $\{L_{jk}(R)\}$ (j fixed), and this shows both that it is strictly invariant, and that the invariant vector is unique up to constant factors. (We consider here only nonnegative vectors: some criteria which apply also to complex vectors will be described at the end of § 5).

REMARK. The reader who is familiar with the probability context will note that in the case of a stochastic matrix this proof replaces Derman's proof of uniqueness using the "time-reversed" chain (see [4]). A careful scrutiny of Derman's proof shows that it hinges on the equality F_{ij} (1) = 1 for a recurrent Markov chain, which, from a purely analytical point of view, can be regarded as a special case of the preceding argument (put $x_j = 1$ and use the right equations). Thus both proofs can be brought back to the minimality property. The use of the time-reversed chain is essentially a device for showing that the property $F_{ij}(1) = 1$ is equivalent to uniqueness, and for transferring the discussion from right to left invariant vectors.

COROLLARY 3. *For* $< R$, *the quantities* $L_{ij}(r)$ *are finite, and*

$$L_{ik}(r)L_{kj}(r) \leqq L_{ij}(r)$$

with equality if $r = R$ *and the matrix is R-recurrent.*

Proof. The finiteness of the $L_{ij}(r)$ was discussed in the preamble to Lemma 4.1. It is evident from equation (13) that for fixed i, the $L_{ik}(r)$ form an r-subinvariant vector, and the corollary follows on substituting $x_k = L_{ik}(r)$ in (15), and from Corollary 2 above.

By extending the arguments in Vere-Jones [17], it is possible to show that the inequality in Corollary 3 is strict if the matrix is r-transient.

COROLLARY 4. *The following criteria define the parameter R and the property of r-transience (and hence of r-recurrence) in terms of solutions to inequalities* (1):

Criterion I. The value R is the greatest value of r for which there exist nonzero r-subinvariant vectors.

Criterion II. The matrix is r-transient if and only if the nonzero r-subinvariant vectors are not unique (up to constant factors).

Criterion II appears to be the best that is available in general. If, however, (as in the stochastic case) there exists a nontrivial right r-invariant vector, say $\{y_k\}$, and the vector is normalized so that $y_i = 1$, the quantities $z_k = y_k - F_{ki}(r)$ satisfy the equalities

$$r \sum_{k \in J - \{i\}} t_{jk} z_k = z_j \ (j \in J - \{i\}) .$$

In this case, therefore, a necessary condition for the matrix to be r-transient is that these equations have a nonzero, nonnegative solution with $z_k \leqq y_k$. The condition is also sufficient, for if such a solution exists, the quantities $\theta_k = y_k - z_k \, (k \neq i)$, $\theta_i = y_i = 1$, form a second r-subinvariant vector for T, which is therefore r-transient by criterion II.

This property corresponds to a well-known criterion for the transience of a Markov chain (see Feller [5], p. 365).

To complete this section, we summarize the results concerning subinvariant vectors in the form of a single theorem (cf. Pruitt, [10], Theorem 1).

THEOREM 4.1. *If the matrix T is r-transient and infinite dimensional, there are infinitely many linearly independent r-subinvariant vectors. If the matrix is R-recurrent, there are unique left and right R-subinvariant vectors which are in fact R-invariant and proportional to the vectors $\{L_{ij}(R)\}$ (i fixed, $j \in J$), $\{F_{ij}(R)\}$ (j fixed, $i \in J$) respectively. If $r > R$, there are no nontrivial r-subinvariant vectors.*

5. Superinvariant vectors. By analogy with the preceding section, we call a nonnegative solution to the inequalities (2) a *superinvariant vector*, or, when more precision is needed, a *left* (or *right*) *r-superinvariant vector*.

Except for finite-dimensional matrices, the theory of superinvariant vectors does not form a precise mirror-image of the discussion in the preceding section. Consider first the region $0 < r \leqq R$. The "mirror-image" result would be that there were no r-superinvariant solutions in this region. This, however, is not true in general. Interesting counter-examples are provided by the branching-process matrices (see [11]), which possess nonnegative eigenvectors for all values in the range $1 \leqq r \leqq R$. In general we can only prove results for superinvariant vectors subject to certain restrictions. The nature of these restrictions will become clear from the lemma below.

LEMMA 5.1. *If T is r-transient, there are no nonzero left r-superinvariant vectors satisfying the condition $\sum_k x_k T_{kj}(r) < \infty$ for any j.*

Proof. Suppose that $\{x_k\}$ is a nonnegative r-superinvariant vector satisfying the condition $\sum_k x_k T_{kj}(r) < \infty$ for some j. Then we have

$$r \sum_k x_k t_{kj} \geqq x_j$$

and for $n = 1, 2, \cdots$

$$r^{n+1} \sum_k x_k t_{kj}^{(n+1)} \geqq r^n \sum_k x_k t_{kj}^{(n)} .$$

Summing over n, from 0 to N, we obtain

$$\sum_k x_k \Big(\sum_{n=0}^{N+1} t_{kj}^{(n)} r^n \Big) \geqq x_j + \sum_k x_k \Big(\sum_{n=0}^{N} t_{kj}^{(n)} r^n \Big) .$$

As $N \rightarrow \infty$, both of the sums in this inequality tend to the same finite, limit $\sum_k x_k T_{kj}(r)$, so that $x_j \leqq 0$, and hence $x_j = 0$.

To complete the proof it is sufficient to show that if the sum $\sum_k x_k T_{kj}(r)$ is convergent for some value of j, then it must necessarily converge for all values of j. This follows from irreducibility after summing over equations (8), and we omit the details.

COROLLARY. *If T is r-transient, there are no nonzero left r-superinvariant vectors satisfying the condition $\sum x_k \beta_k < \infty$ for any nonzero right r-subinvariant vector $\{\beta_k\}$.*

Proof. From Lemma 4.1, $\{\beta_k\}$ must be strictly positive, and so there is no loss of generality in taking, say, $\beta_i = 1$, so that $\beta_k \geqq F_{ki}(r)$ for all k. Since also $T_{ki}(r) = F_{ki}(r)/(1 - F_{ii}(r))$ if T is r-transient (equations (3) and (4)), it follows that $\sum_k x_k \beta_k < \infty$ implies

$$\sum x_k T_{ki}(r) < \infty ,$$

and so the corollary follows directly from the lemma.

Since the vector $\{T_{ki}(r)\}$ (i fixed) is itself an r-subinvariant vector, the lemma and the corollary are in fact equivalent.

The next stage in the discussion is to consider the case when $r = R$ and T is R-recurrent. Here we follow Šidák [12].

LEMMA 5.2. *If $\{y_k\}$ is any nonzero, right r-subinvariant vector and $\{x_k\}$ is a left r-superinvariant vector such that $\sum_k x_k y_k < \infty$, then either $\{x_k\}$ is the zero vector, or both $\{x_k\}$ and $\{y_k\}$ are positive and strictly r-invariant.*

Proof. Consider the inequalities

$$\sum_k x_k y_k \leqq r \sum_k x_k \sum_h t_{kh} y_h = r \sum_h y_h \sum_k x_k t_{kh} \leqq \sum_h x_h y_h .$$

Since the extreme members are equal, and $y_h > 0$ for all h (Lemma 4.1), we must have $r \sum_k x_k t_{kh} = x_h$ for all h. If $\{x_h\}$ is not the zero vector, it must then be positive, and applying the same argument again, $r \sum_h t_{kh} y_h = y_k$ for all k.

COROLLARY. *Suppose that T is R-recurrent, and $\{\alpha_k\}$, $\{\beta_k\}$ denote respectively the unique left and right R-invariant vectors specified in Theorem 4.1. Then there are no nontrivial left R-superinvariant vectors satisfying the condition $\sum x_k \beta_k < \infty$, except, perhaps, the vector $x_k = \alpha_k$. Similarly, there are no nontrivial right R-superinvariant vectors satisfying the condition $\sum \alpha_k y_k < \infty$ except, perhaps, the vector $y_k = \beta_k$.*

The proof follows directly from the lemma above and the fact that when T is R-recurrent the left and right R-invariant vectors are unique.

In fact, the only case when there exist nontrivial eigenvectors satisfying conditions such as $\sum_k x_k \beta_k < \infty$ is when T is R-positive. This proposition follows from the next lemma and its corollaries.

LEMMA 5.3. *A necessary and sufficient condition for the convergence of the series $\sum_k L_{ik}(R) F_{kj}(R)$ is that one (and hence all) of the first moments $F'_{ii}(R)$ should be finite.*

Proof. Consider first the case $i = j$. Replacing the terms in equation (10) by the appropriate expressions involving $F_{ik}(z)$ and $L_{ik}(z)$, we obtain

$$\sum_k L_{ik}(z) F_{ki}(z) = z F'_{ii}(z) - F_{ii}(z)[1 - F_{ii}(z)] . \tag{16}$$

Putting $z = r$ (real and positive) and letting $r \uparrow R$, the terms in the left-hand sum are individually finite, and from positivity the series converges or diverges according as the right side converges or diverges. Since $F_{ii}(R) \leqq 1$, this is equivalent to the condition of the lemma. To complete the proof, it is sufficient to show that as i and j vary, the series $\sum_k L_{ik}(R)F_{kj}(R)$ converge or diverge together. This again follows from irreducibility (either from (13)-(14), or by using the inequalities in Corollary 3 to Lemma 4.1) and we omit the details.

COROLLARY. *If T is R-recurrent, the series $\sum_k \alpha_k\beta_k$ converges if and only if T is R-positive, and in this case, as $n \to \infty$ through the appropriate residue class,*

$$\lim t_{ij}^{(n)} R^n = d\beta_i\alpha_j / \sum_k \alpha_k\beta_k \ . \tag{17}$$

Proof. The first statement of the corollary follows directly from the lemma, for when T is R-recurrent we can put $\alpha_k/\alpha_i = L_{ik}(R)$ and $\beta_k/\beta_i = F_{ki}(R)$, and the first moment $F'_{ii}(R)$ is finite if and only if T is R-positive.

Further, since $F_{ii}(R) = 1$ if T is R-recurrent we have from (16), $RF'_{ii}(R) = \sum_k \alpha_k\beta_k/\alpha_i\beta_i$. (17) now follows from Theorem C when $i = j$. For $i \neq j$, a simple Abelian argument applied to (4) yields

$$\lim_{n\to\infty} t_{ij}^{nd+\nu} R^{(nd+\nu)} = L_{ij}(R) \lim_{n\to\infty} t_{ii}^{(nd)} R^{nd} \ ,$$

where the integer $\nu, 1 \leqq \nu \leqq d$ is determined by the periodic subclasses of i and j. Substituting $L_{ij}(R) = \alpha_j/\alpha_i$ we obtain the general case of (17). This concludes the proof of the corollary.

Note that we have effectively proved the following condition for R-positivity:

Criterion III. A necessary and sufficient condition to ensure that $r = R$ and that T be R-positive is the existence of nonzero, nonnegative left and right r-invariant vectors $\{\alpha_k\}$ and $\{\beta_k\}$ such that $\sum_k \alpha_k\beta_k < \infty$.

Indeed, we know from Theorem 4.1 that if such vectors exist, then $r \leqq R$. On the other hand, it follows from Lemma 5.1 that T cannot be r-transient, so that we must have $r = R$ and T R-recurrent; then the fact that T is R-positive follows from the corollary above. Note that by Lemma 5.2, the condition of the criterion can be weakened to allow one of the vectors to be r-subinvariant and the other r-superinvariant.

Taken in conjunction with criteria I and II following Lemma 4.1, this result shows that all the matrix properties introduced in the first section in terms of convergence behaviour, can also be defined in terms of solutions to the inequalities (1) and (2).

Finally, we consider the existence of r-superinvariant vectors for $r > R$. In this case, by Lemma 2.1, we can always choose an integer N such that $\sum_1^N l_{ii}^{(n)} r^n > 1$. It is now easy to construct an r-superinvariant vector. Put $x_i = 1, x_k = \sum_{n=1}^N l_{ik}^{(n)} r^n$ $(k \neq i)$. Then we have

$$r \sum_k x_k t_{kj} = r t_{ij} + r \sum_{k \neq i} \left(\sum_{n=1}^{N} l_{ik}^{(n)} r^n \right) t_{kj} = \sum_{n=1}^{N+1} l_{ij}^{(n)} r^n \geqq x_j \qquad (j \neq i)$$

and

$$r \sum_k x_k t_{ki} = \sum_{n=1}^{N+1} l_{ii}^{(n)} r^n > 1 = x_i \, .$$

Since $\sum_k l_{ik}^{(n)} F_{kj}(R) \leqq \sum_k t_{ik}^{(n)} F_{kj}(R) \leqq R^{-n} F_{ij}(R) < \infty$, the vectors constructed in this way certainly satisfy the condition $\sum_k x_k \beta_k < \infty$, where $\beta_k = F_{kj}(R)$. It also seems certain that when T is infinite-dimensional, they must include an infinite set of linearly independent elements. However, I have not been able to find a proof of this assertion.

The results we have obtained are summarized in the following theorem and its corollary.

THEOREM 5.1. *If T is r-transient, there are no nonzero left r-superinvariant vectors satisfying the condition $\sum_k x_k \beta_k < \infty$ for any nonzero right r-subinvariant vector $\{\beta_k\}$. If T is R-recurrent, $\beta_k = F_{kj}(R)$ (j fixed, $k \in J$), $\alpha_k = L_{ik}(R)$ (i fixed, $k \in J$), there are no nonzero left R-superinvariant vectors satisfying the condition $\sum x_k \beta_k < \infty$, except the solution $x_k = \alpha_k$ when T is R-positive. Corresponding dual results hold with the roles of left and right vectors interchanged. r-superinvariant vectors exist for all values of r such that $r > R$.*

COROLLARY. *Let $\beta_k = F_{kj}(R)$ (j fixed, $k \in J$). Then there exist left r-superinvariant vectors satisfying the condition $\sum x_k \beta_k < \infty$ if and only if either $r > R$ or $r = R$ and T is R-positive.*

We shall not enter into any detailed discussion of the difficult problem of determining when there exist r-invariant or r-superinvariant vectors for $r \leqq R$, which do not satisfy the condition of the theorem.[1] In this regard, Pruitt [10] has remarked that the criterion developed by Harris and Veech [6], [15] for the existence of stationary measures in a transient Markov chain carries over, mutatis

[1] *Note added in proof.* For a discussion in terms of boundary theory, see Mrs. S. C. Moy "Ergodic properties of expectation matrices" (to appear in J. Math. Mech.). This paper also gives a new proof of some of the results of [**17**] (including the periodic case) and some results along the lines of §6 of the present paper.

mutandis, to the context of any r-transient matrix. We mention only that, in the case there exist no r-invariant vectors, there may exist no r-superinvariant vectors either, and that in all examples known to us, if an r-invariant vector exists for some value $r_0 < R$, then r-invariant vectors exist for all values of r in the range $r_0 \leqq r \leqq R$.

The results of Theorems 4.1 and 5.1 can be put together in the form of the following "minimax" theorem.

THEOREM 5.2. *Let T be any irreducible, nonnegative matrix with convergence parameter R, $\alpha_k = L_{ik}(R)$, $\beta_k = F_{kj}(R)$. Then*

$$\sup_{x \geqq 0} \{\inf_i (Tx)_i / x_i\} = \inf_{\underset{\sim}{x} \geqq 0} \{\sup_i (Tx)_i / x_i\} = 1/R$$

provided that in the left hand member the limiting operations are taken over vectors $\underset{\sim}{x} \geqq 0$ satisfying the supplementary condition $\sum x_k \beta_k < \infty$.

This relation seems first to have been pointed out (for finite-dimensional matrices) by Wielandt [20]; it is related to the minimax theorem in the theory of games (see, for example, [2]).

The results of this section also have implication for eigenvectors with complex-valued components. This follows from the observation that if $\{x_j\}$ is an eigenvector associated with the eigenvalue $1/z$, the vector with components $|x_j|$ is r-superinvariant, where $r = |z|$. Thus from Lemmas 5.1 and 5.2, respectively, we obtain the following two corollaries, which will be used in § 7.

LEMMA 5.4. *If $|z| = r$ and T is r-transient, there are no non-zero left eigenvectors $\{x_k\}$ which are associated with the eigenvalue $1/z$ and also satisfy the condition*

$$\sum |x_k| \beta_k < \infty$$

for any nontrivial right r-subinvariant $\{\beta_k\}$.

LEMMA 5.5. *If T is R-positive, the solution $x_k = \alpha_k$ specified by Theorem 4.1 is up to a constant factor, the unique left eigenvector which is associated with the eigenvalue $1/R$ and satisfies the condition $\sum |x_k| \beta_k < \infty$, where β_k is the right eigenvector specified by Theorem 4.1.*

From Lemma 5.4 we obtain in turn the following uniqueness criterion for the nonhomogeneous equations (11).

LEMMA 5.6. *If* $|z| = r$ *and* T *is* r*-transient, equations* (11) *have a unique solution* (*namely* $x_k = T_{ik}(z)$) *satisfying the condition of Lemma* 5.4.

Proof. Uniqueness follows immediately from Lemma 5.4 and it is only necessary to prove that the condition of the lemma, is satisfied by $x_k = T_{ik}(z)$. This follows from Lemma 3.2, which asserts the convergence of the series $\sum_k T_{ik}(r)T_{kj}(r)$ (and hence of the series

$$\sum_k |T_{ik}(z)| T_{kj}(r))$$

and from the observation that $\beta_k = T_{kj}(r)$ is a right r-subinvariant vector.

The statement in the preamble to Lemma 3.3 can be obtained by appealing first to Lemma 5.6, and then to its dual.

6. Convergence of the sums $P_j(n; r)$, $Q_i(n; r)$ and $S(n; r)$. Only two principal modes of behaviour are open to the individual sequences $\{t_{ij}^{(n)}R^n\}$: they either tend to finite positive limits or to zero. (We exclude for the meanwhile the possibility of periodic behaviour, which will be considered separately in § 7). The behaviour of a sequence of sums such as $\{P_j(n; r)\}$ $(r \leqq R)$ may be more complicated. Even if the vector $\{u_j\}$ is nonnegative, it is not necessarily the case that the sequence tends to a limit, and when the limit does exist it may be infinite. Examples to illustrate all these possibilities are easily constructed using the "renewal" matrices described by Chung ([3], p. 41–42). The main purpose of this section is to find conditions that will ensure the convergence of such sequences of sums to finite limits.

We start with a consideration of the sums $P_j(n; r)$, assuming that $r \leqq R$ and that T has period $d = 1$. The behaviour of these sums is clarified by the observation that they can always be written in the form of a convolution: if $P_j(z) = \sum_{n=0}^{\infty} P_j(n; z) = \sum_k u_k T_{kj}(z)$, then from equations (3) and (4) we have

$$P_j(z) = \Phi_j(z) T_{jj}(z) \tag{18}$$

where

$$\Phi_j(z) = u_j + \sum_{k \neq j} u_k F_{kj}(z) .$$

The behaviour of the terms $t_{jj}^{(n)}$ figuring in the second part of the convolution is well-known; hence by imposing simple conditions on the terms in the first part it is possible to ensure regular behaviour of the convolution as a whole.

LEMMA 6.1. *Suppose that* T *is aperiodic,* $r \leqq R$*, and that the* *vector* $\{u_k\}$ *has nonnegative elements. Then a sufficient condition to ensure that the sums* $P_j(n; r)$ *are finite and tend to finite limits is that* $\sum_k u_k F_{kj}(r) < \infty$ *for some (and hence for all)* j.

Proof. Since the terms $\{t_{jj}^{(n)} r^n\}$ tend to a finite limit (which may, of course, be zero) if $r \leqq R$, a basic Abelian theorem asserts that the terms in the convolution will tend to a limit whenever the terms in the other sequence entering into the convolution have a finite sum. Recalling the representation of $\Phi(z)$, this is in fact precisely the condition of the lemma; "Fubini's theorem" shows that the individual terms must also be finite. A standard irreducibility argument now completes the proof of the lemma.

A condition equivalent to that of the lemma is that $\sum u_k \beta_k < \infty$ for some nonzero r-subinvariant vector β_k. For if β_k is any such vector, then from Lemma 4.1, $F_{kj}(r) \leqq \beta_k/\beta_j$ and so $\sum u_k \beta_k < \infty$ implies $\sum_k u_k F_{kj}(r) < \infty$; conversely, if the condition of the lemma is satisfied, we simply put $\beta_k = F_{kj}(r)$. In the sequel, we shall prefer this alternative form of the condition.

The argument of the lemma can be extended to evaluate the limit, for the same Abelian theorem ensures that (under the conditions of the lemma)

$$\lim_{n\to\infty} P_j(n; r) = \Phi_j(r) \lim_{n\to\infty} (t_{jj}^{(n)} r^n) .$$

The only case which presents any real interest is that in which $r = R$ and T is R-positive, for in all other cases the limit will be zero. In this case, if $\{\alpha_k\}$ and $\{\beta_k\}$ denote the unique R-invariant vectors specified by Theorem 4.1, we have $F_{kj}(R) = \beta_k/\beta_j$ and

$$\lim_{n\to\infty} t_{jj}^{(n)} R^n = \alpha_j \beta_j / \sum \alpha_k \beta_k ,$$

so that

$$\text{(19)} \quad \lim_{n\to\infty} P_j(n; R) = (u_j + \sum_{k\neq j} u_k \beta_k/\beta_j)(\alpha_j\beta_j/\sum\alpha_k\beta_k) = \alpha_j(\sum u_k\beta_k/\sum \alpha_k\beta_k) .$$

In this case, the condition of the lemma is also necessary, for otherwise, taking into account that $\{u_k\}$ is a nonnegative vector, it is easily shown that the terms in the convolution must diverge to infinity.

We can now formulate a more comprehensive theorem.

THEOREM 6.1. *Let* T *be aperiodic,* $r < R$. *Then sufficient conditions to ensure the convergence to finite limits of the sequences* $\{P_j(n; r)\}$, $\{Q_i(n; r)\}$ *are respectively the convergence of the series* $\sum |u_k| \beta_k$ *for some nonzero right* r*-subinvariant vector* $\{\beta_k\}$, *and the convergence of the series* $\sum \alpha_k |\nu_k|$ *for some nonzero left* r*-subinvari-*

ant vector $\{\alpha_k\}$. When the conditions are fulfilled.

(20) $$\lim_{n\to\infty} P_j(n; r) = \lim_{n\to\infty}\left(\sum_k u_k t_{kj}^{(n)} r^n\right) = \sum_k u_k\left(\lim_{n\to\infty} t_{kj}^{(n)} r^n\right)$$

(21) $$\lim_{n\to\infty} Q_i(n; r) = \lim_{n\to\infty}\left(\sum_k t_{ik}^{(n)} v_k r^n\right) = \sum_k \left(\lim_{n\to\infty} t_{ik}^{(n)} r^n\right) v_k .$$

If the vectors are nonnegative and the matrix is R-positive, $r = R$, the conditions are necessary as well as sufficient.

Proof. It follows as before that the convergence of the sum $\sum |u_k| \beta_k$ is equivalent to the convergence of the sums $\sum_k |u_k| F_{kj}(r)$. It remains to prove that this condition implies the convergence of the sequence $\{P_j(n; r)\}$ (where the vector $\{u_k\}$ is not necessarily nonnegative) and to justify the interchange of limits in (20). The dual results follow by exactly analogous arguments, and we shall not give the details.

To establish the first proposition we write the vector $\{u_k\}$ as the sum of real and imaginary parts, and then write each of these as the difference of two nonnegative vectors. It is easy to verify that the sum $\sum_k |u_k| F_{kj}(r)$ converges if and only if the corresponding sums for each component vector converge. Thus Lemma 6.1 applies to each part separately, and the result follows on recombining the four parts after passing to the limit.

Consider next the interchange of limits in (20). This is trivial (since both sides are zero) except in the case that T is R-positive and $r = R$. Then, evaluating the right hand side of (20) from the result $\lim_{n\to\infty} t_{ij}^{(n)} R^n = \alpha_j\beta_i / \sum_k \alpha_k\beta_k$ (Lemma 5.3, Corollary 1), we have

$$\sum_k u_k\left(\lim_{n\to\infty} t_{kj}^{(n)} R^n\right) = \left(\sum_k u_k\beta_k \Big/ \sum_k \alpha_k\beta_k\right)\alpha_j$$

and comparing this with (19) we obtain the desired result.

The necessity of the condition of the theorem when $r = R$, T is R-positive and the vector $\{u_k\}$ is nonnegative, has already been mentioned, and this completes the proof of the theorem.

The next result is concerned with the possibility of extending Theorem 6.1 to the sequence of double sums $\{S(n; r)\}$. Theorem 6.1 suggests that this sequence will be convergent whenever both

$$\sum_k u_k F_{kj}(r) < \infty \quad \text{and} \quad \sum_k L_{ik}^{(r)} v_k < \infty,$$

but in fact it appears that stronger conditions may be necessary. The best result that we have been able to obtain is stated in the theorem below.

THEOREM 6.2. *Sufficient conditions for the convergence of the sequence of double sums $S(n; r)$ $(r \leqq R)$ are the existence of left and right r-subinvariant vectors $\{\alpha_k\}$, $\{\beta_k\}$ respectively such that*

(i) $\sum |u_k| \beta_k < \infty$

(ii) $\sum \alpha_k |v_k| < \infty$

(iii) *either* (a) $|u_k| \leqq C\alpha_k$ *for some* $C < \infty, k \in J$

or (b) $|v_k| \leqq C'\beta_k$ *for some* $C' < \infty, k \in J$.

When these conditions are fulfilled,

$$\lim_{n\to\infty} S(n; r) = \lim_{n\to\infty} \Big(\sum_i \sum_j u_i t_{ij}^{(n)} v_j r^n\Big) = \sum_i \sum_j u_i \Big(\lim_{n\to\infty} t_{ij}^{(n)} r^n\Big) v_j \,. \tag{22}$$

If $r = R$, the matrix is R-positive, and the vectors are nonnegative, conditions (i) *and* (ii) *are necessary.*

Proof. Suppose that conditions (i), (ii) and (iiia) are satisfied; the case when conditions (i), (ii) and (iiib) are satisfied will then follow on interchanging the roles of left and right vectors.

As in the proof of Theorem 6.1, we note that the conditions on the vectors $\{u_k\}$, $\{v_k\}$ imply similar conditions on each of the four nonnegative parts into which these vectors can be decomposed. It is therefore sufficient to suppose that the vectors $\{u_k\}$ and $\{v_k\}$ are nonnegative.

From condition (iiia),

$$P_j(n; r) = \sum_i u_i t_{ij}^{(n)} r^n \leqq C \sum_i \alpha_i t_{ij}^{(n)} r^n \leqq C\alpha_j \,.$$

Consequently the sums $\sum_j P_j(n; r)v_j$ are dominated by the sum $\sum \alpha_j v_j$, which is convergent by (ii). The sums $\sum_j P_i(n; r)v_j$ are therefore uniformly convergent with respect to n, and letting $n \to \infty$ under the summation sign, we obtain

$$\lim_{n\to\infty} \sum_j P_j(n; r)v_j = \sum_j (\lim_{n\to\infty} P_j(n; r))v_j \leqq C \sum_j \alpha_j v_j < \infty$$

where from condition (i) and Theorem 6.1 the limits of the $P_j(n; r)$ exist and are finite. Since all terms in the summations are nonnegative, the quantities $\sum_j P_j(n; r)v_j$ can be identified with the double sums $S(n; r)$; and on evaluating the limits $\lim_{n\to\infty} P_j(n; r)$ from Theorem 6.1 (equation (20)) and again using nonnegativity, the quantity

$$\sum_j \Big[\lim_{n\to\infty} P_j(n; r)\Big] v_j$$

can be identified with the right hand side of (22).

It remains to prove the necessity of conditions (i) and (ii) when $r = R$ and the matrix is R-positive. Since the vectors $\{u_k\}$ and $\{v_k\}$ are assumed to be nonzero, there exists an index j for which $v_j > 0$. For this value of j, the limit of the sum $P_j(n; R)$ (which must exist, though it may be infinite—see the remark preceding Theorem 6.1) must in fact be finite; then from the necessity part of Theorem 6.1 it follows that there must exist a nonnegative right R-subinvariant $\{\beta_k\}$ such that $\sum_k u_k\beta_k < \infty$. Thus condition (i) holds, and a dual argument establishes condition (ii).

REMARK. If $r = R$ and the matrix is R-positive, the left and right subinvariant vectors are unique, invariant, and satisfy the condition $\sum \alpha_k\beta_k < \infty$. In this case, therefore, condition (iiia) implies condition (i), and condition (iiib) implies condition (ii).

COROLLARY. *If P is the stochastic matrix associated with an irreducible, aperiodic, positive-recurrent Markov chain, the matrix iterates tend weakly (and therefore also strongly) to their ergodic limit.*[1]

Proof. If the matrix is stochastic, the vector $\{1, 1, \cdots\}$ is right invariant, so that condition (i) is satisfied by any l_1 vector $\underset{\sim}{u}$ and condition (iiia) by any m-vector $\underset{\sim}{v}$. Since the chain is assumed to be positive recurrent, condition (iiia) implies condition (ii), as in the remark. Hence, adapting an obvious notation, the quantities $(P^n\underset{\sim}{u},\underset{\sim}{v})$ tend to their ergodic limits for all $\underset{\sim}{v} \in m$, i.e. the vectors $P\underset{\sim}{u}^n$ tend weakly to their ergodic limits. The equivalence of weak and strong convergence in l_1 is well-known, and this completes the proof.

The proof of Theorem 6.2 was suggested by the proof of an analogous property for continuous time chains given by Kendall and Reuter [7] (see especially pp. 130-132).

7. Periodic matrices and eigenvalues.

In this section we briefly summarize the eigenvalue properties of periodic matrices, and indicate the changes that must be made to the limit theorems of the preceding section in the periodic case.

Consider first the structure of an eigenvector when T is periodic. There is no loss of generality in supposing here that the corresponding eigenvalue is equal to unity.

[1] The terms weak and strong convergence refer to convergence of operators on the sequence space l_1; it is easily verified that any stochastic matrix defines such an operator.

LEMMA 7.1. *Let T have period d, periodic subclasses $C_\nu(\nu = 1, \cdots, d)$, and suppose that $\underset{\sim}{\alpha} = \sum_{\nu=1}^{d} \underset{\sim}{\alpha}_\nu$ where the components of $\underset{\sim}{\alpha}_\nu$ are zero outside C_ν and*

$$T\underset{\sim}{\alpha}_\nu = \underset{\sim}{\alpha}_{(\nu+1) \bmod (d)} \tag{23}$$

for each $m = 1, 2, \cdots, d$.

Moreover, the vector $\underset{\sim}{\alpha}^{(m)} = \sum_{\nu=1}^{d} \underset{\sim}{\alpha}_\nu e^{2\pi m\nu i/d}$ is an eigenvector associated with the eigenvalue $e^{2\pi mi/d}$.

Proof. It is sufficient to set $(\underset{\sim}{\alpha}_\nu)_k = \alpha_k(k \in C_\nu)$, $(\underset{\sim}{\alpha}_\nu)_k = O(k \notin C_\nu)$. Equation (23) then follows from the definition of the periodic subclasses and the equation $T\underset{\sim}{\alpha} = \underset{\sim}{\alpha}$, and the other assertions are trivial to verify.

COROLLARY. *If T is R-recurrent with period d, each of the points $\lambda_m = (1/R)\, e^{2\pi mi/d}$ is an eigenvalue of T.*

If T is R-positive, more precise results are available (Pruitt, [10] Sidák [12]-[14]).

THEOREM 7.1. *Let T have period d, periodic subclasses C_ν, and suppose that T is R-positive with left and right R-invariant vectors $\{\alpha_k\}$ and $\{\beta_k\}$. Then*

$$\sum_{k\in C_1} \alpha_k\beta_k = \sum_{k\in C_2} \alpha_k\beta_k = \cdots = \sum_{k\in C_\nu} \alpha_k\beta_k < \infty \tag{24}$$

and for each of the eigenvalues λ_m the eigenvector constructed as in Lemma 7.1 is the unique eigenvector satisfying the condition

$$\sum |x_k| \beta_k < \infty ;$$

these are the only eigenvectors satisfying this condition and associated with eigenvalues on the boundary of the disc $|\lambda| = 1/R$, or outside it.

Proof. The proof follows readily from our earlier results, in particular Lemmas 5.4 and 5.5, and we omit the details.

COROLLARY. *If T is finite-dimensional, the conclusions of the Perron-Frobenius theorem hold.*

Proof. To establish the Perron-Frobenius theorem, we have to incorporate in some way the compactness property of operators on finite-dimensional spaces. The most obvious way of doing this is to recall that any singularity of the resolvent is necessarily a pole. It then follows, in particular, that $1/R$ is a pole, and so the matrix must be R-positive (in the other two cases the singularity is worse than a pole). Moreover, the pole must be simple (i.e. $1/R$ must be a simple zero of the characteristic equation) because

$$\lim_{z \to R} (z - R)T_{ij}(z) = -(1/F'_{ii}(R))F_{ij}(R) < \infty$$

(note that the inequality $F_{ij}(R) < \infty$ makes essential use of irreducibility). The remaining assertions are subsumed under the theorem above.

Next we indicate the changes that must be made in Theorems 6.1 and 6.2 when the matrix is periodic. The only essential differences occur when the matrix is R-positive, and we shall restrict attention to this case.

Consider first the behaviour of the sum $P_j(n; R) \equiv P_j(n)$. This sum will take one of d different forms according as n is of the form $md + 1$, $md + 2, \cdots, md$. If $n = \nu \bmod d$, and if $j \in C_\alpha$, the summation defining $P_j(n)$ is effectively over the set of states C_β, where $\beta = (\alpha - \nu) \pmod d$, all the remaining terms vanishing. The appropriate generalization of Theorem 6.1 consists therefore of d separate statements, one for each value of ν. Denoting the left and right invariant vectors of the R-positive matrix T by $\{\alpha_k\}$, $\{\beta_k\}$ respectively, we have

THEOREM 7.2. *If T is R-positive and has period $d > 1$, and if $j \in C_\alpha$, the necessary and sufficient condition that the sequence $P_j(md+\nu)$ should tend to a finite limit as $m \to \infty$ is that $\sum_{k \in C_\beta} |u_k| \beta_k < \infty$, where $\beta = (\alpha - \nu)$ (mod d); moreover, when the condition is satisfied,*

$$\lim_{m\to\infty} P_j(md + \nu) = \sum_k u_k\Big(\lim_{m\to\infty} t_{kj}^{(md+\nu)} R^{md+\nu}\Big) = \frac{\sum_{k \in C_\beta} u_k\beta_k}{\sum_{k \in C_\beta} \alpha_k\beta_k}\alpha_j .$$

COROLLARY. *Under the same conditions, the necessary and sufficient condition that the $C - 1$ limit of the sequence $P_j(n)$ should exist and be finite is that $\sum_{k \in J} u_k\beta_k < \infty$; the value of the limit when it exists is equal to $(\sum_{k \in J} u_k\beta_k / \sum_{k \in J} \alpha_k\beta_k)\alpha_j$.*

Finally, we state an extended form of Theorem 6.2, putting $S(n; R) \equiv S(n)$.

THEOREM 7.3. *Suppose that T is R-positive and has period $d \geqq 1$, and also that the conditions of Theorem 6.2 are satisfied. Then for $\nu = 0, 1, \cdots, d - 1$*

$$\lim_{m\to\infty} S(md + \nu) = d\Big(\sum_{\alpha=1}^{d} U_\alpha V_{\alpha+\nu}\Big)\Big/\sum_{k \in J} \alpha_k\beta_k ,$$

where $U_\alpha = \sum_{k \in C_\alpha} u_k \beta_k$ *and* $V_\alpha = \sum_{k \in C_\alpha} \alpha_k v_k$.

COROLLARY. *Under the same conditions,*

$$C - 1 \lim_{n \to \infty} S(n) = \Big(\sum_{k \in J} u_k \beta_k\Big)\Big(\sum_{k \in J} \alpha_k v_k\Big) \Big/ \sum_{k \in J} \alpha_k \beta_k .$$

Acknowledgments. I should like to thank Professor D.G. Kendall for his help during the early stages of this investigation (part of which was included in the author's D. Phil. thesis), and Mr. E. Seneta for several helpful discussions and for bringing to my notice a number of important references. I am also grateful to the referee for a number of suggestions which have clarified the original presentation.

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Received October 4, 1965, and in revised form October 26, 1966.

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