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A NONIMBEDDING THEOREM OF NILPOTENT LIE ALGEBRAS

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There are many similarities between groups of prime power order and nilpotent Lie algebras. Here we present a nonimbedding theorem in nilpotent Lie algebras which is an analogue of a nonimbedding theorem of Burnside in groups of prime power order.

Burnside in [2] proved the following two theorems:

THEOREM B1. *A nonabelian group whose center is cyclic cannot be the derived group of a p-group.*

THEOREM B2. *A nonabelian group, the index of whose derived group is p^2, cannot be the derived group of a p-group.*

Hobby in [3] proved the following analogues of the theorems of Burnside:

THEOREM H1. *If H is nonabelian group whose center is cyclic, then H cannot be the Frattini subgroup of any p-group.*

THEOREM H2. *A nonabelian group, the index of whose derived group is p^2, cannot be the Frattini subgroup of any p-group.*

The purpose of this note is to establish the analogues of the theorems of Burnside in Lie algebras. The main result is the following Theorem 1. The Lie algebras which we consider here are finite dimensional over an arbitrary field F. The Frattini subalgebra $\phi(M)$ of a Lie algebra M is defined as the intersection of all maximal subalgebras of M. We also show that in a nilpotent Lie algebra N, $\phi(N)$ coincides with the derived algebra of N. Hence, the analogues of Hobby's theorems in Lie algebras are the same as the analogues of Burnside's theorems in Lie algebras.

THEOREM 1. *A nonabelian Lie algebra L whose center is one dimensional cannot be any N_i, $i \geqq 1$, of a nilpotent Lie algebra N where $N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_t \supset 0$ is the lower central series of N.*

Proof. Suppose the contrary, i.e., $L = N_i$ for some i, $1 \leqq i < t$, in N. Since L is nonabelian, $L \neq N_t$. Let z be a basis of the center of L, denoted by $Z(L)$. The following Jacobi identity

$$[[u, z], x] + [[z, x], u] + [[x, u], z] = 0$$

holds for every $u \in N$ and every $x \in L$. Since $z \in Z(L)$, the second term of the identity zero. The third term of the identity is also zero since L is N_i and N_i is an ideal in N. Hence, we have $[[u, z], x] = 0$ for every $x \in L$ and every $u \in N$, i.e., $[u, z] \in Z(L)$ and $[u, z] = a_u z$ where $a_u \in F$.

There are two cases: (1) If $a_u \neq 0$ for some $u \in N$, then the lower central series of N never reaches zero, that is a contradiction to N being nilpotent. (2) The case $[u, z] = 0$ for every $u \in N$. Then $z \in Z(N)$, i.e., $Z(L) \subseteqq Z(N)$. Since $N/Z(L)$ is nilpotent, we have $(N/Z(L)) \supset (N_1/Z(L)) \supset \cdots \supset (N_i/Z(L)) = (L/Z(L)) \supset \cdots \supset (N_t/Z(L)) \supseteqq 0$ where we have $N_t = Z(L)$ since $N_t \subseteqq Z(N)$ and $L = N_i$ and the dimension of $Z(L)$ is one. There is a nonzero $\bar{y} \in Z(N/Z(L)) \cap (L/Z(L))$, i.e., $\bar{y} \in (N_{t-1}/Z(L))$. Then $[\bar{y}, \bar{v}] = \bar{0}$ for every $\bar{v} \in N/Z(L)$, i.e., $[y, v] = a_{vy} z$ where $a_{vy} \in F, \bar{y} = y + Z(L)$ and $\bar{v} = v + Z(L)$. Let w be any element in N, by using Jacobi identity, we have

$$[y, [v, w]] = [a_{vy} z, w] + [v, a_{wy} z] = 0 ,$$

i.e., y commutes with every element in N_1. In particular, y commutes with every element in L. That contradicts the dimension of $Z(L)$ being one. Hence, the proof is completed.

THEOREM 2. *A nonabelian Lie algebra L, the dimension of (L/L_1) is 2, cannot be any $N_i, i \geqq 1$, of a nilpotent Lie algebra N where $N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_t \supset 0$ is the lower contral series of N.*

Proof. Suppose the contrary, i.e., L is some N_i, $t < i \leqq 1$. Then L is nilpotent. We claim that the dimension of L/L_1, denoted by $\dim (L/L_1)$, is 2 implying that $\dim (L_1/L_2) = 1$. Suppose $\dim (L_1/L_2) > 1$, then there exist linearly independent vectors $\bar{x}$ and $\bar{y}$ in $\bar{L}_1 = L_1/L_2$, and there also exist linearly independent vectors $\bar{u}$ and $\bar{v}$ in a complement $\bar{C}$ of $\bar{L}_1$ in $\bar{L}$ such that $[\bar{u}, \bar{v}] = \bar{x}$. Similarly, there exist $\bar{u}', \bar{v}' \in \bar{C}$ such that $[\bar{u}', \bar{v}'] = \bar{y}$. Since $\dim \bar{C} = 2$, $[\bar{u}', \bar{v}'] = a[\bar{u}, \bar{v}]$ where $a \in F$. This contradicts the linear independence of $\bar{x}$ and $\bar{y}$. Hence, $\dim (L_1/L_2) = 1$.

Since L_2 is a characteristic ideal of L, L_2 is an ideal in N. Then, the Lie algebra N/L_2 contains L/L_2 as a term in its lower central series. Since the center of L/L_2 is one dimensional and L/L_2 is nonabelian, this is impossible by Theorem 1.

THEOREM 3. *If N is a nilpotent Lie algebra, then $\phi(N) = N_1$.*

Proof. If N is abelian then $\phi(N) = N_1 = 0$. Consider that N is nilpotent and nonablelian: Let $u_1, u_2, \cdots, u_k$ be a basis a complementary subspace U of N_1 in N, it is easy to verify that k must be $\geqq 2$, and let $U_i = ((u_1)) + \cdots + ((u_{i-1})) + ((u_{i+1})) + \cdots + ((u_k)) + N_1, i = 1, 2, \cdots, k$, where the sums are direct sums of vector spaces. Clearly, each U_i is a maximal subalgebra of N. Then $\phi(N) \subseteqq \bigcap_{i=1}^{k} U_i = N_1$.

Now, we show that $\phi(N) \supseteqq N_1$. Let M_α be a maximal subalgebra of N. It follows from Proposition 3 on p. 56 in [1] that every maximal subalgebra in a nilpotent Lie algebra is an ideal. Hence, M_α is an indeal in N. Let x be a nonzero vector in N and $x \notin M_\alpha$, then the direct sum of the vector spaces $((x))$ and M_α constitutes a subalgebra. Since M_α is maximal, $((x)) + M_\alpha = N$. Since M_α is an ideal of N and since N/M_α is of dimension one and since N/M_α is nilpotent, we have

$$N/M_\alpha \supset N_1 M_\alpha = \bar{0} ,$$

i.e., $M_\alpha \supseteqq N_1$ for any maximal subalgebra M_α. Consequently, $\phi(N) = \bigcap_\alpha N_\alpha \supseteqq N_1$, and $\phi(N) = N_1$.

COROLLARY 1. *If L is a nonabelian Lie algebra whose center is one dimensional, then L cannot be the Frattini subalgebra of any nilpotent Lie algebra.*

It follows from Theorem 3 and Theorem 1.

COROLLARY 2. *A nonabelian Lie algebra L,* $\dim (L/L_1) = 2$, *cannot be the Frattini subalgebra of any nilpotent Lie algebra.*

It follows from Theorem 3 and Theorem 2.

Our Theorem 1 and Theorem 2 contain the analogues of Theorem B1 and Theorem B2 respectively. Corollary 1 and Corollary 2 of Theorem 3 are the analogues of Theorem H1 and Theorem H2 respectively.

REMARK. The following example shows that for each integer $n \geqq 3$ there is a nonabelian nilpotent Lie algebra L of dimension n whose center is one dimensional (also, the dimension of L/L_1 is 2): Let $L = ((x_1, x_2, \cdots, x_n))$ with a bilinear anti-symmetric bracket multiplication such that $[x_1, x_i] = x_{i+1}$ for $i = 2, 3, \cdots, n-1$, and all other products are zero.

References

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