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LOCAL ISOMORPHISM OF COMPACT CONNECTED LIE GROUPS

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It is shown that two nonisomorphic compact connected Lie groups can be covering groups of each other. This is examined in detail and is related to the question of determining all the compact connected Lie groups belonging to a fixed Lie algebra.

Given a Lie group G it is natural to ask: "What are all the groups locally isomorphic to G?" The answer to this is well known. If $\bar{G}$ denotes the simply connected covering group of G, then any group locally isomorphic to G is obtained by forming the quotient group $\bar{G}/\Gamma$ where Γ is a discrete subgroup of the center of $\bar{G}$. (For this see e.g. [3]). We shall say that a Lie group G_1 covers a Lie group G_2 if there exists a continuous homomorphism of G_1 onto G_2 with discrete kernel. We can then pose the question: "Let G be a Lie group and let $G_1, G_2, \cdots$ be Lie groups such that any group locally isomorphic to G is isomorphic to one and only one of the G_i. Order the set $G_1, G_2, \cdots$ by setting $G_i \geqq G_j$ if G_i covers G_j. How can one describe the structure of this ordered set?" In the case when G is compact connected we give a precise answer to this question.

One surprising result of the investigation is that two compact connected Lie groups can cover each other but not be isomorphic. The simplest example of this is provided by the groups $U(5)$ and $U(5)/\Gamma_2$, where $U(5)$ denotes the 5×5 unitary group and Γ_2 is the subgroup of $U(5)$ consisting of the two matrices $\pm I$. $I =$ identity matrix.

Closely related to this example is the:

PROPOSITION. In $U(n)$ let Γ_k denote the subgroup of all diagonal matrices λI where λ is a complex number such that $\lambda^k = 1$. Then $U(n)/\Gamma_{k_1}$ and $U(n)/\Gamma_{k_2}$ are isomorphic if and only if $k_1 \equiv \pm k_2 \bmod n$.

1. Notation.

If G is a compact connected Lie then $\tilde{G}$ shall denote the covering group of G of the form $H_1 \times H_2 \times \cdots \times H_t \times T^n$ where each H_i is a compact connected simply connected group in one of the Cartan series A_n, B_n, C_n, D_n or is a compact connected simply connected exceptional group, and T^n is the n-dimensional torus group. $T^n = \{(z_1, z_2, \cdots, z_n) \mid z_i \in \boldsymbol{C}$ and $\mid z_i \mid = 1\}$.

If K is a subgroup of the center of $H_1 \times H_2 \times \cdots \times H_t$ and φ is a homomorphism of K into T^n, then (K, φ) denotes the subgroup of $\tilde{G}$ consisting of all elements $(g, \varphi(g))$ where g ranges over K.

2. **Proposition.** If G is a compact connected Lie group and G_1 is a compact connected Lie group locally isomorphic to G, then there exists a subgroup K of the center of $H_1 \times \cdots \times H_t$ and a homomorphism $\varphi: K \to T^n$ such that G_1 is isomorphic to $\tilde{G}/(K, \varphi)$.

Proof. If suffices to show that any group $\tilde{G}/\Gamma$, where Γ is a discrete subgroup of the center of $\tilde{G}$, is isomorphic to a group of the form $\tilde{G}/(K, \varphi)$. So given Γ let $p(\Gamma) = \{g \in T^n \mid (1, g) \in \Gamma\}$. Here 1 denotes the identity element of $H_1 \times H_2 \times \cdots \times H_t$. Then $T^n/p(\Gamma)$ is isomorphic to T^n so there exists a homomorphism f of T^n onto itself with Kernel $f = p(\Gamma)$. Define a homomorphism $\tilde{f}: \tilde{G} \to \tilde{G}$ by $\tilde{f} \mid T^n = f$ and $\tilde{f} \mid H_1 \times \cdots \times H_t =$ the identity map. Then $\tilde{f}$ is a homomorphism of $\tilde{G}$ onto itself, $\tilde{f}(\Gamma)$ is a group of the form (K, φ) and $\tilde{f}^{(-1)}(K, \varphi) = \Gamma$. So $\tilde{f}$ gives an isomorphism of $\tilde{G}/\Gamma$ onto $\tilde{G}/(K, \varphi)$.

3. **Corollary.** *If G is a compact connected Lie group then to within isomorphism of Lie groups there exist only finitely many compact connected Lie groups locally isomorphic to G.*

Proof. In $\tilde{G}$ there are only finitely many subgroups of the form (K, φ).

4. **Definition.** A subgroup of $\tilde{G}$ of the form (K, φ) shall be called a *special subgroup of* $\tilde{G}$.

5. **Proposition.** Let G be a compact connected Lie group and let $(K_1, \varphi_1)(K_2, \varphi_2)$ be two special subgroups of $\tilde{G}$. Let $h: \tilde{G}/(K_1, \varphi_1) \to \tilde{G}/(K_2, \varphi_2)$ be a covering homomorphism and let $\pi_i: \tilde{G} \to \tilde{G}/(K_i, \varphi_i)$ be the projection. Then there is a unique covering homomorphism $\tilde{h}: \tilde{G} \to \tilde{G}$ such that the diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{h}} & \tilde{G} \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \tilde{G}/(K_1, \varphi_1) & \xrightarrow{h} & \tilde{G}/(K_2, \varphi_2) \end{array}$$

is commutative. If h is an isomorphism then $\tilde{h}$ is an automorphism of $\tilde{G}$.

Proof. π_i maps T^n isomorphically onto the identity component of the center of $\tilde{G}/(K_i, \varphi_i)$. So $\tilde{h} \mid T^n$ can be defined by $\tilde{h} \mid T^n = \pi_2^{-1} \circ h \circ \pi_1$.

To define $\tilde{h} \mid H_1 \times \cdots \times H_t$ observe that $\pi_2^{-1} \circ h \circ \pi_1$ is well-defined in a neighborhood of the identity of $\tilde{G}$. Thus $\pi_2^{-1} \circ h \circ \pi_1$ gives a local homomorphism of $H_1 \times H_2 \times \cdots \times H_t$ into $\tilde{G}$ and since $H_1 \times \cdots \times H_t$

is simply connected this extends to a homomorphism of $H_1 \times \cdots \times H_t$ into $\tilde{G}$.

Hence $\tilde{h}$ is defined on all of $\tilde{G}$. Since $\tilde{G}$ is connected and locally one has $h\pi_1 = \pi_2\tilde{h}$, one has this globally.

The uniqueness of $\tilde{h}$ follows because $\tilde{G}$ is connected and locally $\tilde{h} = \pi_2^{-1} \circ h \circ \pi_1$.

If h is an isomorphism let $\eta = h^{-1}$. Then $\tilde{h}\tilde{\eta}$ and $\tilde{\eta}\tilde{h}$ are locally the identity mapping so this holds globally and $\tilde{\eta} = (\tilde{h})^{-1}$.

REMARK. If Γ_1 and Γ_2 are arbitrary discrete central subgroups of $\tilde{G}$ and $h\colon \tilde{G}/\Gamma_1 \to \tilde{G}/\Gamma_2$ is a covering homomorphism then it is not in general true that h can be lifted to a homomorphism $\tilde{h}\colon \tilde{G} \to \tilde{G}$.

6. **Corollary.** *$\tilde{G}/(K_1, \varphi_1)$ covers (is isomorphic to) $\tilde{G}/(K_2, \varphi_2)$ if and only if there exists an automorphism α of $H_1 \times \cdots \times H_t$ and a covering homomorphism β of T^n onto itself (and an automorphism β of T^n) such that α maps K_1 into (onto) K_2 and the diagram*

$$\begin{array}{ccc} K_1 & \xrightarrow{\varphi_1} & T^n \\ {\scriptstyle \alpha \mid K_1}\downarrow & & \downarrow{\scriptstyle \beta} \\ K_2 & \xrightarrow{\varphi_2} & T^n \end{array}$$

is commutative.

Proof. By applying the previous proposition and the fact that any homomorphism of $\tilde{G}$ onto itself is of the form $\alpha \times \beta$ where α is an automorphism of $H_1 \times \cdots \times H_t$ and β is a covering homomorphism of T^n onto itself.

7. **Definition.** A *quasi-ordered set* is a set S together with a transitive relation $\geqq$, such that for all $x \in S$, $x \geqq x$. Two quasi ordered sets S_1 and S_2 are isomorphic if there exists a one-to-one and onto mapping of sets $f\colon S_1 \to S_2$ such that for all $x, y \in S_1$, $x \geqq y$ if and only if $f(x) \geqq f(y)$.

8. **Definition.** Let G be a compact connected Lie group and let $G_1, G_2, \cdots, G_l$ be compact connected Lie groups such that any compact connected Lie group locally isomorphic to G is isomorphic to one and only one of the G_i. Define a transitive relation on the G_i by setting $G_i \geqq G_j$ if G_i covers G_j. The set $G_1, G_2, \cdots, G_l$ together with the relation $\geqq$ is called the *local isomorphism system of G*.

9. **Definition.** Let G be a compact connected Lie group and let $(K_1, \varphi_1), (K_2, \varphi_2), \cdots, (K_r, \varphi_r)$ be the special subgroups of $\tilde{G}$. Define an

equivalence relation $\sim$ on the (K_i, φ_i) by setting $(K_i, \varphi_i) \sim (K_j, \varphi_j)$ if there exist automorphisms α, β of $H_1 \times \cdots \times H_t$ and T^n such that α maps K_1 onto K_2 and the diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\varphi_1} & T^n \\ \alpha\downarrow & & \downarrow\beta \\ K_2 & \xrightarrow{\varphi_2} & T^n \end{array}$$

is commutative. Denote the equivalence class of (K_i, φ_i) by $[(K_i, \varphi_i)]$. On the set of equivalence classes define a transitive relation $\geqq$ by $[(K_i, \varphi_i)] \geqq [(K_j, \varphi_j)]$ if there exists an automorphism α of $H_1 \times \cdots \times H_t$ and a covering homomorphism β of T^n onto itself such that α maps K_1 into K_2 and the diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\varphi_1} & T^n \\ \alpha\downarrow & & \downarrow\beta \\ K_2 & \xrightarrow{\varphi_2} & T^n \end{array}$$

is commutative. The set of equivalence class $[(K_i, \varphi_i)]$ together with this transitive relation is the *special subgroup system of* $\tilde{G}$.

10. **Theorem.** *Let G be a compact connected Lie group. Then the local isomorphism system of G and the special subgroup system of $\tilde{G}$ are isomorphic quasi-ordered sets.*

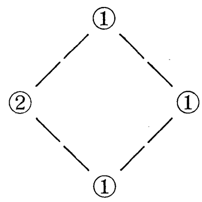
Proof. Let $\mathscr{L} = \{G_1, G_2, \cdots, G_l\}$ be the local isomorphism system of G and let $\mathscr{S}$ be the special subgroup system of $\tilde{G}$. Map $\mathscr{S}$ to $\mathscr{L}$ by $[(K, \varphi)] \rightarrow$ that G_i which is isomorphic to $\tilde{G}/(K, \varphi)$. By Proposition 2 this map is onto and by Corollary 6 it is one-to-one.

REMARKS. The above theorem can be used to make actual computations. In doing a computation the inner automorphisms of $H_1 \times H_2 \times \cdots \times H_t$ are irrelevant since these have no effect on the (K, φ) because K is a central subgroup of $H_1 \times \cdots \times H_t$. Thus only the group Aut $(H_1 \times \cdots \times H_t)$/Inner Aut $(H_1 \times \cdots \times H_t)$ plays a rôle. (Here Aut ( ) denotes the group of automorphisms of ( )). The group Aut $(H_1 \times \cdots \times H_t)$/Inner Aut $(H_1 \times \cdots \times H_t)$ is a finite group whose structure may be read off the Dynkin diagram of $H_1 \times \cdots \times H_t$: Aut $(H_1 \times \cdots \times H_t)$/Inner Aut $(H_1 \times \cdots \times H_t)$ is isomorphic to the group of symmetries of the Dynkin diagram. For this see [4].

Aut (T^n) is isomorphic to the group of all $n \times n$ matrices with integer entries and determinant $= \pm 1$. The semi-group of all covering homomorphisms of T^n onto itself is isomorphic to the semi-group of all $n \times n$ matrices with integer entries and nonzero determinant.

If S is a quasi-ordered set then the structure of S is conveniently represented by a diagram as follows:

Define an equivalence relation on S by setting $x \sim y$ if $x \geqq y$ and $y \geqq x$. The set of equivalence classes forms a partially ordered set which may be diagrammed in the usual fashion (e.g. see [2]). To diagram S diagram the associated partially ordered set and indicate the number of elements of S in each equivalence class. For example

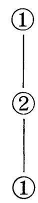


is the diagram of a quasi-ordered set with five elements x_1, x_2, x_3, x_4, x_5 and $\geqq$ defined by $x_1 \geqq x_i$, $x_i \geqq x_i$, $x_i \geqq x_5$ for all i, and $x_2 \geqq x_3$, $x_3 \geqq x_2$.

EXAMPLES. If $G = U(n)$, then $\widetilde{G} = SU(n) \times S^1$ with a covering homomorphism $SU(n) \times S^1 \rightarrow U(n)$ given by $(a, \lambda) \rightarrow a\lambda$. The center of $SU(n)$ is all diagonal matrices of the form μI where $\mu^n = 1$. This is a cyclic group of order n. $SU(n)$ has one outer automorphism and this is given by $(a_{ij}) \rightarrow (\overline{a_{ij}})$. $\overline{a_{ij}}$ denotes the complex conjugate of a_{ij}. S^1 has one automorphism given by $\lambda \rightarrow \bar{\lambda}$. Let Γ_k denote the subgroup of $U(n)$ consisting of all diagonal matrices θI where $\theta^k = 1$. Let $\omega = \cos 2\pi/n + i \sin 2\pi/n$. The inverse image of Γ_k under the covering homomorphism of $SU(n) \times S^1$ onto $U(n)$ is all pairs of the form $(\omega^i I, \theta\omega^{-i})$. Mapping $SU(n) \times S^1$ onto itself by $(a, \lambda) \rightarrow (a, \lambda^{-k})$ the pair $(\omega^i I, \theta\omega^{-i})$ goes to $(\omega^i I, \omega^{ki})$. Let Γ denote the center of $SU(n)$ then $U(n)/\Gamma_k$ is isomorphic to $SU(n) \times S^1/(\Gamma, \varphi_k)$ where φ_k is the homomorphism of Γ into S^1 given by $\varphi_k(\omega I) = \omega^k$. From this and the above remark about the automorphisms of $SU(n) \times S^1$ it now follows that $U(n)/\Gamma_{k_1}$ and $U(n)/\Gamma_{k_2}$ are isomorphic if and only if $k_1 \equiv \pm k_2 \bmod n$. Thus $U(n)$ covers precisely $[n/2] + 1$ distinct Lie groups. ($[n/2] =$ integer part of $n/2$).

If $n = 5$ let $\Gamma =$ center of $SU(5)$ and let $\omega = \cos 2\pi/5 + i \sin 2\pi/5$. There are 5 distinct homomorphisms $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ of Γ into S^1 with φ_i determined by $\varphi_i(\omega I) = \omega^i$. Thus in $SU(5) \times S^1$ there are six special subgroups: the special subgroup consisting of just the identity element and the (Γ, φ_i). The automorphism of $SU(5) \times S^1$ given by $(a, \lambda) \rightarrow (a, \bar{\lambda})$ takes (Γ, φ_1) onto (Γ, φ_4) and takes (Γ, φ_2) onto (Γ, φ_3). Hence the special subgroup system of $SU(5) \times S^1$ has four distinct elements: $[(1, 1)], [(\Gamma, \varphi_0)], [(\Gamma, \varphi_1)], [(\Gamma, \varphi_2)]$. $[(\Gamma, \varphi_1)] \geqq [(\Gamma, \varphi_2)]$ because $(a, \lambda) \rightarrow (a, \lambda^2)$ takes (Γ, φ_1) into (Γ, φ_2). Also $[(\Gamma, \varphi_2)] \geqq [(\Gamma, \varphi_1)]$ because

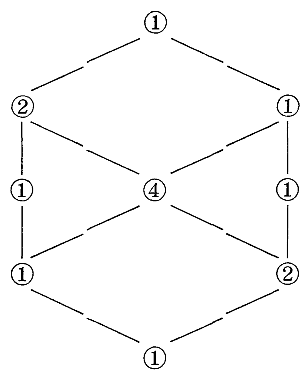
$(a, \lambda) \rightarrow (a, \lambda^2)$ takes (Γ, φ_2) into (Γ, φ_1). So the diagram of the local isomorphism system of $U(5)$ is:



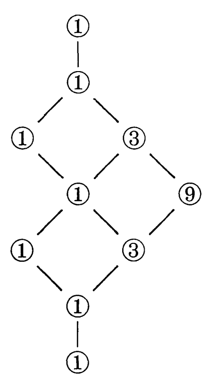
At the top is the group $SU(5) \times S^1$. At the bottom is the group $SU(5)/\Gamma \times S^1$. In the middle are the two groups $U(5)$, $U(5)/\{\pm I\}$.

Other computations can be made similarly.

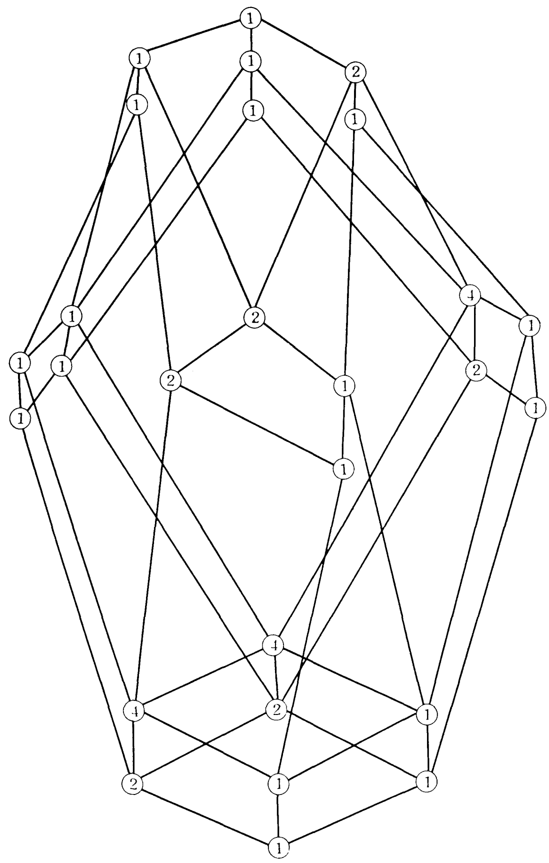
The local isomorphism system of $U(15)$ is:



The local isomorphism system of $U(27)$ is:



The local isomorphism system of $U(30)$ is:



If two compact connected Lie groups cover each other, then as differentiable manifolds they are diffeomorphic. This holds because any compact connected Lie group is diffeomorphic to the Cartesian product of its maximal connected semi-simple subgroup and the identity component of its center. (This can be proved by applying the fact that a differentiable principal fibre bundle over a circle with connected structural group is trivial.)

Connected semi-simple Lie groups can be diffeomorphic as differentiable manifolds but not isomorphic as Lie groups. Examples of this are given in [1]. For compact connected simple (i.e. the only normal subgroups are discrete) Lie groups homotopy equivalence of the underlying differentiable manifolds implies isomorphism [1].

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