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ESTIMATES FOR THE TRANSFINITE DIAMETER WITH APPLICATIONS TO CONFORMAL MAPPING

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Let $f(z)$ be a member of the family S of functions regular and univalent in the open unit disk whose Taylor expansion is of the form: $f(z) = z + a_2z^2 + \cdots$. Let D_w be the image of the unit disk under the mapping: $w = f(z)$. An inequality for the transfinite diameter of n compact sets in the plane $\{T_i\}_1^n$ is established, generalizing a result of Renngli:

$$d(T_1 \cap T_2)\cdot d(T_1 \cup T_2) \leqq d(T_1)\cdot d(T_2)\,.$$

This inequality is applied to derive covering theorems for D_w relative to a class of curves issuing from $w = 0$, arcs on the circle: $|w| = R$ as well as other point sets.

I. Preliminary considerations.

DEFINITION (1.1). Let E be a compact set in the plane. Set:

$$V(z_1, \cdots, z_n) = \prod_{k>l}^{n} (z_k - z_l) \qquad n \geqq 2\,, \quad z_i \in E\,,$$

$$V_n = V_n(E) = \max_{z_1,\cdots,z_n \in E} |\, V(z_1, \cdots, z_n)\,|$$

and

$$d_n = d_n(E) = V_n^{2/n(n-1)}\,.$$

The transfinite diameter of E is then defined by: $d = d(E) = \lim_{n\to\infty} d_n$.

A full discussion of the transfinite diameter and related constants can be found in [2, Chapter 7].

The following is a theorem of Hayman [3]:

THEOREM (1.2). *Suppose $f(z)$ is a function meromorphic in the unit disk with a simple pole of residue k at the origin, i.e., the expansion of $f(z)$ about the origin is of the form:*

$$f(z) = \frac{k}{z} + a_0 + a_1 z + \cdots\,.$$

Let D_w denote the image of $|z| < 1$ under the mapping $w = f(z)$ and let E_w denote the complement of D_w in the w-plane. Then: $d(E_w) \leqq k$ with equality if and only if $f(z)$ is univalent.

Using Hayman's theorem is easy to prove the following:

THEOREM (1.3). *Let* $w(z) = kz + a_2z^2 + a_3z^3 + \cdots$ *be a function univalent in* $|z| < 1$ *and* D_w *the image of* $|z| < 1$ *under* $w(z)$. *Then the complement of the image of* D_w *under the mapping*: $\zeta = 1/w$, *which we denote by* E_ζ, *has transfinite diameter*: $1/k$. *In particular, if* $w(z) = z + a_2z^2 + \cdots$ *then* $d(E_\zeta) = 1$.

We will need to know the transfinite diameter of several specific sets.

LEMMA (1.4). *Let* E *be the set union of*:

(i) *an arc of central angle* θ, $0 \leqq \theta \leqq 2\pi$ *lying on* $|w| = 1$ *with midpoint*: $w = 1$.

(ii) *a linear segment* $[a, b]$, $0 \leqq a \leqq 1 \leqq b$. *Then the transfinite diameter of* E *expressed as a function of* a, b *and* θ *is given by*

$$d(E) = \frac{\cos^2\frac{\theta}{4}\left[(1+b)\left(1+a^2-2a\cos\frac{\theta}{2}\right)^{1/2} + (1+a)\left(1+b^2-2b\cos\frac{\theta}{2}\right)^{1/2}\right]}{2\left[(1+a)+\left(1+a^2-2a\cos\frac{\theta}{2}\right)^{1/2}\right]\times\left[(1+b)-\left(1+b^2-2b\cos\frac{\theta}{2}\right)^{1/2}\right]}$$

where positive roots are taken throughout.

Proof. A univalent mapping, $w = f(z)$, of $|z| < 1$ onto the complement of E with a simple pole at $z = 0$ will be constructed. According to Theorem (1.2) the residue of the mapping function is the transfinite diameter of E. Define:

$$w_1(z) = (z + \alpha)/(1 + \alpha z)$$

where:

$$\alpha = \frac{d - c + \csc\frac{\theta}{4}}{c} - \left[\left(\frac{d - c + \csc\frac{\theta}{4}}{c}\right)^2 - 1\right]^{1/2}, \quad d > 1\,, \quad 2c - d > 0\,.$$

Define:

$$w_2 = \frac{1}{2}\left(w_1 + \frac{1}{w_1}\right) \qquad w_3 = c(w_2 + 1) - d$$

$$w_4 = (w_3^2 - 1)^{1/2} \qquad w_5 = \frac{\cot\frac{\theta}{4} + w_4}{\cot\frac{\theta}{4} - w_4}\,.$$

The composition of these five mappings is given by:

$$w(z) = \frac{\cot\frac{\theta}{4} + \left\{\left[\frac{1}{2}c\left(\frac{z+\alpha}{1+\alpha z} + \frac{1+\alpha z}{z+\alpha} + 2\right) - d\right]^2 - 1\right\}^{1/2}}{\cot\frac{\theta}{4} - \left\{\left[\frac{1}{2}c\left(\frac{z+\alpha}{1+\alpha z} + \frac{1+\alpha z}{z+\alpha} + 2\right) - d\right]^2 - 1\right\}^{1/2}}.$$

$w(z)$ maps $|z| < 1$ onto the exterior of E (upon proper choice of the parameters c and d, to be made presently); it has a simple pole at the origin of residue:

$$\frac{c}{\csc\frac{\theta}{4} + 2(d-c)\sec^2\frac{\theta}{4} + \tan\frac{\theta}{4}\sec\frac{\theta}{4}(d^2+1-2cd)}.$$

This is the transfinite diameter of E. To express it in terms of a, b and θ we note that the point $w = b$ is the image of $w_2 = 1$, and the point $w = a$ is the image of $w_2 = -1$. Using this to solve for c and d we find:

$$d = \frac{\left[a^2 + 1 - 2a\cos\frac{\theta}{2}\right]^{1/2}}{(a+1)\sin\frac{\theta}{4}}$$

$$c = \frac{\left[a^2 + 1 - 2a\cos\frac{\theta}{2}\right]^{1/2}}{2(a+1)\sin\frac{\theta}{4}} + \frac{\left[b^2 + 1 - 2b\cos\frac{\theta}{2}\right]^{1/2}}{2(b+1)\sin\frac{\theta}{4}}.$$

Substituting these values in the above expression for the residue we arrive at the expression given in the statement of the lemma.

When $a = b = 1$ the set E is simply an arc of central angle θ on the unit circle. Using the lemma we find: $d(1, 1, \theta) = \sin\theta/4$.

LEMMA (1.5). *Let E be the set union of two linear segments issuing from the origin at an angle $2\pi\alpha$, $0 < \alpha \leqq 1/2$, each of length*: $4\alpha^\alpha(1-\alpha)^{1-\alpha}$. *Then*: $d(E) = 1$.

Proof. The mapping of $|z| < 1$ onto the exterior of E is given by the Schwarz-Christoffel formula:

$$w = c\cdot\int_0^z \frac{(z+1)^{1-2\alpha}(z-1)^{2\alpha-1}(z-1+2\alpha-2[\alpha^2-\alpha]^{1/2}) \times (z-1+2\alpha+2[\alpha^2-\alpha]^{1/2})}{z^2}\,dz$$

$$= c\cdot\frac{(z+1)^{2-2\alpha}(z-1)^{2\alpha}}{z}.$$

The residue of this function (the transfinite diameter of E) is c. Noting that the map carries $z = 1 - 2\alpha + 2(\alpha^2 - \alpha)^{1/2}$ onto $w = 4\alpha^\alpha(1 - \alpha)^{1-\alpha}e^{i\pi\alpha}$ we find that $d(E) = |c| = |e^{i\pi\alpha}/(-1)^\alpha| = 1$.

Finally, we describe two types of symmetrization.

Steiner symmetrization of a plane set E with respect to a straight line l in the plane transforms E into a set E' characterized by the following:

(i) E' is symmetric with respect to l.

(ii) Any straight line orthogonal to l that intersects one of the sets E or E' also intersects the other. Both intersections have the same linear measure, and

(iii) The intersection with E' consists of just one line segment, and may degenerate to a point.

Circular symmetrization of a plane set E with respect to the positive real axis transforms E into a set E' characterized by the following:

(i) E' is symmetric with respect to the real axis.

(ii) Any circle $|z| = r$, $0 \leqq r < \infty$ that intersects one of the sets E or E' also intersects the other. Both intersections have the same linear measure, and

(iii) The intersection with E' consists of just one arc with its midpoint on the positive real axis, and may degenerate to a point.

The following theorem describes the effect of these symmetrizations on the transfinite diameter [5; p. 6 and Note A]:

THEOREM (1.6). *Neither Steiner nor circular symmetrization increase the transfinite diameter.*

II. Estimates for the transfinite diameter.

A recent result of Renngli [6] is the following:

THEOREM (2.1). *If T_1 and T_2 are compact sets in the plane, then*

$$d(T_1 \cup T_2)\cdot d(T_1 \cap T_2) \leqq d(T_1)\cdot d(T_2) .$$

We will now generalize this to obtain an inequality for n compact sets.

THEOREM (2.2). *If $T_1, T_2, \cdots, T_n$ are compact sets in the plane, let C_k be the set of all points contained in at least k of the T_j's. Then:*

$$\prod_{k=1}^{n} d(C_k) \leqq \prod_{k=1}^{n} d(T_k) . \tag{1}$$

Proof. For $n = 1$ this is a triviality. For $n = 2$ it is identical with Renngli's result:

$$d(T_1 \cup T_2)\cdot d(T_1 \cap T_2) \leqq d(T_1)\cdot d(T_2) .$$

Suppose the theorem is already established for $n-1$ sets. Let B_k be the set of all points lying in at least k of the sets $T_1, T_2, \cdots, T_{n-1}$. Obviously: $B_{n-1} \subset B_{n-2} \subset \cdots \subset B_1$. Also:

$$C_n = B_{n-1} \cap T_n\,, \qquad C_1 = B_1 \cup T_n\,, \tag{2}$$

$$C_k = B_k \cup \{B_{k-1} \cap T_n\} \qquad (k = 2, 3, \cdots, n-1)\,. \tag{3}$$

If $d(B_{n-1} \cap T_n) = d(C_n) = 0$, (1) is certainly true.
If $d(B_{n-1} \cap T_n) \neq 0$, then, *a fortiori*,

$$d(B_k \cap T_n) \neq 0 \qquad (k = 1, 2, \cdots, n-1)\,.$$

By (2), (3) and Renngli's inequality:

$$d(C_n) = d(B_{n-1} \cap T_n)$$

$$d(C_k)\cdot d(B_k \cap T_n) = d(C_k)\cdot d(B_k \cap B_{k-1} \cap T_n) \leqq d(B_k)\cdot d(B_{k-1} \cap T_n) \qquad (k = 2, \cdots, n-1)$$

$$d(C_1)\cdot d(B_1 \cap T_n) \leqq d(B_1)\cdot d(T_n)\,.$$

Multiplying these inequalities and dividing both sides by $\prod_{k=1}^{n} d(B_k \cap T_n)$ yields

$$\prod_{k=1}^{n} d(C_k) \leqq \prod_{k=1}^{n-1} d(B_k) d(T_n)$$

and the theorem is proved, since by the induction hypothesis

$$\prod_{k=1}^{n-1} d(B_k) \leqq \prod_{k=1}^{n-1} d(T_k)\,.$$

DEFINITION (2.3). A point set T will be called a broken ray provided

(i) for every $r \geqq 0$ there is a point $z \in T$ such that: $|z| = r$.

(ii) the set of numbers $r \geqq 0$ for which there is more than one point $z \in T$ such that: $|z| = r$ is a set of measure zero.

DEFINITION (2.4). Let T be a subset of a broken ray. The point sets: $\eta_1 T, \eta_2 T, \cdots, \eta_n T$ where $\{\eta_k\}_1^n$ are the n-th roots of unity, will be called symmetric images of T. The point set: $\{\bigcup_{k=1}^{n} \eta_k \cdot T\}$ will be called the set of n-fold symmetry generated by T and will be denoted by $T^{(n)}$. Subsets of $T^{(n)}$ will be denoted by $\tilde{T}^{(n)}$.

DEFINITION (2.5). Let T be a subset of a broken ray, $T^{(n)}$ the set of n-fold symmetry generated by T and $\tilde{T}^{(n)}$ a subset of $T^{(n)}$. We define the circular projection of $\tilde{T}^{(n)}$ as a subset, $\tilde{\tau}^{(n)}$, of the set of n-fold symmetry, $\tau^{(n)}$, generated by the positive real axis, τ. A point $z = \eta_k \cdot r$ will belong to the projection $\tilde{\tau}^{(n)}$ if and only if there is a point: $\zeta \in \eta_k \cdot T \cap \tilde{T}^{(n)}$ such that $|\zeta| = r$.

DEFINITION (2.6). Let $\tilde{\tau}^{(n)}$ be a set such as described in definition (2.5). We will use the symbol l_k to denote the measure of the set of real numbers r, $0 \leqq r < \infty$ such that at least k of the symmetric images of r lie in $\tilde{\tau}^{(n)}$.

REMARK (2.7). Let L denote the linear measure of $\tilde{\tau}^{(n)}$; that is, the sum of the linear measures of the n legs of $\tilde{\tau}^{(n)}$. Then

$$\sum_{k=1}^{n} l_k = L .$$

The reason is that if I is a set of real numbers which have symmetric images on exactly k legs of $\tilde{\tau}^{(n)}$ the measure of I is included in: $l_1, l_2, \cdots, l_k$; that is, it is counted k times in: $\sum_{k=1}^{n} l_k$.

The following theorem of Fekete is essential to our work [2; page 259].

THEOREM (2.8). *Let E be a compact set and $p(z)$ a polynomial of degree n:*

$$p(z) = z^n + c_1 z^{n-1} + \cdots + c_n.$$

Let E_0 be the set of all points z such that $p(z)$ lies in E; we will call E_0 a root set of E. Then: $d(E_0) = d(E)^{1/n}$.

THEOREM (2.9). *Suppose $\tilde{T}^{(n)}$ is a subset of a set of n-fold symmetry with: $d(\tilde{T}^{(n)}) = 1$, and $\tilde{\tau}^{(n)}$ its circular projection. If l_k ($k = 1, 2, \cdots, n$) represent the measures defined in* (2.6), *then:*

$$\prod_{k=1}^{n} l_k \leqq 4 .$$

Equality occurs when $\tilde{T}^{(n)}$ is itself a set of n-fold symmetry, consisting of a single component and identical with its circular projection: $\tilde{T}^{(n)} = \tilde{\tau}^{(n)}$.

Proof. Let $T_k = \eta_k \cdot \tilde{T}^{(n)}$, ($k = 1, 2, \cdots, n$). Clearly:

$$d(T_k) = d(\tilde{T}^{(n)}) = 1 \qquad (k = 1, 2, \cdots, n) \tag{4}$$

since the transfinite diameter is unaffected by rigid motions.

Let C_k be the set of all points contained in at least k of the T_j's; that is, the set of all points z such that at least k of the symmetric images of z lie in $\tilde{T}^{(n)}$. Each of the sets C_k is a set of n-fold symmetry.

Let γ_k be the circular projection of C_k. In view of our description of the sets C_k it is not difficult to see that the measure of a leg of γ_k is l_k.

Let B_k be the set of which C_k is the root set with respect to the polynomial $p(z) = z^n$. Since C_k is a set of n-fold symmetry B_k is a subset of a single broken ray. Let β_k be the set of which γ_k is the root set with respect to the polynomial $p(z) = z^n$. As above, β_k will be a subset of a single broken ray; in this case the positive real axis.

Since γ_k is the circular projection of C_k it follows that β_k is the circular projection of B_k. When $n = 1$ circular projection is the same transformation as circular symmetrization. Therefore:

$$\begin{aligned} d(C_k) &= d(B_k)^{1/n} && \text{by Theorem (2.8)} \\ &\geqq d(\beta_k)^{1/n} && \text{by Theorem (1.6)} \\ &\geqq \left[\frac{(l_k)^n}{4}\right]^{1/n} = \frac{l_k}{\sqrt[n]{4}} \end{aligned}$$

since β_k has linear measure no less than: $(l_k)^n$. So finally we have:

$$\begin{aligned} 1 = d(\tilde{T}^{(n)}) &= \prod_{k=1}^{n} d(T_k) && \text{by (4)} \\ &\geqq \prod_{k=1}^{n} d(C_k) && \text{by Theorem (2.2)} \\ &\geqq \prod_{k=1}^{n} \frac{l_k}{\sqrt[n]{4}} = \frac{1}{4}\prod_{k=1}^{n} l_k && \text{by (5)}\,. \end{aligned}$$

This is the desired result: $4 \geqq \prod_{k=1}^{n} l_k$.

This theorem contains as a special case a result of G. Szegö [7]; in our notation his result reads: Suppose that $\tilde{T}^{(n)} = \tilde{\tau}^{(n)}$ (i.e., it consists of straight line segments) and that $\tilde{T}^{(n)}$ is a connected set. Then $\prod_{k=1}^{n} L_k \leqq 4$ where L_k is the linear measure of the k-th leg of $\tilde{T}^{(n)}$, $(k = 1, 2, \cdots, n)$.

Proof. In this case: $L_k = l_k$.

The next theorem establishes bounds on the content of a set lying on a circle as a function of the radius and the transfinite diameter of the set.

THEOREM (2.10). *Let $A'_1, A'_2, \cdots, A'_n$, $A'_k \supseteqq A'_{k+1}$ be a nested sequence of arcs on the circle $|z| = R$ where the central angle swept out by*

A'_k is θ_k, $0 < \theta_k \leqq 2\pi/n$. Let $\eta_1, \eta_2, \cdots, \eta_n$ denote the n-th roots of unity and let $\alpha(i)$ be a mapping of the set of integers $\{1, 2, \cdots, n\}$ onto itself. Define:

$$A_k = \eta_{\alpha(k)} A'_k \qquad (k = 1, 2, \cdots, n)$$

and let: $A = A_1 \cup A_2 \cup \cdots \cup A_n$. *Then*:

$$\prod_{k=1}^{n} \sin \frac{n\theta_k}{4} \leqq \left[\frac{d(A)}{R}\right]^{n^2} .$$

Proof. $d(A) = d(\eta_k \cdot A)$ $(k = 1, 2, \cdots, n)$. Therefore:

$$[d(A)]^n = \prod_{k=1}^{n} d(\eta_k \cdot A) . \tag{6}$$

Let C_k be the set of all points contained in at least k of the sets: $\eta_j \cdot A$. It follows from our hypothesis that the sets A'_k are nested that:

$$C_k = \eta_1 \cdot A_k \cup \eta_2 A_k \cup \cdots \cup \eta_n A_k$$

for each k, $1 \leqq k \leqq n$. Thus C_k is the root set with respect to the polynomial $w(z) = z^n$ of an arc on the circle $|w| = R^n$ of central angle $n \cdot \theta_k$. The transfinite diameter of such an arc is, by virtue of the equality: $d(c \cdot E) = |c| \cdot d(E)$ (c a constant) given by: $R^n \cdot \sin(n \cdot \theta_k/4)$. Therefore by Theorem (2.8):

$$d(C_k) = (R^n \cdot \sin(n\theta_k/4))^{1/n} . \tag{7}$$

Also, by virtue of Theorem (2.2) we have that:

$$\prod_{k=1}^{n} d(\eta_k \cdot A) \geqq \prod_{k=1}^{n} d(C_k) . \tag{8}$$

Combining inequalities (6), (7) and (8) we conclude:

$$[d(A)]^n \geqq \prod_{k=1}^{n} [R^n \cdot \sin(n\,\theta_k/4)]^{1/n}$$

or

$$[d(A)/R]^{n^2} \geqq \prod_{k=1}^{n} \sin(n\theta_k/4)$$

as claimed.

III. Covering theorems. The class of functions regular and univalent in $|z| < 1$ whose expansion is of the form: $f(z) = z + a_2 z^2 + \cdots$ will be denoted by S. Let D_w be the image of the unit disk under the mapping $w = f(z) \in S$. A classical result of Koebe and Bieberbach states that D_w contains the disk $|w| < 1/4$ irrespective of the mapping function $w = f(z)$ [2; page 41]. G. Szegö later noted that [8]: If α, β are two values lying in the complement of D_w and if the segment connecting α and β passes through the origin, then: $|\alpha| + |\beta| \geqq 1$.

Generalizing these results, Michael Fekete made the following conjecture: Given n rays issuing from the origin $w = 0$ at equal angles $2\pi/n$, let L denote the linear measure of the intersection of these rays with D_w. Then: $L \geqq n \cdot \sqrt[n]{1/4}$. The theorems of Koebe-Bieberbach and Szegö are the cases $n = 1$ and $n = 2$. For arbitrary n the inequality was proved in 1964 by Marcus [4].

Our first theorem in this section further generalizes these results by considering a more general class of curves issuing from the origin in place of the n rays of Fekete's conjecture. The results of the preceding section will be used to prove this as well as various other covering theorems for the class S.

THEOREM (3.1). *Let $f(z) \in S$ and let D_w be the image of the disk $|z| < 1$ under the mapping $w = f(z)$. Let $S^{(n)}$ be a set of n-fold symmetry generated by an arbitrary broken ray; $\tilde{S}^{(n)}$, a subset of $S^{(n)}$ defined by: $\tilde{S}^{(n)} = D_w \cap S^{(n)}$ and $\tilde{\sigma}^{(n)}$ the circular projection of $\tilde{S}^{(n)}$. Denote by L the linear measure of $\tilde{\sigma}^{(n)}$. Then $L \geqq n \cdot \sqrt[n]{1/4}$.*

Proof. Let E_ζ represent the image of the complement of D_w under the transformation: $\zeta = 1/w$. Then by Theorem (1.3) it follows that: $d(E_\zeta) = 1$. Let $T^{(n)}$ denote the set of n-fold symmetry that is the image of $S^{(n)}$ under the transformation $\zeta = 1/w$ and let $\tilde{T}^{(n)}$ denote the subset of $T^{(n)}$ defined by: $\tilde{T}^{(n)} = E_\zeta \cap T^{(n)}$. Denote by $\tilde{\tau}^{(n)}$ the circular projection of $\tilde{T}^{(n)}$. It is clear from the definition of the sets involved that $\tilde{T}^{(n)}$ is the complement with respect to $T^{(n)}$ of the image of $\tilde{S}^{(n)}$ under the transformation $\zeta = 1/w$ and consequently, that $\tilde{\tau}^{(n)}$ is the complement with respect to $\tau^{(n)} = \sigma^{(n)}$ of the image of $\tilde{\sigma}^{(n)}$ under the transformation: $\zeta = 1/w$.

Let $l_1, l_2, \cdots, l_n$ be measures defined on $\tilde{\tau}^{(n)}$ as in definition (2.6); let $h_1, h_2, \cdots, h_n$ be measures defined on $\tilde{\sigma}^{(n)}$ in the same way. Since $d(E_\zeta) = 1$ it follows by Theorem (2.9) that: $\prod_{k=1}^{n} l_k \leqq 4$. The points that contribute to the measure l_{n-k+1} are points in the complement of the image of the set of points contributing to h_k under $\zeta = 1/w$. For fixed h_k, the measure l_{n-k+1} is minimized when the set whose measure is h_k is the segment $[0, h_k]$ in which case: $l_{n-k+1} = 1/h_k$. Thus:

$$\prod_{k=1}^{n} l_k \geqq \prod_{k=1}^{n} \frac{1}{h_k}$$

and so:

$$4 \geqq \prod_{k=1}^{n} \frac{1}{h_k} \quad \text{or:} \quad \left(\prod_{k=1}^{n} h_k\right)^{1/n} \geqq \sqrt[n]{1/4}\,.$$

Since the arithmetic mean exceeds the geometric mean:

$$\frac{1}{n}\sum_{k=1}^{n} h_k \geqq \sqrt[n]{1/4}\,.$$

According to Remark (2.7): $\sum_{k=1}^{n} h_k = L$, the linear measure of $\tilde{\sigma}^{(n)}$. Thus: $L \geqq n\cdot\sqrt[n]{1/4}$ as claimed.

THEOREM (3.2) *Let $w(z) \in S$ and D_w the image of $|z| < 1$ under $w(z)$. Suppose $D_w \cap \{|w| = R\}$ consists of n disjoint arcs $\{B_k\}_1^n$ where*

(i) *The angle subtended by the arc separating B_k and B_{k+1} is no greater than: $2\pi/n$.*

(ii) *If $\{A_k^*\}_1^n$ are the n arcs in the complement of $\bigcup_{k=1}^n B_k$ with respect to the circle $|w| = R$ the related set of arcs: $\{\eta_k\cdot A_k^*\}_1^n$ are nested.*

Let the endpoints of the arc B_k be given by: $R\cdot e^{i\theta_{2k-1}}$ and $R\cdot e^{i\theta_{2k}}$ $(k = 1, 2, \cdots, n)$.

Then:

$$\prod_{k=1}^{n} \sin\,[n(\theta_{2k+1} - \theta_{2k})/4] \leqq R^{n^2}\,, \quad \theta_{2n+1} = \theta_1 + 2\pi\,.$$

Proof. Let A_k^* be the arc lying between B_k and B_{k+1}. The central angle subtended by A_k^* is: $\theta_{2k+1} - \theta_{2k}$ which by hypothesis is no greater than $2\pi/n$. Let A_k be the image of A_k^* under the transformation $\zeta = 1/w$. The arcs A_k^* all lie in the complement of D_w. Hence: $A = \bigcup_{k=1}^n A_k \subseteqq E_\zeta$ and so $d(A) \leqq d(E_\zeta) = 1$. The sets A_k lie on the circle: $|\zeta| = 1/R$. The central angle subtended by A_k is $\theta_{2k+1} - \theta_{2k}$; the same as that subtended by A_k^*. Finally, the arcs A_k have the nested property hypothesized for the sets A_k^*. Since all this is so, Theorem (2.10) is applicable; therefore:

$$\prod_{k=1}^{n} \sin\frac{n(\theta_{2k+1} - \theta_{2k})}{4} \leqq [d(A)/(1/R)]^{n^2} \leqq R^{n^2}$$

as claimed.

This past theorem takes no account of the fact that the complement of D_w is a continuum containing the point at infinity. A sharpened version which takes this into account is the following:

$$d(0, 1, \theta_3 - \theta_2)\cdot\prod_{k=2}^{n} \sin\frac{n(\theta_{2k+1} - \theta_{2k})}{4} \leqq R^{n^2}$$

where $d(a, b, \theta)$ is as defined in §1. Actually, both Theorems (3.1) and (3.2) are generalized (in a sense, combined) in the following theorem, which takes the above fact into account. The techniques used to prove the theorem are essentially the same as those of the foregoing proofs and so just a statement of the result will be given.

THEOREM (3.3). *Let $f(z) \in S$ and D_w be the image of $|z| < 1$ under $w = f(z)$. Let C be a circle of radius R, $0 < R < \infty$ and n an arbitrary natural number. Let $\{B_n\}_1^n$ be a sequence of arcs on the circle C satisfying the conditions of Theorem* (3.2), *$S^{(n)}$ a set of n-fold symmetry generated by a broken ray and $\tilde{S}^{(n)}$ a subset of $S^{(n)}$ defined by: $\tilde{S}^{(n)} = S^{(n)} \cap D_w \cap \{|w| \leqq R\}$. Let $\tilde{\sigma}^{(n)}$ denote the circular projection of $\tilde{S}^{(n)}$ and $\{h_k\}_1^n$ a sequence of measures on $\tilde{\sigma}^{(n)}$ such as defined in definition* (2.6).

Then:

$$d\left(0, \left[\frac{R}{h_n}\right]^n, n[\theta_3 - \theta_2]\right) \cdot \prod_{k=2}^{n} d\left(1, \left[\frac{R}{h_{n-k+1}}\right]^n, n[\theta_{2k+1} - \theta_{2k}]\right) \leqq R^{n^2} .$$

One final application will be given.

THEOREM (3.4). *Let $f(z) \in S$ and D_w the image of the disk $|z| < 1$ under $w = f(z)$. Let L_1, L_2 denote straight lines intersecting at $w = 0$ at an angle of $\pi\alpha$, $0 < \alpha < 1$. Let $L = L(D_w \cap \{L_1 \cap L_2\}$ denote the linear measure of $D_w \cap \{L_1 \cup L_2\}$. Then*:

$$L \geqq \frac{2}{\alpha^{\alpha/2}(1 - \alpha)^{(1-\alpha)/2}} .$$

Proof. There is no loss in generality in assuming L_1 and L_2 are symmetric images of one-another with respect to the real axis.

A set of four points on the four legs determined by $L_1 \cup L_2$, each lying at a distance r_0 from the origin, will be called a "radially symmetric set"; the points themselves will be called radially symmetric images of one-another and of the point $w = r_0$.

We define h_k $(k = 1, 2, 3, 4)$ as the measure of the set of real numbers r, $0 \leqq r < \infty$ such that at least k of the radially symmetric images of r (in $L_1 \cup L_2$) lie in D_w. Then:

$$L(D_w \cap \{L_1 \cup L_2\}) = \textstyle\sum_{k=1}^{4} h_k . \tag{9}$$

Map by $\zeta = 1/w$ and let E_ζ represent the complement of the image of D_w under this map. Then $d(E_\zeta) = 1$. Notice that $L_1 \cup L_2$ is mappped onto itself. Let l_k be the measure of the set of real numbers r such that at least k of the radially symmetric images of r (in $L_1 \cup L_2$) lie in E_ζ. Then:

$$\prod_{k=1}^{4} l_k \geqq \prod_{k=1}^{4} \frac{1}{h_k} . \tag{10}$$

Let $T_1 = E_\zeta \cap \{L_1 \cup L_2\}$; let T_2 be the reflection of T_1 in the imaginary axis; let T_3 be the reflection of T_2 in the real axis; let T_4 be the reflection of T_3 in the imaginary axis. Clearly:

$$d(T_1) = d(T_2) = d(T_3) = d(T_4)\,. \tag{11}$$

Let C_k be the set of all points contained in at least k of the T_j's. The set C_k is a radially symmetric set; that is, it consists of all radially symmetric images of those points ζ such that at least k of radially symmetric images of ζ lie in T_1. Thus the measure of a leg of C_k is l_k. Let B_k be the set consisting of four segments lying on the four rays determined by $L_1 \cup L_2$, each of length l_k, the intersection of the four being the point $\zeta = 0$. Since the shift of segments that transforms C_k into B_k can only bring extremal points closer together, it follows that: $d(C_k) \geqq d(B_k)$. Using the mapping lemma (1.5) and Fekete's theorem (2.8) the transfinite diameter of B_k can be calculated:

$$d(B_k) = \frac{l_k}{2\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}}\,.$$

We have

$$\begin{aligned}
1 = d(E_\zeta) &\geqq d(T_1) && \text{since: } T_1 \subseteqq E_\zeta \\
&= \left[\prod_{k=1}^{4} d(T_k)\right]^{1/4} \geqq \left[\prod_{k=1}^{4} d(C_k)\right]^{1/4} && \text{by Theorem (2.2)} \\
&\geqq \left[\prod_{k=1}^{4} d(B_k)\right]^{1/4} = \left[\prod_{k=1}^{4} \frac{l_k}{2\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}}\right]^{1/4} \\
&\geqq \frac{1}{2\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}} \left[\prod_{k=1}^{4} \frac{1}{h_k}\right]^{1/4} \\
&\geqq \frac{1}{2\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}} \cdot \frac{4}{\sum_{k=1}^{4} h_k}
\end{aligned}$$

since the arithmetic mean exceeds the geometric mean;

$$= [2/(\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2})]\cdot(1/L)\,.$$

This sequence of inequalities means:

$$L \geqq [2/(\alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2})]\,.$$

REMARK. When $\alpha = 1/2$ that is, when $L_1 \cup L_2$ is a set of 4-fold symmetry, the result of the theorem reads: $L \geqq 2/(1/4)^{1/4} = 4(1/4)^{1/4}$ in agreement with Theorem (3.1).

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