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ABELIAN p-GROUPS DETERMINED BY THEIR ULM SEQUENCES

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Ulm's theorem asserts that, within the class of all reduced countable abelian p-groups, a group is determined, up to isomorphism, by its Ulm sequence. Although this theorem fails in general for uncountable groups, there are classes of uncountable abelian p-groups whose members are determined within the class by their Ulm sequences. Kolettis has shown that the class of direct sums of countable p-groups has this property. Here it is shown that the class of those abelian p-groups for which the Ulm type is finite and all the Ulm factors except the last are direct sums of cyclic groups, is another such class.

Let G be a reduced abelian p-group. Define the subgroup G^α for each ordinal α as follows. Set $G^0 = G$. Proceeding inductively, if $\alpha = \beta + 1$, define G^α to be the subgroup of those elements in G^β which have infinite height in G^β; if α is a limit ordinal, define $G^\alpha = \bigcap_{\beta<\alpha} G^\beta$. Since G is reduced, there is a first ordinal τ such that $G^\tau = 0$; this ordinal τ is called the *Ulm type* of G. The *Ulm factors* of G are defined to be the factor groups $G_\alpha = G^\alpha/G^{\alpha+1}$ $(\alpha > \tau)$. And the sequence of Ulm factors G_α $(\alpha < \tau)$ is called the *Ulm sequence* of G. Two groups G and H have *isomorphic Ulm sequences* if $G_\alpha \cong H_\alpha$ for every α.

Our theorem is now the following:[1]

If G is a reduced abelian p-group having finite Ulm type n and such that its first $n-1$ Ulm factors $G_0, \cdots, G_{n-2}$ are direct sums of cyclic groups, and if H is any other abelian p-group whose Ulm sequence is isomorphic to that of G, then $H \cong G$.

It should be noted that in this theorem no assumption is made on the last Ulm factor G_{n-1} of G.

Neither the assumption of finite Ulm type nor the assumption that all but the last Ulm factor are direct sums of cyclic groups can be dropped from the hypotheses of the theorem. For example, if G is any countable p-group whose Ulm type is infinite, then there is an uncountable p-group whose Ulm sequence is isomorphic to that of G. Moreover, it is known that there are nonisomorphic p-groups of Ulm type 2 having isomorphic Ulm sequences.

[1] Since completing this paper, I have learned that P. Hill and C. Megibben have obtained similar results.

Throughout the proof of the theorem, the usual notation and terminology, for the most part, is used.[2] If x is an element of a group, then the cyclic subgroup generated by x is denoted by $[x]$. The symbol $\oplus$ will be used to designate the direct sum of a pair of subgroups, while the symbol $\sum$ will designate the direct sum of a family of subgroups.

The proof begins with the following common generalization of Kolettis' theorem [5] and a theorem of Zippin [6].

(a) *Let G and H be reduced abelian p-groups which are direct sums of countable groups, and whose Ulm sequences are isomorphic. If σ is an ordinal, and f is an isomorphism of G^σ onto H^σ, then f extends to an isomorphism of G onto H.*

Proof. Write $G = \sum_{i \in I} G_i$ and $H = \sum_{j \in J} H_j$ where each G_i and each H_j is countable. For every subset $K \subseteqq I$ and every subset $L \subseteqq J$ set $G(K) = \sum_{i \in K} G_i$ and $H(L) = \sum_{j \in L} H_j$.

Let α be an ordinal less than σ. For each $i \in I$ and each $j \in J$ pick sequences of elements $\{b_{i,\alpha,n}\}_{n<\infty}$ in G_i^α and $\{c_{j,\alpha,m}\}_{m<\infty}$ in H_j^α such that

$$G_i^\alpha / G_i^{\alpha+1} = \sum_{n<\infty} [b_{i,\alpha,n} + G_i^{\alpha+1}]$$

and

$$H_j^\alpha / H_j^{\alpha+1} = \sum_{m<\infty} [c_{j,\alpha,m} + H_j^{\alpha+1}] \,.$$

Then

$$\begin{aligned} G^\alpha / G^{\alpha+1} &= \sum_{i,n} [b_{i,\alpha,n} + G^{\alpha+1}] \\ &\cong H^\alpha / H^{\alpha+1} = \sum_{j,m} [c_{j,\alpha,m} + H^{\alpha+1}] \,, \end{aligned}$$

and consequently there exists a one-to-one function Φ_α which maps the set $\{b_{i,\alpha,n} \mid i \in I; n = 1, 2, \cdots\}$ onto the set $\{c_{j,\alpha,m} \mid j \in J; m = 1, 2, \cdots\}$ in such a way that $b_{i,\alpha,n} + G^{\alpha+1}$ and $\Phi_\alpha(b_{i,\alpha,n}) + H^{\alpha+1}$ have the same order.

We now construct two sequences of subsets $I_\nu \subseteqq I$ and $J_\nu \subseteqq J$ such that the following conditions hold for each ordinal ν:

(i) I_ν and J_ν are countable and nonempty.

(ii) $I_\nu \cap I_\mu = J_\nu \cap J_\mu = \varnothing$ for $\mu < \nu$.

Let I^ν denote the set-union of the I_μ $(\mu < \nu)$, and let J^ν denote the set-union of the J_μ $(\mu < \nu)$.

[2] See, for example, Fuchs [2] or Kaplansky [4].

(iii) If $i \in I_\nu$ and $x \in G_i^\sigma$, then $f(x) \in H(J^{\nu+1})$. Conversely, if $j \in J_\nu$ and $y \in H_j^\sigma$, then $f^{-1}(y) \in G(I^{\nu+1})$.

(iv) If $i \in I_\nu$, then $\Phi_\alpha(b_{i,\alpha,n}) \in H(J_\nu)$ for all α and n. Conversely, if $j \in J_\nu$, then $\Phi_\alpha^{-1}(c_{j,\alpha,m}) \in G(I_\nu)$ for all α and m.

Suppose the sets I_μ and J_μ have been constructed for each $\mu < \nu$, and suppose that $I^\nu \neq I$. Pick two sequences of countable subsets $M_1 \subseteqq M_2 \subseteqq M_3 \subseteqq \cdots \subseteqq I - I^\nu$ and $N_1 \subseteqq N_2 \subseteqq N_3 \subseteqq \cdots \subseteqq J - J^\nu$ such that $M_1 \neq \varnothing$ and such that the following hold for each positive integer q:

$$\begin{aligned} &f(G(M_q)^\sigma) \subseteqq H(N_q) \oplus H(J^\nu) , && \\ &\Phi_\alpha(b_{i,\alpha,n}) \in H(N_q) && \text{if } i \in M_q , \\ &f^{-1}(H(N_q)^\sigma) \subseteqq G(M_{q+1}) \oplus G(I^\nu) , && \\ &\Phi_\alpha^{-1}(c_{j,\alpha,m}) \in G(M_{q+1}) && \text{if } j \in N_q . \end{aligned}$$

Then if $I_\nu = M_1 \cup M_2 \cup M_3 \cup \cdots$ and $J_\nu = N_1 \cup N_2 \cup N_3 \cup \cdots$, it is clear that I_ν and J_ν satisfy (i)-(iv).

Since all the sets I_ν are nonempty, there exists an ordinal λ such that $I^\lambda = I$ and $J^\lambda = J$. Consequently, $G = G(I^\lambda)$ and $H = H(J^\lambda)$, and for each $\nu < \lambda$,

$$(1) \qquad G(I^{\nu+1}) = G(I^\nu) \oplus G(I_\nu) \quad \text{and} \quad H(J^{\nu+1}) = H(J^\nu) \oplus H(J_\nu) .$$

Suppose that there is an isomorphism g_ν of $G(I^\nu)$ onto $H(J^\nu)$ such that the restrictions of g_ν and f to $G(I^\nu)^\sigma$ are equal. Let f_ν denote the restriction of f to $G(I_\nu)^\sigma$, and let π_0 denote the projection of $G(I^{\nu+1})$ onto $G(I_\nu)$, π_1 denote the projection of $H(J^{\nu+1})$ onto $H(J_\nu)$, and π_2 denote the projection of $H(J^{\nu+1})$ onto $H(J^\nu)$, as determined by the decompositions (1). If $x \in G(I_\nu)^\sigma$, then by (iii) there exist elements $y \in H(J_\nu)$ and $z \in H(J^\nu)$ such that $f(x) = y + z$. Then $f^{-1}(y) = x - f^{-1}(z)$, and as $f^{-1}(z) \in G(I^\nu)$, it follows that $\pi_0 f^{-1} \pi_1 f(x) = x$. Similarly, if $y \in H(J_\nu)^\sigma$, then $\pi_1 f \pi_0 f^{-1}(y) = y$. Therefore $\pi_1 f_\nu$ is an isomorphism of $G(I_\nu)^\sigma$ onto $H(J_\nu)^\sigma$. Moreover, for each $\alpha < \sigma$, the αth Ulm factors of $G(I_\nu)$ and $H(J_\nu)$ are isomorphic by (iv), and hence $G(I_\nu) \cong H(J_\nu)$ by Ulm's theorem. By Zippin's theorem [6, § 8], the mapping $\pi_1 f_\nu$ extends to an isomorphism h_1 of $G(I_\nu)$ onto $H(J_\nu)$. And inasmuch as $\pi_2 f_\nu$ is a homomorphism of $G(I_\nu)^\sigma$ into $H(J^\nu)^\sigma$, this homomorphism extends to a homomorphism h_2 of $G(I_\nu)$ into $H(J^\nu)$ by [1, 1.2]. Consequently, if $h = h_1 + h_2$, and if $g_{\nu+1}$ is the direct sum of g_ν and h, then $g_{\nu+1}$ is an isomorphism of $G(I^{\nu+1})$ onto $H(JI^{\nu+1})$ whose restriction to $G(I^{\nu+1})^\sigma$ equals the restriction of f to $G(I^{\nu+1})^\sigma$. A transfinite induction now completes the proof of (a).

(b) *Let G be a reduced abelian p-group, and let B be a basic* *subgroup of* G^1. *Then there is a subgroup* S *such that* $S + G^1 = G$ *and* $S^1 = S \cap G^1 = B$.

Proof. Since G^1/B is divisible, it is a direct summand of G/B, say $G/B = G^1/B \oplus S/B$, where S is a subgroup of G containing B. Then $S + G^1 = G$, and $S \cap G^1 = B$. Suppose $b \in B$, and n is any positive integer. Then as $b \in G^1$, there are elements $c \in S$ and $u \in G^1$ such that $b = p^n c + p^n u$. Therefore $p^n c \in G^1$, and consequently $p^n c \in B$. Hence $p^n u \in B$, and inasmuch as B is pure in G^1, there is an element $d \in B$ such that $p^n u = p^n d$. Thus $b = p^n(c + d)$, and it follows that $B \subseteqq S^1 \subseteqq S \cap G^1$.

The next lemma is a special case of the principal result of Irwin-Richman [3].

(c) *If* G *is an abelian* p-*group for which* G/G^1 *and* G^1 *are direct sums of cyclic groups, then* G *is a direct sum of countable groups.*

(d) *If* G *and* H *are reduced abelian* p-*groups such that* $G/G^1 \cong H/H^1$ *and* G/G^1 *is a direct sum of cyclic groups, and if* f *is an isomorphism of* G^1 *onto* H^1, *then* f *extends to an isomorphism of* G *onto* H.

Proof. Let B be a basic subgroup of G^1, and let C be that basic subgroup of H^1 which is the image of B under f. By (b) there are subgroups S of G and T of H such that $S + G^1 = G$, $S \cap G^1 = S^1 = B$, $T + H^1 = H$, and $T \cap H^1 = T^1 = C$. Now

$$S/S^1 = S/B \cong G/G^1 \cong H/H^1 \cong T/C = T/T^1 ,$$

and hence by (c) and (a) there is an isomorphism φ of S onto T such that the restriction of φ to B is the same as the restriction of f to B. If A is a complete set of coset representatives of B in S, then each element $x \in G$ is uniquely of the form $x = a + u$ where $a \in A$ and $u \in G^1$, and the mapping g defined by $g(x) = \varphi(a) + f(u)$ is the desired isomorphism of G onto H which extends f.

The theorem now follows directly from (*d*).

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