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ON l-SIMPLICIAL CONVEXITY IN VECTOR SPACES

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The paper is concerned with a generalized type of convexity, which is called l-simplicial convexity. The name is derived from the simplex with l vertices, an l-simplicial convex set being the union of all $(i-1)$-simplexes with vertices in another set, i varying between 1 and l. The basic space is a linear one.

For convex sets the l-order (which is a natural number associated to an l-simplicial convex set) is a decreasing function of l. Several inequalities between l- and k-orders are established. In doing this the case of a convex set and that of a non convex set are distinguished.

Besides these inequalities, the main result of the paper is the proof of non monotonicity of the l-order, given by an example in a 34-dimensional linear space.

Let us consider a real vector space and recall some notations and definitions, given in [2].

The convex cover (hull) of a set M is $\mathscr{C}(M)$. We denote $\mathscr{C}(\{x_1, \cdots, x_s\})$ by $\mathscr{C}(x_1, \cdots, x_s)$, $x_1, \cdots, x_s$ being not necessarily distinct points in the space.

The operation $\mathscr{S}_l$, called l-simplicial convex cover, and defined by

$$\mathscr{S}_l(M) = \{\cup \mathscr{C}(p_1, \cdots, p_l) : p_j \in M, 1 \leqq j \leqq l\}$$

(for arbitrary set M and natural number $l \geqq 2$) will play a significant role throughout the paper. The operation $\mathscr{S}_l$ is studied in [1], where is denoted by con_l. It is easy to verify the following elementary properties of this operation:

(1) $\mathscr{S}_{mn} = \mathscr{S}_m \circ \mathscr{S}_n$,

(2) $\mathscr{S}_{m^p} = \mathscr{S}_m^p$, and

(3) $\mathscr{S}_m(M) \subset \mathscr{C}(M)$,

for arbitrary $m, n, p \geqq 2$ and M. $\mathscr{S}_l(M)$ is an increasing[1] function of l. It is also an increasing function of M with respect to the inclusion ordering. Let us denote $\mathscr{S}_m(\{x_1, \cdots, x_s\})$ by $\mathscr{S}_m(x_1, \cdots, x_s)$.

A set K is said to be *l-simplicial convex* if there exists a subset M such that $K = \mathscr{S}_l(M)$.

The *l-order* (*l-simplicial convexity order*) of an l-simplicial convex set K is

$$\omega_l(K) = \sup_M \min \{k: \mathscr{S}_l^k(M) = K\} .$$

[1] By increasing and decreasing functions we mean here not necessarily strictly increasing and strictly decreasing ones.

A set K is said to be *simplicial convex* if there exists a number l such that K is l-simplicial convex.

The *degree* of a simplicial convex set K is

$$\delta(K) = \min\{l\colon K \text{ is } l\text{-simplicial convex}\}\ .$$

The *order* (*simplicial convexity order*) of a simplicial convex set K is

$$\Omega(K) = \sup_l \omega_l(K)\ .$$

The *power* of a simplicial convex set K of finite order is

$$\Delta(K) = \min\{l\colon \Omega(K) = \omega_l(K)\}\ .$$

It is proved in § 1 of both [1] and [2] that

$$\mathscr{E}(M) = \mathscr{S}_l^{[\log_l n]+1}(M)$$

in an n-dimensional vector space.

2. Relations between l-simplicial and k-simplicial convexity orders. In this section some inequalities concerning l-order and k-order of a same set will be established.

THEOREM 1. *The l-order and k-order of a convex set K satisfy the inequalities*:

$$[\log_k(l^{\omega_l(K)-1}+1)] \leqq \omega_k(K) \leqq [\log_k(l^{\omega_l(K)}-1)]+1\ .$$

Proof. Consider a set M such that

$$\mathscr{S}_k(M) = K\ .$$

If $k < l^q$,

$$K = \mathscr{S}_k(M) \subset \mathscr{S}_l^q(M)\ .$$

Since K is convex the inverse inclusion holds too and $K = \mathscr{S}_l^q(M)$.

Let $x \in K$. Since

$$\mathscr{S}_l^{\omega_l(K)}(M) = K\ ,$$

x belongs to a simplex with vertices $x_1, \cdots, x_s \in M$ ($s \leqq l^{\omega_l(K)}$). There exists a linear manifold of dimension $l^{\omega_l(K)} - 1$ containing $\mathscr{E}(x_1, \cdots, x_s)$. Following the last remark of § 1,

$$\mathscr{E}(x_1, \cdots, x_s) = \mathscr{S}_k^{[\log_k(l^{\omega_l(K)}-1)]+1}(x_1, \cdots, x_s) \subset \mathscr{S}_k^{[\log_k(l^{\omega_l(K)}-1)]+1}(M)\ ,$$

hence

$$K \subset \mathscr{S}_k^{[\log_k(l^{\omega_l(K)}-1)]+1}(M) .$$

The inverse inclusion holds owing to the convexity of K, whence

$$\mathscr{S}_k^{[\log_k(l^{\omega_l(K)}-1)]+1}(M) = K .$$

Thus

$$\omega_k(K) \leqq [\log_k (l^{\omega_l(K)} - 1)] + 1 .$$

The symmetry in l and k gives

$$\omega_l(K) \leqq [\log_l (k^{\omega_k(K)} - 1)] + 1 \leqq \log_l (k^{\omega_k(K)} - 1) + 1 ,$$

i.e.

$$l^{\omega_l(K)-1} \leqq k^{\omega_k(K)} - 1 .$$

It follows that

$$[\log_k (l^{\omega_l(K)-1} + 1)] \leqq \log_k (l^{\omega_l(K)-1} + 1) \leqq \omega_k(K)$$

and both inequalities are obtained.

In [2], we have established that in general, k-simplicial convexity does not imply l-simplicial convexity for $l < k$. However, if k is a power of l this implication holds.

THEOREM 2. *For all natural numbers $k, l, q \geqq 2$ satisfying $k = l^q$, non convexity and k-simplicial convexity imply l-simplicial convexity; also the k-simplicial and l-simplicial convexity orders verify the inequalities*

$$q\omega_k \leqq \omega_l \leqq q\omega_k + q - 1 .$$

Proof. Let M be such that

$$\mathscr{S}_k^{\omega_k(C)}(M) = C ,$$

where C is a non convex, k-simplicial convex set. Then

$$\mathscr{S}_l^{q\omega_k(C)} = C ,$$

whence C is l-simplicial convex and

$$\omega_l(C) \geqq q\omega_k(C) .$$

To prove the other inequality, suppose that there is a set M' such that

$$\mathscr{S}_l^{m}(M') = C$$

with

$$m \geqq q\omega_k(C) + q .$$

Then, either $m = q\omega_k(C) + q$ and

$$\mathscr{S}_k^{\omega_k(C)+1}(M') = C ,$$

or $m > q\omega_k(C) + q$ and

$$\mathscr{S}_k^{\omega_k(C)+1}(\mathscr{S}_l^{m-q\omega_k(C)-q}(M')) = C ,$$

both impossible. Following Theorem 5 of [2],

$$\omega_l(C) = \sup_M \{m: \mathscr{S}_l^m(M) = C\} .$$

Hence

$$\omega_l(C) \leqq q\omega_k(C) + q - 1 ,$$

which concludes the theorem.

A different inequality is obtained, if $k = l^q$, for a convex set. We have, indeed, by Theorem 1, $\omega_k \leqq \log_k (l^{\omega_l} - 1) + 1$, i.e.

$$k^{\omega_k - 1} \leqq l^{\omega_l} - 1$$

and $\omega_k \geqq \log_k (l^{\omega_l - 1} + 1)$, i.e.

$$k^{\omega_k} \geqq l^{\omega_l - 1} + 1 .$$

Hence

$$l^{q(\omega_k - 1)} \leqq l^{\omega_l} - 1 ,$$

whence

$$q(\omega_k - 1) < \omega_l$$

and

$$l^{q\omega_k} \geqq l^{\omega_l - 1} + 1 ,$$

whence

$$q\omega_k > \omega_l - 1 .$$

Therefore

$$q\omega_k - q + 1 \leqq \omega_l \leqq q\omega_k .$$

These inequalities and those of Theorem 2 show that, for arbitrary sets, k-simplicial convexity implies l-simplicial convexity (for $k = l^q$) and

$$q\omega_k - q + 1 \leqq \omega_l \leqq q\omega_k + q - 1 .$$

3. Monotonicity of $\omega_l(K)$ for convex K. It is proved in § 4 of [2] that $\omega_k \leqq \omega_l$ if k is a multiple of l. Moreover, we shall prove that for convex K, $\omega_k(K) \leqq \omega_l(K)$ if $k \geqq l$.

THEOREM 3. *For a convex set K, the l-simplicial convexity order is decreasing on l and $\Omega(K) = \omega_2(K)$.*

Proof. Prove that $\omega_l(K)$ is a decreasing function of l. Suppose, on the contrary, that $i > j$ and

$$\omega_i(K) > \omega_j(K) .$$

It follows that

$$\omega_i(K) - 1 \geqq \omega_j(K)$$

and

$$i^{\omega_i(K)-1} > j^{\omega_i(K)-1} \geqq j^{\omega_j(K)} .$$

But, from Theorem 1,

$$\omega_i(K) \leqq \log_i (j^{\omega_j(K)} - 1) + 1 ,$$

whence

$$i^{\omega_i(K)-1} \leqq j^{\omega_j(K)} - 1 < j^{\omega_j(K)} .$$

The contradiction shows that $\omega_l(K)$ is decreasing of l, for convex K. Since K is 2-simplicial convex,

$$\Omega(K) = \omega_{\delta(K)}(K) = \omega_2(K) .$$

4. Non monotonicity of ω_l. It may be conjectured that ω_l is in general a decreasing function of l, i.e. $\omega_l(C)$ is also decreasing for non convex C. Then the inequality of Theorem 11 would be trivially implied by Theorem 6, both of [2], and Ω and δ would equal respectively ω_δ and Δ.

On the other hand one can believe that Theorem 2 can be obtained from two more general inequalities, for non convex sets, like

$$(*) \qquad [\log_k (l^{\omega_l} + 1)]-1 \leqq \omega_k \leqq [\log_k (l^{\omega_l+1} - 1)]$$

in the same way as, for convex sets,

$$q\omega_k - q + 1 \leqq \omega_l \leqq q\omega_k \qquad (k = l^q)$$

is implied by Theorem 1.

It should be noted that each inequality of (*) would imply that ω_l is decreasing. If, on the contrary $i < j$ and $\omega_i < \omega_j$, then

$$i^{\omega_i+1} < j^{\omega_j}$$

which contradicts the two inequalities

$$j^{\omega_j} \leqq i^{\omega_i+1} - 1$$

and

$$\omega_j \leqq [\log_j (i^{\omega_i+1} - 1)] .$$

Also,

$$i^{\omega_i+1} < j^{\omega_j}$$

contradicts the two inequalities

$$j^{\omega_j} + 1 \leqq i^{\omega_i+1}$$

and

$$[\log_i (j^{\omega_j} + 1)] - 1 \leqq \omega_i .$$

The conjecture that ω_l is, in general, a decreasing function of l (and with it also relation $(*)$) is disproved by the following counter-example.

PROPOSITION. Let $\mathscr{V}$ be a 34-dimensional real vector space and C be the union of the subsimplexes with 18 vertices of a simplex with 35 vertices in $\mathscr{V}$. Then C is both 2-simplicial convex and 3-simplicial convex, $\omega_2(C) = 1$ and $\omega_3(C) = 2$. Also $\Omega(C) = 2$ and $\Delta(C) = 3$.

Note. This proposition and its proof, are the simplest that we could find to provide our counter-example. We ask for simpler ones. In fact, we have found such a simpler example in a vector space of smaller dimension, but the proof was much more complicated. However it should be interesting to find the smallest dimension of the space in which such an example can be found, even if its proof is difficult.

Proof. Let $S_j^i (i = 1, \cdots, \binom{35}{j})$ be the subsimplexes with j vertices $(j = 2, \cdots, 34)$ of the given simplex S. Thus

$$C = \bigcup_i S_{18}^i .$$

(1) The 2-simplicial and 3-simplicial convexity of C follow from

$$C = \mathscr{S}_2\Big(\bigcup_i S_9^i\Big) = \mathscr{S}_3\Big(\bigcup_i S_6^i\Big) .$$

(2) Prove that $\omega_2(C) = 1$. Suppose that $\mathscr{S}_2^2(M) = C$.
(α) First, we shall prove that

$$\mathscr{S}_2(M) \subset \bigcup_i S_9^i .$$

Suppose there exists

$$y \in \mathscr{S}_2(M) - \bigcup_i S_9^i ;$$

then y belongs to the interior of a simplex S_j^{ia} with $10 \leqq j \leqq 18$.

(a) If $10 \leqq j \leqq 17$, let S_{35-j}^{ib} be the simplex with $35 - j$ vertices disjoint from S_j^{ia}. If $\mathscr{S}_2(M)$ contains also a point z in the interior of a subsimplex S_{l-j}^{ic} of S_{35-j}^{ib}, with $l \geqq 19$, then $\mathscr{E}(y, z)$ intersects the interior of $\mathscr{E}(S_j^{ia} \cup S_{l-j}^{ic})$, which is impossible; hence

$$\mathscr{S}_2(M) \cap S_{35-j}^{ib} \subset \{\cup S_{18-j}^{ic} : S_{18-j}^{ic} \subset S_{35-j}^{ib}\} \subset \{\cup S_8^i : S_8^i \subset S_{35-j}^{ib}\}$$

for $10 \leqq j \leqq 16$ or

$$\mathscr{S}_2(M) \cap S_{18}^{ib} \subset \{x_1, \cdots, x_{35}\} \cap S_{18}^{ib} \subset \{\cup S_8^i : S_8^i \subset S_{18}^{ib}\}$$

and if

$$S_{18}^{id} \subset S_{35-j}^{ib} ,$$

then

$$\mathscr{S}_2(M) \cap S_{18}^{id} \subset \{\cup S_8^i : S_8^i \subset S_{35-j}^{ib}\} \cap S_{18}^{id} = \{\cup S_8^i : S_8^i \subset S_{18}^{id}\} .$$

It follows that

$$\mathscr{S}_2^2(M) \cap S_{18}^{id} \subset \{\cup S_{16}^i : S_{16}^i \subset S_{18}^{id}\} \neq S_{18}^{id} ,$$

whence

$$\mathscr{S}_2^2(M) \not\supset S_{18}^{id} ,$$

absurd.

(b) If $j = 18$, then the segment joining y with a vertex x_s of S that does not belong to S_{18}^{ia}, meets the interior of $\mathscr{E}(S_{18}^{ia} \cup \{x_s\})$, absurd again.

(β) Now, prove that

$$M \not\subset \bigcup_i S_9^i .$$

Suppose that $M \subset \bigcup_i S_4^i$. Then

$$\mathscr{S}_2^2(M) \subset \bigcup_i S_{16}^i ,$$

whence

$$\mathscr{S}_2^2(M) \neq \bigcup_i S_{18}^i ,$$

impossible. Hence there exists $x \in M - \bigcup_i S_4^i$.

(a) If x is an interior point of a simplex $S_j^{i_e}$ with $5 \leqq j \leqq 8$, then, following (α), M does not intersect the interior of any subsimplex $S_l^{i_f} (l \geqq 5)$ of

$$S_{35-j}^{i_g} = \mathscr{E}(\{x_1, \cdots, x_{35}\} - S_j^{i_e}) .$$

Hence, if $S_{18}^{i_h} \subset S_{35-j}^{i_g}$, then

$$\mathscr{S}_2^2(M) \cap S_{18}^{i_h} \subset \mathscr{S}_2^2\{\cup S_4^i : S_4^i \subset S_{18}^{i_h}\} = \{\cup S_{16}^i : S_{16}^i \subset S_{18}^{i_h}\} \neq S_{18}^{i_h} ,$$

whence $\mathscr{S}_2^2(M) \not\supset S_{18}^{i_h}$, absurd.

(b) If x is an interior point of the simplex $S_9^{i_k}$ then, for $x_r \in S_9^{i_k}$, $\mathscr{E}(x, x_r)$ intersects the interior of $\mathscr{E}(S_9^{i_k} \cup \{x_r\})$, absurd.

Combining (α) and (β) we obtain that $M \not\subset \bigcup_i S_9^i$ and $\mathscr{S}_2(M) \subset \bigcup_i S_9^i$, contradicting one other. Therefore any $M \subset \mathscr{V}$ such that $\mathscr{S}_2^2(M) = C$ does not exist. Also, $\mathscr{S}_2^k(M) \neq C$ for $k \geqq 3$, because $\mathscr{S}_2^2(\mathscr{S}_2^{k-2}(M)) \neq C$. Hence $\omega_2(C) = 1$.

(3) Prove that $\omega_3(C) = 2$. Suppose there exist a subset $M \subset \mathscr{V}$ and a number $k \geqq 3$ such that $\mathscr{S}_3^k(M) = C$. Then M must contain the vertices $x_1, \cdots, x_{35}$ of S. But

$$\mathscr{S}_3^k(M) \supset \mathscr{S}_3^3(M) \supset \mathscr{S}_3^3(x_1, \cdots, x_{35}) = \bigcup_i S_{27}^i ,$$

whence $\mathscr{S}_3^k(M) \neq C$, absurd. Because

$$\mathscr{S}_3^2\Big(\bigcup_i S_2^i\Big) = C ,$$

$\omega_3(C) = 2$.

(4) Prove that $\Omega(C) = 2$ and $\Delta(C) = 3$. If C would be 4-simplicial convex, then $2\omega_4(C) \leqq 1$, following Theorem 2, which is not possible. If C is l-simplicial convex, with $l \geqq 5$, then suppose that M and $k \geqq 2$ satisfy $\mathscr{S}_l^k(M) = C$. Then

$$\mathscr{S}_l^k(M) \supset \mathscr{S}_5^2(M) \supset \mathscr{S}_5^2(x_1, \cdots, x_{35}) = \bigcup_i S_{25}^i ,$$

absurd. Hence $\omega_l(C) = 1$.

Thus $\Omega(C) = \omega_3(C) = 2$ and $\Delta(C) = 3$. The proof is complete.

5. Concluding remarks. Most of the results of [2] and of the present paper are purely algebraic and hence valid in vector spaces over arbitrary ordered fields.

Besides the operation $\mathscr{S}_l$, the function ω_l whose study was begun in § 2 of [2] and continued here is of special interest in the investigation of l-simplicial convexity. In §§ 3–4 of [2] we established some elementary facts concerning degree, order, and power. In order to initiate a systematic study of simplicial convexity more information about these three functions should be obtained.

References

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