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AN INEQUALITY FOR GENERALIZED MEANS

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This paper is concerned with the behavior of certain combinations of generalized means of positive real numbers, considered as functions of the index set. It is shown that these combinations are actually superadditive functions (over set unions) of the index set. Several previously established inequalities of this nature are obtained as corollaries of the main theorem, namely, certain results of R. Rado, W. N. Everitt, D. S. Mitrinović and P. M. Vasić, and H. Kestleman.

Let $\{a_i\}_{i=1}^n$ be a sequence of positive real numbers. A result of W. N. Everitt [2] (which generalizes an earlier result of R. Rado [8]; see also Tchakaloff [9], Jacobsthal [5], and Dinghas [1]) states that for any $1 \leqq m < n$, one has

$$
\begin{aligned}
& n\left[\frac{1}{n}\sum_{i=1}^n a_i - \left(\prod_{i=1}^n a_i\right)^{1/n}\right] \\
(1)\qquad & \geqq m\left[\frac{1}{m}\sum_{i=1}^m a_i - \left(\prod_{i=1}^m a_i\right)^{1/m}\right] \\
& \quad + (n-m)\left[\frac{1}{n-m}\sum_{i=m+1}^n a_i - \left(\prod_{i=m+1}^n a_i\right)^{1/(n-m)}\right].
\end{aligned}
$$

That is, n times the difference between the arithmetic mean and the geometric mean, considered as a function of the set

$$\{1, 2, \cdots, n\} = \{1, \cdots, m\} \cup \{m+1, \cdots, n\},$$

is "superadditive" over this set union. In [2] Everitt generalizes this result further by considering differences of more general means.

In a recent paper, [7] Mitrinović and Vasić established an inequality which may be interpreted in this same "superadditive" sense. They considered a ratio of means and restricted their attention to the union.

$$\{1, 2, \cdots, n\} = \{1, \cdots, n-1\} \cup \{n\}.$$

The intention of the present authors is to establish here, by means of a simple argument, an inequality which generalizes and unifies the results of Everitt and Mitrinović and Vasić, and which yields readily the conditions for equality in the result of Mitrinović and Vasić.

Let $\{a_1, a_2, \cdots\}$ and $\{p_1, p_2, \cdots\}$ be infinite sequences of positive numbers. Suppose I is a nonempty finite set of distinct positive integers. Then the mean of order $r(-\infty < r < +\infty)$ of the numbers

$\{a_i\}_{i\in I}$, with weights $\{p_i\}_{i\in I}$, is defined as follows:

$$M_r(a; p, I) = \begin{cases} \left(\dfrac{\sum_{i\in I} p_i a_i^r}{\sum_{i\in I} p_i}\right)^{1/r}, & -\infty < r < +\infty \\ & r \neq 0 \\ \left(\prod_{i\in I} a_i^{p_i}\right)^{1/\sum_{i\in I} p_i}, & r = 0\,. \end{cases},$$

It is known (see, e.g., Hardy, Littlewood, and Pólya [4; p. 15]) that this definition yields a continuous function of r in $-\infty < r < +\infty$. For an illustration of this notation consider again the inequality (1). If $I = \{1, \cdots, m\}$, $J = \{m+1, \cdots, n\}$, and $p_i = 1/n (i = 1, \cdots, n)$, then (1) may be rewritten, using the above notation, as follows:

$$\begin{aligned}\left(\sum_{I\cup J} p\right)&[M_1(a; p, I\cup J) - M_0(a; p, I\cup J)] \\ &\geqq \left(\sum_I p\right)[M_1(a; p, I) - M_0(a; p, I)] \\ &\quad + \left(\sum_J p\right)[M_1(a; p, J) - M_0(a; p, J)]\,,\end{aligned}$$

where sums of the form $\sum_{i\in I} p_i$ have been shortened to $\sum_I p$.

2. The main inequality.

The general inequality referred to in the previous section will now be established.

THEOREM. *Let I and J be nonempty disjoint finite sets of distinct positive integers, and let $\{a_i\}_{i\in I\cup J}$, $\{p_i\}_{i\in I\cup J}$, and $\{q_i\}_{i\in I\cup J}$ be sets of positive real numbers. Suppose $0 < \lambda, \mu$ and $\lambda + \mu \geqq 1$. Then, for any finite real numbers r and s, one has*

$$\begin{aligned}\left(\sum_{I\cup J} p\right)^{\lambda}\left(\sum_{I\cup J} q\right)^{\mu} &M_r^{\lambda r}(a; p, I\cup J) M_s^{\mu s}(a; q, I\cup J) \\ &\geqq \left(\sum_I p\right)^{\lambda}\left(\sum_I q\right)^{\mu} M_r^{\lambda r}(a; p, I) M_s^{\mu s}(a; q, I) \\ &\quad + \left(\sum_J p\right)^{\lambda}\left(\sum_J q\right)^{\mu} M_r^{\lambda r}(a; p, J) M_s^{\mu s}(a; q, J)\,.\end{aligned}$$

If $\lambda + \mu > 1$, then equality never holds; while, if $\lambda + \mu = 1$, then equality holds if and only if the ordered pairs

$$\left(\left(\sum_I q\right) M_s^s(a; q, I), \left(\sum_J q\right) M_s^s(a; q, J)\right)$$

and

$$\left(\left(\sum_I p\right) M_r^r(a; p, I), \left(\sum_J p\right) M_r^r(a; p, J)\right)$$

are proportional.

Proof. Jensen's inequality (see Hardy, Littlewood, and Pólya [4; p. 29]) asserts that, if A_1, A_2, B_1, and B_2 are positive real numbers, then

$$(A_1 + A_2)^\lambda (B_1 + B_2)^\mu \geqq A_1^\lambda B_1^\mu + A_2^\lambda B_2^\mu \,. \tag{2}$$

If $\lambda + \mu > 1$, then equality never holds; while, if $\lambda + \mu = 1$, then equality holds if and only if the ordered pairs (A_1, A_2) and (B_1, B_2) are proportional. The inequality of the theorem follows immediately from (2) upon choosing

$$A_1 = \Big(\sum_I p\Big) M_r^r(a; p, I),\ A_2 = \Big(\sum_J p\Big) M_r^r(a; p, J)\,,$$
$$B_1 = \Big(\sum_I q\Big) M_s^s(a; q, I),\ B_2 = \Big(\sum_J q\Big) M_s^s(a; q, J)\,,$$

and noting that, for instance,

$$\Big(\sum_{I\cup J} p\Big) M_r^r(a; p, I \cup J)$$
$$= \Big(\sum_I p\Big) M_r^r(a; p, I) + \Big(\sum_J p\Big) M_r^r(a; p, J) = A_1 + A_2\,.$$

REMARK 1. If λ and μ are such that

$$\lambda\mu < 0 \quad \text{and} \quad \lambda + \mu = 1\,,$$

then the sense of the inequality of the above theorem is reversed, while the necessary and sufficient condition for equality remains unchanged. This is a consequence of the fact that the sense of inequality (2) is reversed under these assumptions on λ and μ, while the necessary and sufficient condition for equality in (2) remains unchanged (see Hardy, Littlewood, and Pólya [4; p. 24]).

3. Special cases. In the corollaries which follow, as in the theorem above, I and J denote nonempty disjoint finite sets of distinct positive integers, and $\{a_i\}_{i\in I\cup J}$, $\{p_i\}_{i\in I\cup J}$, and $\{q_i\}_{i\in I\cup J}$ are sets of positive real numbers.

The following corollary may be interpreted as a direct generalization of the inequality of Mitrinović and Vasić [7; Th. 3], which is itself given as Corollary 2.

COROLLARY 1. *For any finite real numbers r and s, such that $rs < 0$, one has*

$$\frac{\left(\sum_{I\cup J} p\right)^{s/(s-r)}}{\left(\sum_{I\cup J} q\right)^{r/(s-r)}}\left[\frac{M_r(a;\,p,\,I\cup J)}{M_s(a;\,q,\,I\cup J)}\right]^{rs/(s-r)}$$
$$\geqq \frac{\left(\sum_{I} p\right)^{s/(s-r)}}{\left(\sum_{I} q\right)^{r/(s-r)}}\left[\frac{M_r(a;\,p,\,I)}{M_s(a;\,q,\,I)}\right]^{rs/(s-r)} + \frac{\left(\sum_{J} p\right)^{s/(s-r)}}{\left(\sum_{J} q\right)^{r/(s-r)}}\left[\frac{M_r(a;\,p,\,J)}{M_s(a;\,q,\,J)}\right]^{rs/(s-r)}$$

Equality holds if and only if the ordered pairs

$$\left(\left(\sum_I q\right)M_s^s(a;\,q,\,I),\ \left(\sum_J q\right)M_s^s(a;\,q,\,J)\right)$$

and

$$\left(\left(\sum_I p\right)M_r^r(a;\,p,\,I),\ \left(\sum_J p\right)M_r^r(a;\,p,\,J)\right)$$

are proportional. If $rs > 0$ and $r \neq s$, then the sense of this inequality reverses, while the necessary and sufficient condition for equality remains unchanged.

Proof. When $rs < 0$, one has

$$\frac{s}{s-r} = \frac{1}{1-\dfrac{r}{s}} > 0 \quad \text{and} \quad \frac{-r}{s-r} = \frac{1}{1-\dfrac{s}{r}} > 0\,.$$

Choosing

$$\lambda = \frac{s}{s-r} \quad \text{and} \quad \mu = -\frac{r}{s-r}$$

in the theorem gives the desired result. Also, if $rs > 0$ and $r \neq s$, then the reversal of the sense of this inequality follows from Remark 1, upon choosing λ and μ as above.

Taking $I = \{1, \cdots, n-1\}$ and $J = \{n\}$ in Corollary 1 gives the following inequality of Mitrinović and Vasić [7; Th. 3], together with the necessary and sufficient condition for equality to hold.

COROLLARY 2. *For any finite real numbers r and s, such that $rs < 0$, one has*

$$\frac{\left(\sum_1^n p\right)^{s/(s-r)}}{\left(\sum_1^n q\right)^{r/(s-r)}}\left[\frac{M_r(a;p,I\cup J)}{M_s(a;q,I\cup J)}\right]^{rs/(s-r)}$$
$$\geqq \frac{\left(\sum_1^{n-1} p\right)^{s/(s-r)}}{\left(\sum_1^{n-1} q\right)^{r/(s-r)}}\left[\frac{M_r(a;p,I)}{M_s(a;q,I)}\right]^{rs/(s-r)} + \frac{p_n^{s/(s-r)}}{q_n^{r/(s-r)}},$$

where $I=\{1,\cdots,n-1\}$ *and* $J=\{n\}$. *Equality holds if and only if*

$$\frac{q_n}{p_n}a_n^{s-r}=\frac{\sum_1^{n-1} q}{\sum_1^{n-1} p}\cdot\frac{M_s^s(a;q,I)}{M_r^r(a;p,I)}.$$

If $rs>0$ *and* $r\neq s$, *then the sense of this inequality reverses, while the necessary and sufficient condition for equality remains unchanged.*

The next corollary is a consequence of Corollary 1 and the arithmetic mean-geometric mean inequality. As special cases of this corollary, there will follow inequalities of Mitrinović and Vasić [7; Th. 1] and Kestleman [6].

COROLLARY 3. *For any finite real numbers* r *and* s, *such that* $rs<0$, *one has*

$$\left(\frac{\sum_{I\cup J} p}{\sum_{I\cup J} q}\right)^{\sum_{I\cup J} p}\left[\frac{M_r(a;p,I\cup J)}{M_s(a;q,I\cup J)}\right]^{s\sum_{I\cup J} p}$$
$$\leqq \left(\frac{\sum_I p}{\sum_I q}\right)^{\sum_I p}\left[\frac{M_r(a;p,I)}{M_s(a;q,I)}\right]^{s\sum_I p}$$
$$\cdot\left(\frac{\sum_J p}{\sum_J q}\right)^{\sum_J p}\left[\frac{M_r(a;p,J)}{M_s(a;q,J)}\right]^{s\sum_J p}.$$

Proof. The right-hand side (lower bound) of the inequality of Corollary 1, when divided by $\sum_{I\cup J} p$, may be rewritten as

$$\frac{\sum_I p}{\sum_{I\cup J} p}\cdot\left(\frac{\sum_I p}{\sum_I q}\right)^{r/(s-r)}\left[\frac{M_r(a;p,I)}{M_s(a;q,I)}\right]^{rs/(s-r)}$$
$$+\frac{\sum_J p}{\sum_{I\cup J} p}\cdot\left(\frac{\sum_J p}{\sum_J q}\right)^{r/(s-r)}\left[\frac{M_r(a;p,J)}{M_s(a;q,J)}\right]^{rs/(s-r)}.$$

This quantity, when viewed as an arithmetic mean of two positive numbers, with weights $\sum_I p/\sum_{I\cup J} p$ and $\sum_J p/\sum_{I\cup J} p$, may be bounded below by the geometric mean of the two positive numbers, namely,

$$\left\{\left(\frac{\sum_I p}{\sum_I q}\right)^{r/(s-r)}\left[\frac{M_r(a;\,p,\,I)}{M_s(a;\,q,\,I)}\right]^{rs/(s-r)}\right\}^{\sum_I p/\sum_{I\cup J} p}$$

$$\cdot\left\{\left(\frac{\sum_J p}{\sum_J q}\right)^{r/(s-r)}\left[\frac{M_r(a;\,p,\,J)}{M_s(a;\,q,\,J)}\right]^{rs/(s-r)}\right\}^{\sum_J p/\sum_{I\cup J} p}.$$

Thus, this quantity is bounded above by the left-hand side (upper bound) of the inequality of Corollary 1, divided by $\sum_{I\cup J} p$. Raising both of these quantities to the $((s-r)/r) < 0$ power reverses their order relation and yields the desired inequality.

REMARK 2. In Corollary 3, upon choosing $s = 1$, $I = \{1,\cdots,n-1\}$, $J = \{n\}$, and letting $r \to 0-$, one obtains the inequality of Mitrinović and Vasić [7; Th. 1]. This result in turn implies an inequality of Kestleman [6], as is mentioned in [7].

REMARK 3. In the last remark it was shown how Theorem 1 of [7] follows as a consequence of Corollary 3. However, this result, together with the necessary and sufficient condition for equality, may be obtained more directly by a simple application of the arithmetic mean-geometric mean inequality, as follows. Since

$$M_0(a;\,p,\,I\cup J) = [M_0(a;\,p,\,I)]^{\sum_I p/\sum_{I\cup J} p}\cdot[M_0(a;\,p,\,J)]^{\sum_J p/\sum_{I\cup J} p}$$

and

$$\begin{aligned} M_1(a;\,q,\,I\cup J) &= \left(\frac{\sum_I p}{\sum_{I\cup J} p}\right)\left(\frac{\sum_{I\cup J} p}{\sum_I p}\cdot\frac{\sum_I q}{\sum_{I\cup J} q}\right)M_1(a;\,q,\,I) \\ &\quad + \left(\frac{\sum_J p}{\sum_{I\cup J} p}\right)\left(\frac{\sum_{I\cup J} p}{\sum_J p}\cdot\frac{\sum_J q}{\sum_{I\cup J} q}\right)M_1(a;\,q,\,J) \\ &\geqq \left(\frac{\sum_{I\cup J} p}{\sum_{I\cup J} q}\cdot\frac{\sum_I q}{\sum_I p}M_1(a;\,q,\,I)\right)^{\sum_I p/\sum_{I\cup J} p} \\ &\quad \cdot\left(\frac{\sum_{I\cup J} p}{\sum_{I\cup J} q}\cdot\frac{\sum_J q}{\sum_J p}M_1(a;\,q,\,J)\right)^{\sum_J p/\sum_{I\cup J} p}, \end{aligned}$$

where equality holds in the last inequality if and only if

$$\frac{\sum_I q}{\sum_I p} M_1(a; q, I) = \frac{\sum_J q}{\sum_J p} M_1(a; q, J) , \tag{3}$$

one has, upon division, that

$$\left[\left(\frac{\sum_{I\cup J} p}{\sum_{I\cup J} q}\right)\frac{M_0(a; p, I\cup J)}{M_1(a; q, I\cup J)}\right]^{\sum_{I\cup J} p}$$
$$\leqq \left[\left(\frac{\sum_I p}{\sum_I q}\right)\frac{M_0(a; p, I)}{M_1(a; q, I)}\right]^{\sum_I p} \cdot \left[\left(\frac{\sum_J p}{\sum_J q}\right)\frac{M_0(a; p, J)}{M_1(a; q, J)}\right]^{\sum_J p} ,$$

with equality if and only if (3) holds. This last inequality is that of Theorem 1 of [7] referred to in Remark 2. It is to be noted that the equality condition follows readily from the method of proof.

It will next be shown how an extension of the result of Everitt [2, Th. 1] follows from the inequality of the theorem of § 2.

COROLLARY 4. *Let s be a real number. If $1 < s$, then*

$$\left(\sum_{I\cup J} p\right) M_s(a; p, I\cup J) \geqq \left(\sum_I p\right) M_s(a; p, I)$$
$$+ \left(\sum_J p\right) M_s(a; p, J) ,$$

with equality if and only if

$$M_s(a; p, I) = M_s(a; p, J) .$$

If $s = 1$, then equality always holds. If $s < 1$, then the sense of the inequality is reversed, while the necessary and sufficient condition for equality remains unchanged.

Proof. Suppose first that $1 < s$. Set $q_i = p_i$ for $i \in I \cup J$, $r = 0$, $\lambda = 1 - 1/s$, and $\mu = 1/s$ in the theorem to obtain Corollary 4 when $1 < s$.

If $s = 1$, then the result is immediate, since the left-hand and right-hand sides of the inequality are always equal.

If $s < 1$ and $s \neq 0$, then Remark 1, with $q_i = p_i$ for $i \in I \cup J$, $r = 0$, $\lambda = 1 - 1/s$, and $\mu = 1/s$, gives the desired result (upon noting that, in this case, one has $\lambda\mu < 0$ and $\lambda + \mu = 1$).

Finally, the inequality corresponding to $s = 0$ follows upon letting s tend to zero in the inequality already established for $s < 1$ and $s \neq 0$. The necessary and sufficient condition for equality does not appear to follow from the corresponding condition for $s < 1$ and $s \neq 0$, upon letting s tend to zero. However, its validity is a consequence of the necessary and sufficient condition for equality in the arithmetic mean-geometric mean inequality, since

$$M_0(a; p, I \cup J) = [M_0(a; p, I)]^{\sum_I p / \sum_{I \cup J} p} \cdot [M_0(a; p, J)]^{\sum_J p / \sum_{I \cup J} p} .$$

REMARK 4. The above Corollary 4 shows that Everitt's result (which is the case $s \geqq 0$) also holds when the order of the mean is negative.

Letting $a_i, b_i > 0$, $i \in I$, and $1/\lambda + 1/\mu = 1$, $\lambda > 1$, Hölder's inequality asserts that

$$H(a, b; I) = \left(\sum_I a^\lambda\right)^{1/\lambda}\left(\sum_I b^\mu\right)^{1/\mu} - \sum_I ab \geqq 0 .$$

In fact, Everitt [3] has shown that if $I \cap J = \varnothing$, then

$$H(a, b; I \cup J) \geqq H(a, b, I) + H(a, b, J) .$$

That this last inequality follows from Corollary 1 is proved in the next corollary.

COROLLARY 5. *Let* $a_i, b_i > 0$, $i \in I \cup J$, $I \cap J = \varnothing$. *Then*

$$\begin{aligned} &\left(\sum_{I \cup J} a^\lambda\right)^{1/\lambda}\left(\sum_{I \cup J} b^\mu\right)^{1/\mu} - \sum_{I \cup J} ab \\ &\quad \geqq \left(\sum_I a^\lambda\right)^{1/\lambda}\left(\sum_I b^\mu\right)^{1/\mu} - \sum_I ab \\ &\qquad + \left(\sum_J a^\lambda\right)^{1/\lambda}\left(\sum_J b^\mu\right)^{1/\mu} - \sum_J ab . \end{aligned}$$

Equality holds if and only if the ordered pairs

$$\left(\sum_I b^\mu, \sum_J b^\mu\right) \quad \textit{and} \quad \left(\sum_I a^\lambda, \sum_J a^\lambda\right)$$

are proportional.

Proof. Since $\sum_{I \cup J} ab = \sum_I ab + \sum_J ab$, it suffices to show that

$$\left(\sum_{I \cup J} a^\lambda\right)^{1/\lambda}\left(\sum_{I \cup J} b^\mu\right)^{1/\mu} \geqq \left(\sum_I a^\lambda\right)^{1/\lambda}\left(\sum_I b^\mu\right)^{1/\mu} + \left(\sum_J a^\lambda\right)^{1/\lambda}\left(\sum_J b^\mu\right)^{1/\mu} .$$

This inequality follows immediately from the inequality of Corollary 1 by choosing $r = \lambda$, $s = -\mu$, $p_i = 1$, and $q_i = (a_i b_i)^{-s}$ for $i \in I \cup J$. The necessary and sufficient condition for equality is a consequence of that in Corollary 1.

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