PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 3, 1967

ON THE STRUCTURE OF TOR, II

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The following results are proved:

If A and B are abelian p-groups and the length of A is greater than the length of B, then $\mathrm{Tor}(A, B)$ is a direct sum of countable groups if and only if (i) B is a direct sum of countable groups and (ii) if the β-th Ulm invariant of B is not zero, then every $p^{\beta}A$-high subgroup of A is a direct sum of countable groups.

If β is an ordinal, A is a p-group, and if one $p^{\beta}A$-high subgroup of A is a direct sum of countable groups then every $p^{\beta}A$-high subgroup of A is a direct sum of countable groups.

If A and B are p-groups of cardinality $\leqq \aleph_1$ without elements of infinite height, then $\mathrm{Tor}(A, B)$ is a direct sum of cyclic groups.

For each n with $1 \leqq n < \omega$, there is a p-group G without elements of infinite height such that G is not itself a direct sum of cyclic groups but every subgroup of G having cardinality $\leqq \aleph_n$ is a direct sum of cyclic groups.

If A and B are (abelian) p-groups, when is $\mathrm{Tor}(A, B)$ a direct sum of countable groups (d.s.c. group)? This paper contains a complete answer for this question when A and B have different lengths.

If A and B have the same length the situation is much more complicated. The simplest case occurs when A and B have no elements of infinite height. Then $\mathrm{Tor}(A, B)$ has no elements of infinite height and it is a d.s.c. group if and only if it is a direct sum of cyclic groups (Σ-cyclic). Here, although there is no satisfactory answer to the question, some partial results are obtained. For example it is shown that if A and B are p-groups without elements of infinite height having cardinalities $\leqq \aleph_1$, then $\mathrm{Tor}(A, B)$ is Σ-cyclic.

Finally some examples of strange groups are constructed. Let a p-group be called λ-cyclic, where λ is a cardinal number, if every subgroup with cardinality $< \lambda$ is Σ-cyclic. Every p-group without elements of infinite height is $\aleph_1$-cyclic. For each n with $1 \leqq n < \omega$, a group is constructed which is $\aleph_m$-cyclic but not Σ-cyclic.

These results are obtained by homological methods together with the concept of N-high subgroup due to John Irwin [2].

In diagrams $\rightarrowtail$ denotes a monomorphism and $\twoheadrightarrow$ an epimorphism. An extension $C \rightarrowtail E \twoheadrightarrow A$ is *p^{α}-pure* where p is a prime and α an ordinal number if it belongs to $p^{\alpha}\,\mathrm{Ext}(A, C)$. A monomorphism $f: C \rightarrowtail E$ is p^{α}-pure if the extension $C \rightarrowtail E \twoheadrightarrow \mathrm{Coker}\, f$ is. Similarly $C \subseteqq E$ is a *p^{α}-pure subgroup* of E if the extension

$C \rightarrowtail E \twoheadrightarrow E/C$ is p^α-pure. If E is a p-group, then p^ω-purity coincides with the ordinary concept of purity. More generally, if $C \rightarrowtail E \twoheadrightarrow A$ is p^α-pure then:

(1°) $(p^\beta A)[p] = (C + (p^\beta E)[p])/C$ for all $\beta < \alpha$, and

(2°) $C \cap p^\beta E = p^\beta C$ for all $\beta \leqq \alpha$.

An easy transfinite induction shows that (1°) $\Rightarrow$ (2°). If $\alpha \leqq \omega$, then (2°) $\Rightarrow$ p^α-purity. If A is a divisible p-group then (1°) $\Rightarrow$ p^α-purity for all ordinals α. This last implication holds in certain other situations but not in general. These facts are proved in [8].

If N is a subgroup of the group G, a subgroup H of G is called *N-high in G* if H is maximal with respect to the property $H \cap N = 0$.

PROPOSITION 1. If G is a p-group, $N \subseteqq p^\alpha G$, and H is N-high in G, then H is $p^{\alpha+1}$-pure in G. If $N \subseteqq p^\omega G$, then G/H is divisible.[1]

Proof. We prove first that if G is a p-group, then H is N-high in G if and only if $H \cap N = 0$ and $(G/H)[p] = (H + N[p])/H$. To see this suppose H is N-high in G. Clearly $H \cap N = 0$ and $(H + N[p])/H \subseteqq (G/H)[p]$. Let $0 \neq x \in (G/H)[p]$ and let $g \in G$ map onto x mod H. Then $g \notin H$, $pg \in H$, and by the maximality of H there is a nonzero $a \in H \cap (H + \{g\})$. Thus $a = h + kg$ with k an integer. Moreover p does not divide k for otherwise $a \in H \cap N = 0$. Hence $1 = rk + sp$ for suitable integers r and s and $g = rkg + spg = ra + (spg\text{-}rh)$. Now $spg - rh \in H$ and $pra \in GH \cap N = 0$ so that $x \in (H + N[p])/H$ as desired.

Conversely suppose $(G/H)[p] = (H + N[p])/H$ and $H \cap N = 0$. To show the maximality of H it is enough to show that $(H + \{g\} \cap N \neq 0$ whenever $g \notin H$ but $pg \in H$. If g has these properties then the hypothesis gives $g = a + h$ with $a \in N$ and $h \in H$. Then $a \neq 0$ because $g \notin H$ and $a = g - h \in (H + \{g\}) \cap N$.

Now suppose H is N-high in G and $N \subseteqq p^\alpha G$. Then by the result just proved we have $(G/H)[p] \subseteqq (H + (p^\beta G)[p])/H$ for all $\beta \leqq \alpha$. Since $(p^\beta(G/H))[p] \subseteqq (G/H)[p]$ and $(H + (p^\beta G)[p])/H \subseteqq (p^\beta G/H))[p]$ we have

$$(p^\beta(G/H))[p] = (H + (p^\beta G)[p])/H \qquad \text{for all } \beta \leqq \alpha\ .$$

If $\alpha < \omega$ the discussion preceeding the statement of this proposition shows that H is $p^{\alpha+1}$-pure in G. The same discussion shows $p^{\alpha+1}$-purity for $\alpha \geqq \omega$ once we know that G/H is divisible.

If $N \subseteqq p^\omega G$, then $(G/H)[p] \subseteqq (H + p^\omega G)/H \subseteqq p^\omega(G/H)$ which implis that G/H is divisible.

PROPOSITION 2. If $C \rightarrowtail E \twoheadrightarrow A$ is p^α-pure with $\alpha \geqq \omega$ and B is any p-group, then

[1] The first statement of this proposition and the first statement of [**3**] Theorem 2 read the same, however the term p^α-purity has different meanings in the two places.

$$\mathrm{Tor}\,(C, B) \rightarrowtail \mathrm{Tor}\,(E, B) \twoheadrightarrow \mathrm{Tor}\,(A, B)$$

is exact and p^α-pure.

Proof. The condition $\alpha \geqq \omega$ is needed only to show that $\mathrm{Tor}\,(E, B) \to \mathrm{Tor}\,(A, B)$ is epic. We use the description of $\mathrm{Tor}\,(A, B)$ in terms of generators and relations given in [6, p. 150]. The generators are triples $\langle a, n, b\rangle$ with $n \in Z$ (the group if integers), $a \in A[n]$ and $b \in B[n]$. The relations require $\langle a, n, b\rangle$ to be bilinear as a function of a and b and also require $\langle ka, n, b\rangle = \langle a, kn, b\rangle$ for $k, n \in z$, $a \in A[kn]$, $b \in B[n]$, and $\langle a, n, kb\rangle = \langle a, kn, b\rangle$ for $k, n \in z$, $a \in A[n]$, $b \in B[kn]$.

If $\alpha \geqq 0$, then $C \rightarrowtail E \twoheadrightarrow A$ is p^ω-pure and it follows that each $a \in A[p^n]$ can be lifted to an element $e \in E$ with the same order. Since B is a p-group, $\mathrm{Tor}\,(A, B)$ is generated by the elements $\langle a, p^n, b\rangle$ with $p^n a = 0 = p^n b$. Letting $e \in E[p^n]$ map onto a we have $\langle e, p^n, b\rangle \in \mathrm{Tor}\,(E, B)$ mapping onto $\langle a, p^n, b\rangle$ as required to show that $\mathrm{Tor}\,(E, B) \to \mathrm{Tor}\,(A, B)$ is epic.

The sequence with Tor is now exact because Tor is left-exact.

For a given α, the functor p^α is represented by an exact sequence $Z \rightarrowtail G \twoheadrightarrow H$ (See [7] or [8] for the definitions and details). For a group A let ∂_A: $\mathrm{Tor}\,(H, A) \to A$ be the connecting homomorphism induced by this sequence. We then have

$$\partial_A\langle x, n, a\rangle = (ny)a$$

where y is any element of G mapping onto x. Since $nx = 0$, $ny \in Z$ so that the term $(ny)a$ makes sense.

The extension

$$C \rightarrowtail E \overset{\lambda}{\twoheadrightarrow} A$$

is p^α-pure if and only if there is a map φ: $\mathrm{Tor}\,(H, A) \to E$ such that $\lambda\varphi = \partial_A$.

MacLane shows in [5] that, for groups A, B, C, the group $\mathrm{Tor}\,(\mathrm{Tor}\,(A, B), C)$ is generated by the elements $\langle\langle a, n, b\rangle, n, c\rangle$ with $n \in Z$ $na = nb = nc = 0$. Similarly $\mathrm{Tor}\,(A, \mathrm{Tor}\,(B, C))$ is generated by the elements $\langle a, n, \langle b, n, c\rangle\rangle$. Moreover there is a natual isomorphism

$$\theta\colon \mathrm{Tor}\,(\mathrm{Tor}\,(A, B), C) = \mathrm{Tor}\,(A, \mathrm{Tor}\,(B, C))$$

such that

$$\theta\langle\langle a, n, b\rangle, n, c\rangle = \langle a, n, \langle b, n, c\rangle\rangle\,.$$

For groups A, B we have a diagram

$$\begin{array}{ccc} \operatorname{Tor}(\operatorname{Tor}(H, A), B) & \xrightarrow{\theta} & \operatorname{Tor}(H, \operatorname{Tor}(A, B)) \\ {\scriptstyle \operatorname{Tor}(\partial_A, B)} \downarrow & \swarrow {\scriptstyle \partial_{\operatorname{Tor}(A,B)}} & \\ \operatorname{Tor}(A, B) & & \end{array}$$

This diagram commutes for we have

$$\begin{aligned} \operatorname{Tor}(\partial_A, B)\langle\langle x, n, a\rangle, n, b\rangle &= \langle \partial_A \langle x, n, a\rangle, n, b\rangle \\ &= \langle (ny)a, n, b\rangle \\ &= (ny)\langle a, n, b\rangle \\ &= \partial_{\operatorname{Tor}(A,B)}\langle x, n, \langle a, n, b\rangle\rangle \\ &= \partial_{\operatorname{Tor}(A,B)}\theta\langle\langle x, n, a\rangle, n, b\rangle \;. \end{aligned}$$

Now suppose $C \rightarrowtail E \twoheadrightarrow A$ is p^α-pure with $\lambda: E \rightarrow A$. Hence $\partial_A = \lambda\varphi$. Applying Tor we get $\operatorname{Tor}(\partial_A, B) = \operatorname{Tor}(\lambda, B) \operatorname{Tor}(\varphi, B)$ and therefore $\partial_{\operatorname{Tor}(A,B)} = \operatorname{Tor}(\lambda, B) \operatorname{Tor}(\varphi, B)\theta^{-1}$. Thus the sequence

$$\operatorname{Tor}(C, B) \rightarrowtail \operatorname{Tor}(E, B) \twoheadrightarrow \operatorname{Tor}(A, B)$$

is p^α-pure.

For the purposes of this paper we define the *length* $\lambda(A)$ of the p-group A to be the least ordinal α such that $p^\alpha A = 0$ and ∞ if there is no such ordinal. The symbol ∞ is assumed to be larger than any ordinal. According to [7] $p^\alpha \operatorname{Tor}(A, B) = \operatorname{Tor}(p^\alpha A, p^\alpha B)$ so that the length of $\operatorname{Tor}(A, B)$ is the minimum of the lengths of A and of B. The group A is *p^α-projective* if each p^α-pure extension $C \rightarrowtail E \twoheadrightarrow A$ splits. A d.s.c. group is p^α-projective if and only if it has length $\leqq \alpha$ ([7] or [8]).

In the proofs of the next few theorems we shall refer repeatedly to the following situation. Let β be an infinite ordinal and let M be a $p^\beta A$-high subgroup of A. Then by Proposition 1 the sequence

$$M \rightarrowtail A \twoheadrightarrow A/M$$

is $p^{\beta+1}$-pure and A/M is divisible. If B is a p-group, then by Proposition 2 the sequence

$$(*) \qquad \operatorname{Tor}(M, B) \rightarrowtail \operatorname{Tor}(A, B) \twoheadrightarrow \operatorname{Tor}(A/M, B)$$

is also $p^{\beta+1}$-pure. Moreover if $A/M \neq 0$, then $A/M = \Sigma Z(p^\infty)$ and hence $\operatorname{Tor}(A/M, B) = \Sigma \operatorname{Tor}(Z(p^\infty), B) = \Sigma B$.

In the remainder of the paper we shall use without further reference Kaplansky's theorem [4] that a direct summand of a d.s.c. group is itself one.

PROPOSITION 3. *If β is an infinite countable ordinal, A has a $p^\beta A$-high subgroup which is a d.s.c. group, and B is a countable p-group of length $\leqq \beta + 1$, then $\operatorname{Tor}(A, B)$ is a d.s.c. group.*

Proof. Let M be the $p^\beta A$-high subgroup called for and refer to the $p^{\beta+1}$-pure sequence (*) above. Since B is countable of length $\leqq \beta+1$ it is $p^{\beta+1}$-projective. Hence $\mathrm{Tor}(A/M, B) = \Sigma B$ is also $p^{\beta+1}$-projective and the sequence (*) splits. But then $\mathrm{Tor}(A, B)$ is a direct sum of the d.s.c. groups $\mathrm{Tor}(M, B)$ and $\mathrm{Tor}(A/M, B)$ and is therefore a d.s.c. group. $\mathrm{Tor}(M, B)$ is a d.s.c. group because M is a d.s.c. group, Tor commutes with direct sums and $\mathrm{Tor}(G, B)$ is countable whenever G and B are.

PROPOSITION 4. Let $\mathrm{Tor}(A, B)$ be a d.s.c. group.

(i) If $\lambda(A) > \lambda(B)$, then B is a d.s.c. group.

(ii) If $\lambda(A) \geqq \lambda(B) = \beta+1$ with β an infinite countable ordinal, and B is a d.s.c. group, then every $p^\beta A$-high subgroup of A is a d.s.c. group.

Proof. To show (i) let $\beta = \lambda(B)$. If $\beta < \omega$ then B has bounded order and is clearly a d.s.c. group. Hence suppose $\beta \geqq \omega$. Let M be a $p^\beta A$-high subgroup of A and consider the $p^{\beta+1}$-pure sequence (*). Since $\mathrm{Tor}(A, B)$ is a d.s.c. group of length β it is p^β-projective. According to [8, Proposition 3.1] the $p^{\beta+1}$-purity of (*) then implies that the sequence (*) splits. Hence $\mathrm{Tor}(A/M, B)$ is a d.s.c. group. Since $\lambda(A) > \beta$, $A/M \neq 0$ so that B is a direct summand of $\mathrm{Tor}(A/M, B)$ and therefore a d.s.c. group.

To prove (ii) we again let M be $p^\beta A$-high and refer to (*). Now B is $p^{\beta+1}$-projective so that $\mathrm{Tor}(A, B)$ is $p^{\beta+1}$-projective. Hence the sequence (*) splits. Therefore $\mathrm{Tor}(M, B)$ is a d.s.c. group. Since $\lambda(M) = \beta < \lambda(B)$, M is a d.s.c. group by (i) and the commutativity of Tor.

COROLLARY 5. *If β is an infinite ordinal and the p-group A has one $p^\beta A$-high subgroup which is a d.s.c. group, then every $p^\beta A$-high subgroup of A is a d.s.c. group.*

Proof. If $\lambda(A) \leqq \beta$, then A is the only $p^\beta A$-high subgroup so the result is trivial. Therefore assume $\lambda(A) > \beta$. Next observe that if $\beta > \Omega$, then a $p^\beta A$-high subgroup cannot be a d.s.c. group because it has length β and the length of a d.s.c. group is either $\leqq \Omega$ or is ∞.

If $\beta = \Omega$ and M is $p^\beta A$-high, then $M \rightarrowtail A \twoheadrightarrow A/M$ is in $p^\Omega \mathrm{Ext}(A/M, M)$. If M is a d.s.c. group, then $p^\Omega \mathrm{Ext}(Z(p^\infty), M) = 0$ by [8] Lemma 3.10. Since $A/M = \Sigma Z(p^\infty)$,

$$\mathrm{Ext}(A/M, M) = \Pi\, \mathrm{Ext}(Z(p^\infty), M)$$

so that $p^\Omega \mathrm{Ext}(A/M, M) = 0$. Thus M is a direct summand of A.

Since M is $p^{\Omega}A$-high in A we have $(A/M)[p] \cong (p^{\Omega}A)[p]$ and it follows easily that $p^{\Omega}A$ is the maximal divisible subgroup of A. Hence $M \cong A/p^{\Omega}A$ in this case. Since $A/p^{\Omega}A$ is independent of M it follows that all $p^{\Omega}A$-high subgroups are isomorphic (if one is a d.s.c. group) and hence all are d.s.c. groups.

Finally if β is infinite and countable, let B be a countable p-group of length $\beta + 1$. By Proposition 3 $\operatorname{Tor}(A, B)$ is a d.s.c. group because A has a $p^{\beta}A$-high subgroup which is a d.s.c. group. By Proposition 4 (ii) every $p^{\beta}A$-high subgroup of A is a d.s.c. group.

THEOREM 6. *If $\lambda(A) > \lambda(B) \geqq \omega$, then* $\operatorname{Tor}(A, B)$ *is a d.s.c. group if and only if*

(i) *B is a d.s.c. group, and*

(ii) *if β is an infinite ordinal such that the β-th Ulm invariant of B is $\neq 0$, then every $p^{\beta}A$-high subgroup of A is a d.s.c. group.*

Proof. We need two easy consequences of Ulm's theorem and Zippin's theorem (cf. [1] p. 135):

(1) If B is a d.s.c. group whose β-th Ulm invariant is not zero, then B has a countable direct summand B' of length $\beta + 1$.

(2) If B is a d.s.c. group, then $B = \Sigma B_i$ where i ranges over some index set and B_i is countable of length $\beta_i + 1$.

Suppose $\operatorname{Tor}(A, B)$ is a d.s.c. group. We get (i) by proposition 4(i). Suppose further that β is infinite and the β-th Ulm invariant of B is not zero. Let B' be a countable direct summand of B with length $\beta + 1$ as provided by (1) above. Then $\operatorname{Tor}(A, B')$ is a direct summand of $\operatorname{Tor}(A, B)$, hence a d.s.c. group and (ii) follows from Proposition 4(ii).

Suppose $\lambda(A) > \lambda(B) \geqq \omega$ and (i) (ii) are satisfied. Using (2) above we write $B = \Sigma B_i$ with B_i countable of length $\beta_i + 1$. Then $\operatorname{Tor}(A, B) = \Sigma \operatorname{Tor}(A, B_i)$. If $\beta_i < \omega$, then B_i is a direct sum of cyclic groups so $\operatorname{Tor}(A, B_i)$ is a d.s.c. group. If $\beta_i \geqq \omega$, then $\operatorname{Tor}(A, B_i)$ is a d.s.c. group by Proposition 3. Hence $\operatorname{Tor}(A, B)$ is a d.s.c. group.

In order to continue we must derive further properties of Tor. The inclusions $A' \subseteqq A$, $B' \subseteqq B$ induce a monomorphism $\operatorname{Tor}(A', B') \rightarrowtail \operatorname{Tor}(A, B)$. We shall identify $\operatorname{Tor}(A', B')$ with its image in $\operatorname{Tor}(A, B)$.

LEMMA 7.

(i) *If $A', A'' \subseteqq A$, then* $\operatorname{Tor}(A' \cap A'', B) = \operatorname{Tor}(A', B) \cap \operatorname{Tor}(A'', B)$.

(ii) *If $A' \subseteqq A$ and $B' \subseteqq B$, then*

$$\operatorname{Tor}(A', B') = \operatorname{Tor}(A', B) \cap \operatorname{Tor}(A, B') .$$

(iii) *If* $A', A'' \subseteqq A$ *and* $B', B'' \supseteqq B$, *then*

$$\mathrm{Tor}\,(A' \cap A'', B' \cap B'') = \mathrm{Tor}\,(A', B') \cap \mathrm{Tor}\,(A'', B'') \,.$$

Proof. If $A', A'' \subseteqq A$, then there is a commutative diagram

$$\begin{array}{ccccc} A' \cap A'' & \rightarrowtail & A' & \twoheadrightarrow & A'/A' \cap A'' \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \rightarrowtail & A & \twoheadrightarrow & A/A'' \end{array}$$

with exact rows and monic vertical maps. Applying Tor we get

$$\begin{array}{ccccc} \mathrm{Tor}\,(A' \cap A'', B) & \rightarrowtail & \mathrm{Tor}\,(A', B) & \longrightarrow & \mathrm{Tor}\,(A'/A' \cap A'', B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Tor}\,(A'', B) & \rightarrowtail & \mathrm{Tor}\,(A, B) & \longrightarrow & \mathrm{Tor}\,(A/A'', B) \end{array}$$

with exact rows and monic vertical maps. Conclusion (i) follows from this diagram.

To prove (ii) we note the existence of the commutative diagram

$$\begin{array}{ccccc} \mathrm{Tor}\,(A', B') & \rightarrowtail & \mathrm{Tor}\,(A, B') & \longrightarrow & \mathrm{Tor}\,(A/A', B') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Tor}\,(A', B) & \rightarrowtail & \mathrm{Tor}\,(A, B) & \longrightarrow & \mathrm{Tor}\,(A/A', B) \end{array}$$

with exact rows and monic vertical maps and proceed as before.

For (iii) we have

$$\begin{aligned} &\mathrm{Tor}\,(A' \cap A'', B' \cap B'') \\ &\quad = \mathrm{Tor}\,(A' \cap A'', B) \cap \mathrm{Tor}\,(A, B' \cap B'') \\ &\quad = \mathrm{Tor}\,(A', B) \cap \mathrm{Tor}\,(A'', B) \cap \mathrm{Tor}\,(A, B') \cap \mathrm{Tor}\,(A, B'') \\ &\quad = \mathrm{Tor}\,(A', B') \cap \mathrm{Tor}\,(A'', B'') \end{aligned}$$

using in order (ii), (i), (ii).

Lemma 7 holds for any left exact covariant functor of two variables.

PROPOSITION 8. For $x \in \mathrm{Tor}(A, B)$, there are unique finite subgroups $A_x \subseteqq A$ and $B_x \subseteqq B$ such that

(i) $x \in \mathrm{Tor}\,(A_x, B_x)$ and

(ii) if $x \in \mathrm{Tor}\,(A', B')$ with $A' \subseteqq A$ and $B' \subseteqq B$, then $A_x \subseteqq A'$ and $B_x \subseteqq B'$.

Proof. There exist finite subgroups $G \subseteqq A$, $H \subseteqq B$ such that $x \in \mathrm{Tor}\,(G, H)$. Let $G_1, H_1; G_2, H_2; \cdots; G_n, H_n$ enumerate the pairs of subgroups of G and H respectively such that $x \in \mathrm{Tor}(G_i, H_i)$ for $i = 1, \cdots, n$. By Lemma 7 (iii) we have $x \in \mathrm{Tor}(G_1 \cap \cdots \cap G_n, H_1 \cap \cdots \cap H_n)$. Put $A_x = G_1 \cap \cdots \cap G_n$ and $B_x \in H_1 \cap \cdots \cap H_n$. Thus (i) is satisfied.

If $x \in \mathrm{Tor}(A', B')$ we have by Lemma 7 (iii) that

$$x \in \mathrm{Tor}(G \cap A', H \cap B') .$$

Then $G \cap A' = G_i$ and $H \cap B' = H_i$ for some i so that $A_x \subseteqq A'$ and $B_x \subseteqq B'$ proving (ii).

COROLLARY 9. *If $a \in A$, $b \in B$ have the same order and*

$$\mathrm{Tor}(\{a\}, \{b\}) \subseteqq \mathrm{Tor}(A', B')$$

with $A' \subseteqq A$ and $B' \subseteqq B$, then $a \in A'$ and $b \in B'$.

Proof. If the common order of a and b is n, then $\mathrm{Tor}(\{a\}, \{b\})$ is cyclic of order n. Let x be a generator. Then $x \in \mathrm{Tor}(\{a\} \cup A', \{b\} \cap B')$ by hypothesis and Lemma 7 (iii). If either $\{a\} \cap A'$ or $\{b\} \cap B'$ had order $<n$, then $\mathrm{Tor}(\{a\} \cap A', \{b\} \cap B')$ being cyclic would have order $<n$. Thus x would have order $<n$ contradicting the fact that its order is n. It follows that $\{a\} \cap A' = \{a\}$ and $\{b\} \cap B' = \{b\}$ so that $a \in A'$ and $b \in B'$.

PROPOSITION 10. If $A' \subseteqq A$ and $B' \subseteqq B$ with A, B p-groups, B' has unbounded order, and $\mathrm{Tor}(A', B')$ is pure in $\mathrm{Tor}(A, B)$, then A' is pure in A.

Proof. Let $a \in A' \cap p^n A$. Since B' has unbounded order there is a $b \in p^n B'$ having the same order as a. In [7] it was shown that $p^n \mathrm{Tor}(A, B) = \mathrm{Tor}(p^n A, p^n B)$. Hence

$$\begin{aligned} \mathrm{Tor}(\{a\}, \{b\}) &\subseteqq \mathrm{Tor}(A' \cap p^n A, B' \cap p^n B) \\ &\subseteqq \mathrm{Tor}(A', B') \cap p^n \mathrm{Tor}(A, B) \\ &\supseteqq p^n \mathrm{Tor}(A', B') = \mathrm{Tor}(p^n A', p^n B') . \end{aligned}$$

By Corollary 9 $a \in p^n A'$. Since a and n were arbitrary we have $A' \cap p^n A = p^n A'$ for all n and A' is pure in A.

An indexed family $\{A_\alpha\}_{a>\rho}$ of subgroups of A will be called a *sequence* of subgroups if α ranges over the set of ordinals less then some ordinal ρ and $A_\alpha \subseteqq A_\beta$ whenever $\alpha \leqq \beta < \rho$. If $\{A_\alpha\}_{\alpha<\rho}$ is a sequence of subgroups of A, then $\bigcup A_\alpha$ (the set theoretical union) is the subgroup generated by the A_α.

PROPOSITION 11. If $\{A_\alpha\}_{\alpha<\rho}$ and $\{B_\alpha\}_{\alpha<\rho}$ are sequences of subgroups of A and B respectively, then $\{\mathrm{Tor}(A_\alpha, B_\alpha)\}_{\alpha<\rho}$ is a sequence of subgroups of Tor (A, B) and

$$\bigcup \operatorname{Tor}(A_\alpha, B_\alpha) = \operatorname{Tor}(\bigcup A_\alpha, \bigcup B_\alpha)\ .$$

Proof. It is clear that $\{\operatorname{Tor}(A_\alpha, B_\alpha)\}_{\alpha<\rho}$ is a sequence of subgroups of Tor (A, B). Since $A_\alpha \subseteqq \bigcup A_\alpha$ and $B_\alpha \subseteqq \bigcup B_\alpha$ for all $\alpha < \rho$ we have

$$\bigcup \operatorname{Tor}(A_\alpha, B_\alpha) \subseteqq \operatorname{Tor}(\bigcup A_\alpha, \bigcup B_\alpha)\ .$$

Suppose $x \in \operatorname{Tor}(\bigcup A_\alpha, \bigcup B_\alpha)$ and let A_x, B_x be the subgroups defined by Proposition 8. Then $A_x \subseteqq \bigcup A_\alpha$ and $B_x \subseteqq \bigcup B_\alpha$. Since A_x and B_x are finite, there is a $\beta < \rho$ such that $A_x \subseteqq A_\beta$ and $B_x \subseteqq B_\beta$. Hence $x \in \operatorname{Tor}(A_\beta, B_\beta) \subseteqq \bigcup \operatorname{Tor}(A_\alpha, B_\alpha)$.

By the term Σ-cyclic we shall mean a direct sum of cyclic groups. A p-group is Σ-cyclic if and only if it is a d.s.c. group without elements of infinite height. In view of Proposition 4 if Tor (A, B) is Σ-cyclic and A has elements of infinite height, then B is Σ-cyclic.

The following theorem gives a necessary condition for Tor (A, B) to be Σ-cyclic. The symbol $|A|$ denotes the cardinality of A.

THEOREM 12. *If* Tor (A, B) *is Σ-cyclic and B is not Σ-cyclic, then*

(i) $p^\omega A = 0$, *and*

(ii) *if $A' \subseteqq A$ with $|A'| \geqq |B|$, then A' is contained in a pure subgroup A'' of the same cardinality, such that $p^\omega(A/A'') = 0$ and* Tor $(A/A'', B)$ *is Σ-cyclic.*

Proof. As stated above conclusion (i) follows from Proposition 4.

Recall that if A and B are infinite p-groups, then $|\operatorname{Tor}(A, B)| = |A||B|$. Now let $A' \subseteqq A$ with $|A'| \geqq |B|$ and let $\operatorname{Tor}(A, B) = \Sigma C_i$ with the sum direct and each C_i cyclic.

If G is an infinite subgroup of Tor (A, B), then, since each element has nonzero component in but a finite number of the summands C_i, G is contained in a subgroup $G' = \Sigma_{j\in J} C_j$ where J is a subset of the index set and $|G'| = |G|$. Moreover there are subgroups $A_0 \subseteqq A$, $B_0 \subseteqq B$ such that $|A_0| = |B_0| = |G|$ and $G \subseteqq \operatorname{Tor}(A_0, B_0)$. This is so because each $x \in \operatorname{Tor}(A, B)$ is a finite sum of elements of the form $\langle a, n, b\rangle$.

We define recursively a sequence

$$\begin{aligned}\operatorname{Tor}(A', B) \subseteqq G_1 \subseteqq \operatorname{Tor}(A_1, B) \subseteqq G_2 \subseteqq \cdots \\ \subseteqq G_n \subseteqq \operatorname{Tor}(A_n, B) \subseteqq G_{n+1} \subseteqq \cdots\end{aligned}$$

of subgroups all having the same cardinality such that G_n is the sum of a set of the C_i appearing in the chosen direct sum decomposition of Tor (A, B).

Let $A'' = \bigcup A_n$. Then by Proposition 11,

$$\operatorname{Tor}(A'', B) = \operatorname{Tor} \bigcup (A_n, B) = \bigcup G_n .$$

Hence $\operatorname{Tor}(A'', B)$ is the sum of a set of the C_i and is therefore a direct summand of $\operatorname{Tor}(A, B)$. Hence $\operatorname{Tor}(A'', B)$ is pure in $\operatorname{Tor}(A, B)$. Since B is not Σ-cyclic, it is unbounded and A'' is pure in A by Proposition 10. Hence the sequence

$$\operatorname{Tor}(A'', B) \rightarrowtail \operatorname{Tor}(A, B) \twoheadrightarrow \operatorname{Tor}(A/A'', B)$$

is exact and splits. Thus $\operatorname{Tor}(A/A'', B)$ is Σ-cyclic and $p^{\omega}(A/A'') = 0$ by part (i).

COROLLARY 13. *If A and B are p-groups without elements of infinite height, B is not Σ-cyclic, $|A| > |B|$, and A has greater cardinality than a basic subgroup, then* $\operatorname{Tor}(A, B)$ *is not Σ-cyclic.*

Proof. Suppose $\operatorname{Tor}(A, B)$ is Σ-cyclic and let C be a basic subgroup of A such that $|C| < |A|$. There is a subgroup A' with $C \subseteqq A' \subseteqq A$ and $|A| > |A'| \geqq |B|$. By Theorem 12 there is a subgroup A'' with $A' \subseteqq A'' \subseteqq A$, $|A''| = |A'|$, and $p^{\omega}(A/A'') = 0$. Now A/A'' is divisible because $C \subseteqq A''$ and A/C is divisible. Moreover $A/A'' \neq 0$ because $|A''| < |A|$. Hence $p^{\omega}(A/A'') \neq 0$ contradicting the construction of A''. Therefore $\operatorname{Tor}(A, B)$ is not Σ-cyclic.

LEMMA 14. *Let A be a p-group and let ρ be the least ordinal having the same cardinality as $|A|$. Then there is a sequence $\{A_\alpha\}_{\alpha<\rho}$ of subgroups of A such that $\bigcup_{\alpha<\rho} A_\alpha = A$ and*

(i) *each A_α is pure in A,*
(ii) *$A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if β is a limit ordinal $<\rho$,*
(iii) *$|A_\alpha| = \aleph_0$ if $\alpha < \omega$, and*
(iv) *$|A_\alpha| = |\alpha|$ if $\omega \leqq \alpha < \rho$.*

Proof. Well order A as $\{a_\alpha\}_{\alpha<\rho}$. Let A_0 be a countable pure subgroup of A containing a_0. Suppose A_β has been defined for all $\beta < \alpha$ satisfying (i)–(iv) above and also (v) $a_\gamma \in A_\beta$ if $\gamma < \beta$ and β is a singular ordinal $<\alpha$. If α is a limit ordinal (ii) forces the definition $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Then (i)–(v) follow easily. If $\alpha = \gamma + 1$, then $A_\gamma + \{a_\gamma\}$ has the same cardinality as A_γ and there is a pure subgroup A_α of A having the same cardinality as $A_\gamma + \{a_\gamma\}$. Then (i)–(v) are still satisfied. If $\alpha < \rho$, then $|A_\alpha| = |\alpha| < |A|$ so that the construction can be continued as long as $\alpha < \rho$. Since $a_\alpha \in A_{\alpha+1}$ it follows that $\bigcup_{\alpha<\rho} A_\alpha = A$.

THEOREM 15. *If A and B are p-groups with the same cardinality such that every subgroup of either with smaller cardinality is a Σ-group, then* Tor (A, B) *is Σ-cyclic.*

Proof. Let $\{A_\alpha\}_{\alpha<\rho}$, $\{B_\alpha\}_{\alpha<\rho}$, be sequences of subgroups of A and of B satisfying the conditions of Lemma 14. Since $|A_\alpha| = |B_\alpha| < |A|$ we have A_α and B_α Σ-cyclic for all $\alpha < \rho$. Set $G_\alpha = \text{Tor}(A_\alpha, B_\alpha)$. Since A_α is pure in A and B_α pure in B, G_α is pure in Tor (A, B). Moreover by Proposition 11 and Lemma 14 (ii), $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ whenever α is a limit ordinal $<p$.

Since Tor is left exact, there is an exat sequence

$$G_\alpha \rightarrowtail G_{\alpha+1} \longrightarrow \text{Tor}(A_{\alpha+1}/A_\alpha, B_{\alpha+1}) \oplus \text{Tor}(A_{\alpha+1}, B_{\alpha+1}/B_\alpha) .$$

The term on the right is Σ-cyclic because $A_{\alpha+1}$ and $B_{\alpha+1}$ are Σ-cyclic. Thus $G_{\alpha+1}/G_\alpha$ is Σ-cyclic and therefore $G_{\alpha+1} = G_\alpha \oplus C_\alpha$ with C_α Σ-cyclic because G_α is pure in $G_{\alpha+1}$. Hence we have a sequence $\{G_\alpha\}_{\alpha>\rho}$ of subgroups of Tor (A, B) such that

(1°) $G_\alpha \subseteqq G_{\alpha+1}$ for $\alpha < \rho$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ if α is a limit ordinal $<\rho$;

(2°) $G_{\alpha+1} = G_\alpha \oplus C_\alpha$ with C_α a Σ-group for all $\alpha < \rho$;

(iii) Tor $(A, B) = \bigcup_{\alpha<\rho} G_\alpha$.

It follows that Tor $(A, B) = \Sigma C_\alpha$ and is therefore a Σ-group.

COROLLARY 16. *If A and B are p-groups without elements of infinite height whose cardinality is at most $\aleph_1$, then* Tor (A, B) *is Σ-cyclic.*

If ρ is a cardinal number, call a p-group A ρ-cyclic if every subgroup of cardinal $<\rho$ is Σ-cyclic. Every p-group without elements of infinite height is $\aleph_1$-cyclic. Let ρ^+ be the cardinal next larger than ρ.

COROLLARY 17. *If A and B are ρ-cyclic, then* Tor (A, B) *is a ρ^+-cyclic.*

Proof. Let $G \subseteqq \text{Tor}(A, B)$ with $|G| = \rho$. Since, for $x \in \text{Tor}(A, B)$, there are finite subgroups A', B' of A and B respectively with $x \in \text{Tor}(A', B')$, there are subgroups $A_0 \subseteqq A$, $B_0 \subseteqq B$ such that $|A_0| = |B_0| = |G| = \rho$ and $G \subseteqq \text{Tor}(A_0, B_0)$. Since every subgroup of a direct sum of cyclic groups is one, A_0 and B_0 satisfy the hypotheses of Theorem 15. Hence Tor (A_0, B_0) and therefore G is Σ-cyclic. However G was arbitrary with $|G| = \rho$ so the corollary follows.

For groups $A_1, \cdots, A_n$, define Tor $(A_1, \cdots, A_n)$ inductively as Tor (Tor $(A_1, \cdots, A_{n-1}), A_n$).

LEMMA 18. *If $A_1, \cdots, A_{2n}$ are p-grrups without elements of infinite height, then* Tor $(A_1, \cdots, A_{2n})$ *is* $\aleph_{n+1}$*-cyclic.*

Proof. The proof is by induction. We use the associativity of Tor to show that

$$\text{Tor}(A_1, \cdots, A_{2n}) = \text{Tor}(\text{Tor}(A_1, \cdots, A_m), \text{Tor}(A_{m+1}, \cdots, A_{2^n}))$$

where $m = 2^{n-1}$ and then use Theorem 15 to complete the inductive step.

PROPOSITION 19. For each n with $1 \leqq n < \omega$, there is a p-group G_n without elements of infinite height such that G_n is $\aleph_n$-cyclic but not Σ-cyclic.

Proof. If C is the direct sum of ρ copies of $\Sigma Z(p^n)$ and $\rho \geqq \aleph_0$, then the torsion completion of C has cardinality $\rho^{\aleph_0}$. Hence if $\rho^{\aleph_0} > \rho$, there is a p-group without elements of infinite height which has greater cardinality than a basic subgroup. Since there are arbitrarily large cardinals with this property there exists a sequence $A_1, A_2, \cdots, A_n, \cdots$ of p-groups of increasing cardinality, all without elements of infinite height, and all with greater cardinality than a basic subgroup.

Set $G_n = \text{Tor}(A_1, \cdots, A_{2^{n-1}})$ and $G_1 = A_1$. Then G_n is $\aleph_n$-cyclic by Lemma 18. If A and B are infinite torsion groups $|\text{Tor}(A, B)| = \max\{|A|, |B|\}$ so that $|\text{Tor}(A_1, \cdots, A_k)| = |A_k|$. Thus G_n is not Σ-cyclic by Theorem 15.

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Received June 8, 1966. This work was supported in part by the National Science Foundation grant GP-5691.

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