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ON CONSTRUCTING DISTRIBUTION FUNCTIONS; A BOUNDED DENUMERABLE SPECTRUM WITH n LIMIT POINTS

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If $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ are real sequences with the b_n's all positive, then a theorem of Favard states that there exists a bounded increasing function $\psi(x)$ which is a distribution function for the polynomial set $\{\phi_n\}_{-1}^\infty$ which is recursively defined as follows: $\phi_{-1}(x) \equiv 0, \phi_0(x) \equiv 1$,

$$\text{(1-A)} \qquad \phi_{n+1}(x) = (x - a_n)\phi_n(x) - b_n\phi_{n-1}(x) \qquad (n \geqq 0)\,.$$

This study considers the problem of constructing $\psi(x)$ for certain classes of sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$. The sequences considered all lead to functions $\psi(x)$ which have a bounded denumerable spectrum with n limit points ($1 \leqq n < \infty$).

2. **Notation, preliminaries, and summary.** The following notational conventions will be maintained throughout this paper:

(1) $\{a_n\}_0^\infty$ is a sequence of real numbers.

(2) $\{b_n\}_1^\infty$ is a sequence of positive real numbers.

For each nonnegative integer s,

(3) $\{c_n^{(s)}\}_0^\infty$ is the sequence $\{c_{n+s}\}_{n=0}^\infty$.

(4) $\{\phi_n^{(s)}(x)\}_{-1}^\infty$ is the sequence of monic polynomials defined recursively by $\phi_{-1}^{(s)}(x) = 0$, $\phi_0^{(s)}(x) = 1$, and $\phi_{n+1}^{(s)}(x) = (x - a_n^{(s)})\phi_n^{(s)}(x) - b_n^{(s)}\phi_{n-1}^{(s)}(x)$ $(n \geqq 0)$.

(5) $\psi^{(s)}(x)$ is a bounded increasing function defined on $(-\infty, +\infty)$ and having the property that

$$\int_{-\infty}^{+\infty} \phi_n^{(s)}(x)\phi_m^{(s)}(x)d\psi^{(s)}(x) = \delta_{n,m}\cdot k_n^{(s)}$$

($k_n^{(s)} \neq 0, n = 0, 1, 2, \cdots$). ($\psi^{(s)}(x)$ is known to exist by the above-mentioned theorem of Favard.)

(6) $K^{(s)}(x)$ is the continued fraction given by

$$K^{(s)}(x) = \frac{1\ \ |}{|\ x - a_s} - \frac{b_{1+s}\ \ |}{|\ x - a_{1+s}} - \frac{b_{2+s}\ \ |}{|\ x - a_{2+s}} - \cdots\,.$$

(7) $\mathscr{S}(\psi^{(s)}(x))$ is the spectrum of the distribution function $\psi^{(s)}(x)$, i.e., $\mathscr{S}(\psi^{(s)}(x)) = \{x: -\infty < x < +\infty$ and $\psi^{(s)}(x+\varepsilon) - \psi^{(s)}(x-\varepsilon) > 0$ for all $\varepsilon > 0\}$. In terms of measures, $\mathscr{S}(\psi^{(s)}(x))$ is the support of the positive real measure induced by $\psi^{(s)}(x)$.

(8) We shall say that the polynomials $\phi_n^{(s)}(x)$, the bounded increasing function $\psi^{(s)}(x)$; and the continued fraction $K^{(s)}(x)$ are

associated with the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ if they are related to these sequences by (4), (5), and (6) above.

(9) C will represent the field of complex numbers.

In terms of the techniques which are used, this study is a continuation of the work of Dickinson, Pollak, and Wannier, [11], and that of Goldberg, [13]. It differs from these in considering the nonsymmetric case (i.e., we do not require that the a_i's be zero).

We now state some previous results for reference and comparison. The first of these is by Goldberg and many of our results will be generalizations of it.

THEOREM 2.1. *(Goldberg, [13]). Let $\{b_n\}_1^\infty$ be an arbitrary sequence of positive real numbers with $\lim_{n\to\infty} b_n = 0$. Suppose $\{\phi_n^{(s)}\}_0^\infty$ are the sets of orthogonal polynomials corresponding to $\{a_n^{(s)} = 0\}_0^\infty$ and $\{b_n^{(s)}\}_1^\infty$, then for each $s \geqq 0$*

(i) *$(1/x)K^{(s)}(1/x)$ is a meromorphic function with the series representation*

$$(1/x)K^{(s)}(1/x) = -A^{(s)} + \sum_{n=1}^{\infty} \frac{2A_n^{(s)}}{x^2 - (\alpha_n^{(s)})^2}$$

where $\sum_1^\infty A_n^{(s)}[\alpha_n^{(s)}]^{-2}$.

(ii) *$\mathscr{S}(\psi^{(s)}(x))$ is the closure of the set of poles of $xK^{(s)}(x)$; namely, $x = 0$ and $x = \pm 1/\alpha_n^{(s)}$, $n = 1, 2, \cdots$.*

(iii) *$\psi^{(s)}(x + 0) - \psi^{(s)}(x - 0) = -A_n^{(s)}(\alpha_n^{(s)})^{-2}$, $x = \pm 1/\alpha_n^{(s)}$.*

(iv) *$\psi^{(s)}(0+) - \psi^{(s)}(0-) = -A^{(s)}$.*

(v) *$\{x^n\phi_n^{(s)}(1/x)\}_0^\infty$ converges to an entire function only if $\sum_1^\infty b_i < \infty$.*

Next, in [2], Krein used the theory of completely continuous operators in a Hilbert Space to prove

THEOREM 2.2. *(Krein) Let $\{a_n\}_0^\infty$ be a real sequence and $\{b_n\}_1^\infty$ be a real positive sequence. Then a necessary and sufficient condition that $\mathscr{S}(\psi^{(0)}(x))$ be a bounded denumerable set with its limit points contained in the set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is both $\{a_i\}_0^\infty$ and $\{b_i\}_1^\infty$ be bounded and that $\lim_{i,j\to\infty} g_{ij} = 0$, where g_{ij} is the entry in the ith row and the jth column of the infinite matrix*

$$(A - \alpha_1 I)X(A - \alpha_2 I)X \cdots X(A - \alpha_n I)$$

where

$$A = \left\| \begin{matrix} a_0 & b_1 & 0 & \cdots \\ b_1 & a_1 & b_2 & \cdots \\ 0 & b_2 & a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{matrix} \right\| .$$

We shall also need some well-known results about continued fractions and orthogonal polynomials. These can all be found in Chapter III of Szego's book, [16]. We collect them into the following lemma.

LEMMA 2.3. *The convergents of the continued fraction* $K^{(s)}(x)$ *are the rational functions* $\phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x)$, *and the zeros of the monic polynomials* $\phi_n^{(s)}(x)$ *are real, simple, and interlaced with the zeros of* $\phi_{n-1}^{(1+s)}(x)$.

We now enumerate those conditions which we will impose upon the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$. They are as follows:

(1) the point set $\{a_0, a_1, a_2, \cdots\}$ is bounded and has derived set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ $1 \leqq n < \infty$,

(2) $b_n \to 0$ as $n \to \infty$.

Under these conditions we shall show in § 3 that $K^{(s)}(x)$ is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$. In § 4 we let $n = 1$ and prove a generalization of Theorem 2.1 above. Finally in § 5 we consider $2 \leqq n < \infty$ and again obtain results similar to those of Theorem 2.1

3. **$K^{(s)}(x)$ is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$.** In order to show that $K^{(s)}(x)$ is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$ we need a continued fraction theorem which is due to Worpitsky. We state this for reference.

THEOREM 3.1. (*Worpitsky*, [17], p. 42) *Let* $a_2, a_3, \cdots$ *be complex functions of any variables over a domain* D *in which* $|a_{p+1}| \leqq 1/4$, $p = 1, 2, \cdots$. *Then the following statements hold*:

(i) *The continued fraction* $w = \frac{1 |}{| 1} + \frac{a_2 |}{| 1} + \frac{a_3 |}{| 1} + \cdots$ *converges uniformly over* D.

(ii) *The values of the continued fraction and of its approximates are in the circular domain* $|w - 4/3| \leqq 2/3$.

We now prove our first result.

THEOREM 3.2. *If the real sequences* $\{a_i\}_0^\infty$ *and* $\{b_i\}_1^\infty$ *satisfy the* *following conditions*:

(1) $\{a_0, a_1, \cdots\}$ *is a bounded point set with derived set* $\{\alpha_1, \cdots, \alpha_n\}$ *where* $\alpha_1 < \alpha_2 < \cdots < \alpha_n, 1 \leqq n < \infty$,

(2) $b_n > 0$ $n = 1, 2, \cdots$ *and* $b_n \to 0$ *as* $n \to \infty$, *then for each* $s \geqq 0$ *the continued fraction*

$$K^{(s)}(x) = \frac{1 \mid}{\mid x - a_0^{(s)}} - \frac{b_1^{(s)} \mid}{\mid x - a_1^{(s)}} - \frac{b_2^{(s)} \mid}{\mid x - a_3^{(s)}} - \cdots$$

converges to a function which is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$. *Moreover this convergence is uniform in compact sets which do not contain poles of* $K^{(s)}(x)$.

Proof. Using an equivalence transformation we can rewrite $K^{(s)}(x)$ as follows:

$$K^{(s)}(x) = \frac{1/(x - a_0^{(s)}) \mid}{\mid 1} - \frac{b_1^{(s)}/[(x - a_0^{(s)})(x - a_1^{(s)})] \mid}{\mid 1}$$
$$- \frac{b_2^{(s)}/[(x - a_1^{(s)})(x - a_2^{(s)})] \mid}{\mid 1} - \cdots .$$

Next let $\varepsilon > 0$ be given. Since the derived set of $\{a_0, a_1, \cdots\}$ consists of the finite points $\alpha_1, \cdots, \alpha_n$, there exists N_ε such that for every $i \geqq N_\varepsilon$, $|a_i - \alpha_j| < \varepsilon/2$ for some $j \in \{1, 2, \cdots, n\}$. Also $b_i \to 0$, so there exists M_ε such that for every $i \geqq M_\varepsilon$, $|b_i/(\varepsilon/2)^2| \leqq 1/4$. Therefore, if we restrict x to the domain $D_\varepsilon = \{x: |x - \alpha_j| > \varepsilon, j = 1, 2, \cdots, n\}$, then for each i such that $i \geqq \max\{M_\varepsilon, N_\varepsilon\}$ we have

$$|b_i/(x - a_i)(x - a_{i-1})| \leqq |b_i/(\varepsilon/2)^2| \leqq 1/4 .$$

Now let $L(\varepsilon) = \max\{M_\varepsilon, N_\varepsilon\}$ and set

$$K_{L(\varepsilon)}^{(s)}(x) = \frac{b_L^{(s)} \mid}{\mid x - a_L^{(s)}} - \frac{b_{L+1}^{(s)} \mid}{\mid x - a_{L+1}^{(s)}} - \cdots$$
$$= \frac{b_L^{(s)}/(x - a_L^{(s)}) \mid}{\mid 1} - \frac{b_{L+1}^{(s)}/[(x - a_{L+1})(x - a_L^{(s)})] \mid}{\mid 1} - \cdots .$$

Then by Theorem 3.1 and our remarks above we see that $K_L^{(s)}(x)$ converges uniformly in D and moreover that the values assumed by $K_L^{(s)}(x)$ and its convergents all lie in the set $|w - 4/3| \leqq 2/3$. Since the convergents of $K_L^{(s)}(x)$ are rational functions this means these rational functions have no poles and hence are analytic in D_ε. Thus by the uniform convergence $K_L^{(s)}(x)$ is also analytic in D_ε. But by definition $K_L^{(s)}(x)$ is just the tail end of $K^{(s)}(x)$ and hence $K^{(s)}(x)$ is meromorphic in D_ε. Moreover this holds for each $\varepsilon > 0$, so $K^{(s)}(x)$ is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$.

We now consider the behavior of $K^{(s)}(x)$ for large values of x. To do this we let $K_*^{(s)}(w) = K^{(s)}(1/w)$ and consider w small. We have the following result.

THEOREM 3.3. *For each* $s \geqq 0$ *the function*

$$K_*^{(s)}(w)/w = (1/w)K^{(s)}(1/w)$$

is analytic about the origin and has value 1 *for* $w = 0$.

Proof. By the definition of $K_*^{(s)}(w)$ we have

$$K_*^{(s)}(w)/(w) = \frac{1/(1 - a_0^{(s)}w) \mid}{\mid \quad 1} - \frac{b_1^{(s)}w^2/[(1 - a_0^{(s)}w)(1 - a_1^{(s)}w)] \mid}{\mid \quad 1} - \frac{b_2^{(s)}w^2/[(1 - a_1^{(s)}w)(1 - a_2^{(s)}w)] \mid}{\mid \quad 1} - \cdots .$$

Now conditions (1) and (2) on the sequences $\{a_i\}$ and $\{b_i\}$ imply that both these sequences are bounded. Thus for $|w|$ small enough we have

$$| b_i^{(s)} \cdot w^2/[(1 - a_{i-1}^{(s)}w)(1 - a_i^{(s)}w)] | \leqq 1/4 \qquad i = 1, 2, 3, \cdots .$$

Hence by Worpitsky's theorem, (Theorem 3.1 above), $K_*^{(s)}(w)/w$ is analytic about $w = 0$. Also, by inspection we see that $K_*^{(s)}(w)/w$ has value 1 at $w = 0$.

We now note two results which will aid in the construction of $\psi^{(s)}(x)$ in § 4 and § 5. The first is an important recursion formula which says that for $s \geqq 0$ and $n \geqq 2$ we have

$$\text{(3–A)} \qquad \phi_n^{(s)}(x) = (x - a_0^{(s)})\phi_{n-1}^{(1+s)}(x) - b_1^{(s)} \cdot \phi_{n-2}^{(2+s)}(x) .$$

The proof of this formula is a straight-forward induction argument and is omitted here. The other result is a complex orthogonality relationship which is proven as a lemma.

LEMMA 3.4. *For each* $s \geqq 0$ *and* $0 \leqq p \leqq n$ *there exists* $R > 0$ *such that*

$$\text{(3–B)} \qquad 1/(2\pi i)\int_{|x|=R} x^p \phi_n^{(s)}(x) K^{(s)}(x) dx = \delta_{n,p} \cdot k_n^{(s)}$$
$$(k_n^{(s)} \neq 0, s, n = 0, 1, \cdots) .$$

Proof. From Lemma 2.3 we know that

$$\lim_{n \to \infty} \phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x) = K^{(s)}(x) .$$

Hence if we divide (3–A) by $\phi_{n-1}^{(1+s)}(x)$ and let $n \to \infty$, we obtain

$$1/K^{(s)}(x) = (x - a_0^{(s)}) - b_1^{(s)} \cdot K^{(1+s)}(x) \,. \tag{3–C}$$

Next we combine (3–C) and (3–A) by eliminating $(x - a_0^{(s)})$. After simplication this gives

$$\begin{aligned} &\phi_n^{(s)}(x)K^{(s)}(x) - \phi_{n-1}^{(1+s)}(x) \\ &\quad = b_1^{(s)} \cdot K^{(s)}(x)\{\phi_{n-1}^{(1+s)}(x)K^{(1+s)}(x) - \phi_{n-2}^{(2+s)}(x)\} \,. \end{aligned} \tag{3–D}$$

We now note that the term in parenthesis on the right of (3–D) is just the left side with a change of index. Therefore we can interate this formula, and recalling $\phi_0^{(j)} = 1, \phi_{-1}^{(j)} = 0, j = 1, 2, \cdots$, we obtain

$$\phi_n^{(s)}(x)K^{(s)}(x) - \phi_{n-1}^{(1+s)}(x) = \left\{\prod_{i=1}^{n} b_i^{(s)}\right\}\left\{\prod_{i=0}^{n} K^{(i+s)}(x)\right\} \,. \tag{3–E}$$

Now we multiply (3–E) by $x^p/(2\pi i)$ and integrate about a circle $|x| = R$ in the complex plane. Since $x^p\phi_{n-1}^{(1+s)}(x)$ is analytic, this gives us

$$\frac{1}{2\pi i}\int_{|x|=R} x^p\phi_n^{(s)}(x)K^{(s)}(x)dx = \frac{k_n^{(s)}}{2\pi i}\int_{|x|=R} x^pK^{(s)}(x) \cdots K^{(n+s)}(x)dx \tag{3–F}$$

where $k_n^{(s)} = \prod_{i=1}^{n} b_i^{(s)} \neq 0$ because each b_i is positive. In the integral on the right of (3–F) we now let $x = 1/w$. We also choose R large enough so that each of the functions $K_*^{(i+s)}(w)/w$, $i = 0, 1, \cdots, n$ is analytic in the disc $|w| = 1/R$. This is possible by Theorem 3.3. Then by using the residue theorem and the fact that $K_*^{(i+s)}(w)/w$ has value 1 at $w = 0$, $i = 0, \cdots, n$, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{|x|=R} x^p\phi_n^{(s)}(x)K^{(s)}(x)dx \\ &\quad = \frac{k_n^{(s)}}{2\pi i}\int_{|w|=1/R} w^{n-p-1}\left\{\frac{K_*^{(s)}(w)}{w}\right\} \cdots \left\{\frac{K_*^{(s+n)}(w)}{w}\right\}dw \\ &\quad = k_n^{(s)} \cdot \delta_{n,p} \end{aligned}$$

Sections 4 and 5 will change the above complex orthogonality into a real orthogonality relationship and hence obtain $\psi^{(s)}(x)$.

4. Constructing $\psi^{(s)}(x)$ for one limit point. We now assume that in Theorem 3.2 $n = 1$, i.e. $\alpha_i \to \alpha_1$ as $i \to \infty$. As a further simplification we initially choose $\alpha_1 = 0$ and then later show this restriction is not needed. Now in order to convert (3–B) into a real integral, we first obtain a Mittag-Leffler type expansion for the function $K^{(s)}(x)$. For this we need a theorem of Montel.

THEOREM 4.1. (*Montel*, [14], p. 38) *A necessary and sufficient condition that a meromorphic function $f(z)$ be the limit of rational functions whose zeros and poles are real, simple, and interlaced on the real axis is that $f(z)$ have the form*

$$f(z) = B - Az + \sum_{n=1}^{\infty} A_n(1/(z - \alpha_n) + 1/\alpha_n) ,$$

where the numbers B, A, A_n, and α_n are all real, A and A_n are all of the same sign, and the series $\sum_1^\infty A_n(\alpha_n)^{-2}$ is convergent.

We now obtain our expansion for $K^{(s)}(x)$.

THEOREM 4.2. *With the hypothesis of Theorem 3.2 and with $n = 1, \alpha_1 = 0$, the function $K^{(s)}(x)$ has the form*

$$K^{(s)}(x) = A^{(s)}/x + \sum_{n=1}^{\infty} A_n^{(s)}/(x - \alpha_n^{(s)})$$

where $A^{(s)}$ and $A_n^{(s)}$, $a = 1, 2, \cdots$, are all nonnegative and $\sum_{n=1}^\infty A_n^{(s)} < \infty$.

Proof. From Theorem 3.2 we know that $K^{(s)}(x)$ is meromorphic in the set $C - \{0\}$. Thus $K_*^{(s)}(w) = K^{(s)}(1/w)$ is also meromorphic in $C - \{0\}$. However, by Theorem 3.3, $K_*^{(s)}(w)$ is analytic at $w = 0$, and hence $K_*^{(s)}(w)$ is meromorphic in C. Next from Lemma 2.3

$$K^{(s)}(x) = \lim_{n\to\infty} \frac{\phi_{n-1}^{(1+s)}(x)}{\phi_n^{(s)}(x)} ,$$

so that

$$K_*^{(s)}(w) = \lim_{n\to\infty} \frac{w^n\phi_{n-1}^{(1+s)}(1/w)}{w^n\phi_n^{(s)}(1/w)} .$$

Also from Lemma 2.3 the rational function $\phi_{n-1}^{(1+s)}(z)/\phi_n^{(s)}(z)$ has its zeros and poles interlaced on the real axis. Thus $w^n\phi_{n-1}^{(1+s)}(1/w)/w^n\phi_n^{(s)}(1/w)$ also has its zeros and poles interlaced on the real axis. Now since $K^{(s)}(x)$ is known to converge uniformly on those compact sets of $C - \{0\}$ that exclude the poles of $K^{(s)}(x)$, it follows that $K_*^{(s)}(w)$ converges uniformly on those compact sets of C which exclude poles of $K_*^{(s)}(w)$. Therefore we can apply Theorem 4.1 to $K_*^{(s)}(w)$. This gives

$$K_*^{(s)}(w) = C^{(s)} - B_0^{(s)}w + \sum_1^\infty B_n^{(s)} (1/(w - \beta_n^{(s)}) + 1/\beta_n^{(s)})$$

where $B_0^{(s)}, B_1^{(s)}, \cdots$ are all of the same sign and $\sum_1^\infty B_n^{(s)}[\beta_n^{(s)}]^{-2}$ converges. Next, $K_*^{(s)}(0) = 0$; so $C^{(s)} = 0$. Thus converting to $K^{(s)}(x)$, we obtain

$$K^{(s)}(x) = \frac{-B_0^{(s)}}{x} - \sum_{n=1}^{\infty} \frac{B_n^{(s)}}{[\beta_n^{(s)}]^2} \frac{1}{(x - 1/\beta_n^{(s)})} .$$

Now, letting $A^{(s)} = -B_0^{(s)}$, $A_n^{(s)} = -B_n^{(s)}/[\beta_n^{(s)}]^2$ and $\alpha_n^{(s)} = 1/\beta_n^{(s)}$, gives us

$$K^{(s)}(x) = \frac{A^{(s)}}{x} + \sum_{n=1}^{\infty} \frac{A_n^{(s)}}{x - \beta_n^{(s)}}$$

where $A^{(s)}$, $A_1^{(s)}$, $A_2^{(s)}$, $\cdots$ are all of the same sign and $\sum_1^\infty A_n^{(s)}$ converges. Finally the fact that the $A_i^{(s)}$'s are actually non-negative follows from Hurwitz's theorem and the special interlacing of the zeros of $\phi_{n-1}^{(1+s)}(x)$ with those of $\phi_n^{(s)}(x)$. This interlacing means that $\phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x)$ has only positive residues, and by Hurwitz's theorem the limit function $K^{(s)}(x)$ must have nonnegative residues.

Next, before proving our main theorem of this section, we state a result of Carleman which will give us uniqueness in our result.

THEOREM 4.3. (*Carleman*, [15], p. 59) *If* $\sum_{n=1}^\infty 1/\sqrt{b_n} = \infty$, *then* $\psi^{(s)}(x)$ *is unique when it is normalized by the conditions* $\psi^{(s)}(-\infty) = 0$ *and for* $x \neq -\infty$, $\psi^{(s)}(x-0) = \psi^{(s)}(x)$.

We now obtain $\psi^{(s)}(x)$.

THEOREM 4.4. *Let the real sequences* $\{a_i\}_0^\infty$ *and* $\{b_i\}_1^\infty$ *satisfy the following restrictions*:

(1) $a_i \to 0$ *as* $i \to \infty$.

(2) $b_i > 0$ *for each* i *and* $b_i \to 0$ *as* $i \to \infty$. *Then if* $\psi^{(s)}$ *and* $K^{(s)}(x)$ *are defined by* (5) *and* (6) *of* § 2, *we have*

(i)

$$K^{(s)}(x) = \frac{A^{(s)}}{x} + \sum_{n=1}^{\infty} A_n^{(s)}/(x - \alpha_n^{(s)})$$

where the $A_n^{(s)}$ *and* $A^{(s)}$ *are all nonnegative and* $\sum_1^\infty A_n^{(s)} < \infty$.

(ii) $\psi^{(s)}(x)$ *is a unique jump function and* $\mathscr{S}(\psi^{(s)}(x))$ *is contained in the closure of the set of poles of* $K^{(s)}(x)$.

(iii) $\psi^{(s)}(x+0) - \psi^{(s)}(x-0) = A_n^{(s)}$ *for* $x = \alpha_n^{(s)}$.

(iv) $\psi^{(s)}(0+) - \psi^{(s)}(0-) = A^{(s)}$.

Proof. (i) is just Theorem 4.2. To prove the remainder of the theorem, we first note that condition (2) together with Theorem 4.3 guarantees that $\psi^{(s)}(x)$ is essentially unique. Therefore we need only show that a jump function with jump $A_n^{(s)}$ at $\alpha_n^{(s)}$, $n = 1, 2, \cdots$ and jump $A^{(s)}$ at $x = 0$ is actually a distribution function for the polynomial set $\{\phi_n^{(s)}(x)\}_0^\infty$. Thus let $\psi^{(s)}(x)$ be such a jump function and consider $\int_{-\infty}^{+\infty} x^p \phi_n^{(s)}(x) d\psi^{(s)}(x)$. By the definition of the Riemann-Stieltjes integral we have

$$\int_{-\infty}^{+\infty} x^p \phi_n^{(s)}(x) d\psi^{(s)}(x)$$
$$= \delta_{po} \cdot \phi_n^{(s)}(0) + \sum_{j=1}^{\infty} A_j^{(s)} [\alpha_j^{(s)}]^p \phi_n^{(s)}(\alpha_j^{(s)}) ,$$

provided the series on the right converges. To show this convergence and to evaluate the sum, we combine Lemma 3.4 and Theorem 4.2. This gives us

$$\frac{1}{2\pi i} \int_{|x|=R} x^p \phi_n^{(s)}(x) \left\{ \frac{A^{(s)}}{x} + \sum_{j=1}^{\infty} \frac{A_j^{(s)}}{x - \alpha_j^{(s)}} \right\} dx = \delta_{n,p} \cdot k_n^{(s)} .$$

Now R was chosen so that $|w| \leqq 1/R$ contained no singularities of $K_*^{(s)}(w)$. Thus $|x| < R$ contains all the singularities of $K^{(s)}(x)$. Also the sum converges uniformly and absolutely on $|x| = R$, so we can integrate the series term-by-term. We use the residue theorem and obtain

$$\delta_{po} \cdot \phi_n^{(s)}(0) + \sum_{j=1}^{\infty} A_j^{(s)} [\alpha_j^{(s)}]^p \phi_n^{(s)}(\alpha_j^{(s)}) = \delta_{p,n} \cdot k_n^{(s)} .$$

Since this sum is the same as our expression for $\int_{-\infty}^{+\infty} x^p \phi_n^{(s)}(x) d\psi^{(s)}(x)$ we see that

$$\int_{-\infty}^{+\infty} x^p \phi_n^{(s)}(x) d\psi^{(s)}(x) = \delta_{n,p} \cdot k_n^{(s)} ,$$
$$k_n^{(s)} \neq 0 , \quad n = 0, 1, 2, \cdots , \quad s \geqq 0 .$$

Thus with respect to $\psi^{(s)}(x)$, $\phi_n^{(s)}(x)$ is orthogonal to x^p, $p = 0, 1, \cdots, n - 1$. Hence by definition, $\psi^{(s)}(x)$ is a distribution function for the polynomial set $\{\phi_n^{(s)}(x)\}_0^\infty$.

It is of interest to note that the converse of Theorem 4.4 also holds. This follows immediately from Krein's result which we labeled Theorem 2.2. Thus suppose that the bounded increasing function $\psi(x)$ has a bounded spectrum with its only limit point at zero. Then by Theorem 2.2, $\lim_{(i,j)\to(\infty,\infty)} g_{ij} = 0$ where g_{ij} is the entry in the ith row and jth column of the infinite matrix

$$\left\| \begin{matrix} a_0 & b_1 & 0 & \cdots \\ b_1 & a_1 & b_2 & \cdots \\ 0 & b_2 & a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{matrix} \right\| .$$

Thus we see $b_i \to 0$ and $a_i \to 0$. Combining these comments with Theorem 4.4 gives

THEOREM 4.5. *A necessary and sufficient condition that $\mathscr{S}(\psi^{(s)}(x))$ be a bounded set with a single limit point at zero is that the sequence of rational functions $\phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x)$ converges to a function $K^{(s)}(x)$ which has an expansion of the form*

$$K^{(s)}(x) = A^{(s)}/x + \sum_1 A_n^{(s)}/(x - \alpha_n^{(s)})$$

where $A_n^{(s)} > 0$ for each n and $\sum_1^\infty A_n^{(s)} < \infty$. Moreover $\psi^{(s)}(x)$ is a jump function with jumps $A_n^{(s)}$ at $\alpha_n^{(s)}$ and $A^{(s)}$ at $x = 0$.

We now show that in Theorem 4.5 the limit point of the spectrum does not have to be a $x = 0$.

COROLLARY 4.6. *A necessary and sufficient condition that $\mathscr{S}(\psi^{(s)}(x))$ be a bounded set with a single limit point at $x = a \neq \infty$ is that the sequence of rational functions $\phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x)$ converges to a function $K^{(s)}(x)$ which has an expansion of the form*

$$K^{(s)}(x) = \frac{A^{(s)}}{x - a} + \sum_{n=1}^{\infty} \frac{A_n^{(s)}}{x - (a + \alpha_n^{(s)})} ,$$

where $A^{(s)} \geqq 0$ and $A_n^{(s)} > 0$ for each n with $\sum_1^\infty A_n^{(s)} < \infty$. Then $\psi^{(s)}(x)$ is a jump function with jumps of $A_n^{(s)}$ at $\alpha_n^{(s)} + a$ and $A^{(s)}$ at $x = a$.

Proof. By Theorem 2.2 $\mathscr{S}(\psi^{(s)}(x))$ is a bounded set with a single limit point at $x = a$ if and only if $a_i \to a$ and $b_i \to 0$. Now

$$\phi_{n+1}^{(s)}(x) = (x - a_n)\phi_n^{(s)}(x) - b_n\phi_{n-1}^{(s)}(x) ,$$

so that

$$\phi_{n+1}^{(s)}(x + a) = (x - (a_n - a))\phi_n^{(s)}(x + a) - b_n\phi_{n-1}^{(s)}(x + a) .$$

Thus the sequence of polynomials $\{\phi_n^{(s)}(x + a)\}_0^\infty$ is associated with the real sequences $\{a_n - a\}_{n=0}^{\infty}$ and $\{b_n\}_1^{\infty}$. Hence if $a_n \to a$ and $b_n \to 0$, then by Theorem 2.9 the polynomials $\phi_n^{(s)}(x + a)$ are orthogonal with respect to a function $\psi_1^{(s)}(x)$ which has a bounded spectrum with its only limit point at zero. Now

$$\int_{-\infty}^{+\infty} \phi_n^{(s)}(x)\phi_m^{(s)}(x)d\psi_1^{(s)}(x - a)$$
$$= \int_{-\infty}^{+\infty} \phi_n^{(s)}(x + a)\phi_m^{(s)}(x + a)d\psi_1^{(s)}(x) ,$$

so the polynomials $\{\phi_n^{(s)}(x)\}_0^{\infty}$ are orthogonal with respect to $\psi_1^{(s)}(x - a)$. But $b_i \to 0$ implies that $\psi^{(s)}(x)$ is unique, so $\psi^{(s)}(x) = \psi_1^{(s)}(x - a)$. Also by Theorem 4.4 $\phi_{n-1}^{(1+s)}(x + a)/\phi_n^{(s)}(x + a)$ converges to a function $K_1^{(s)}(x)$ which has an expansion of the form

$$K_1^{(s)}(x) = \frac{A^{(s)}}{x} + \sum_1^{\infty} \frac{A_n^{(s)}}{x - \alpha_n^{(s)}} .$$

Consequently, $\phi_{n-1}^{(1+s)}(x)/\phi_n^{(s)}(x)$ converges to a function $K^{(s)}(x)$ which has the form

$$K^{(s)}(x) = K_1^{(s)}(x - a) = \frac{A^{(s)}}{x - a} + \sum_1^{\infty} \frac{A_n^{(s)}}{x - (\alpha_n^{(s)} + a)} .$$

Moreover, $\psi_1^{(s)}(x)$ has a jump of $A_n^{(s)}$ at $x = \alpha_n^{(s)}$, so $\psi^{(s)}(x) = \psi_1^{(s)}(x - a)$ has the same jump at $x = \alpha_n^{(s)} + a$. This proves the corollary.

We now turn to an interesting convergence question. We have shown that if $a_i \to 0$ and $b_i \to 0$ then

$$K_*^{(s)}(w) = \lim_{n\to\infty} w \cdot \frac{w^{n-1}\phi_{n-1}^{(1+s)}(1/w)}{w^n\phi_n^{(s)}(1/w)}$$

is a meromorphic function. Hence we ask, when does the sequence $\{F_n^{(s)}(w) = w^n\phi_n^{(s)}(1/w)\}$ of polynomials converge to an entire function so that the numerators and denominators of the rational functions which converge to $K_*^{(s)}(w)$ will themselves converge to entire functions. The techniques of this section were used by Dickinson, Pollak, and Wannier, [11], when they considered the special case $a_i = 0$, $i = 0, 1, \cdots$ and $\sum_1^{\infty} b_i < \infty$.

THEOREM 4.7. *If* $\sum_0^{\infty} |a_i| < \infty$ *and if* $\sum_1^{\infty} b_i < \infty$ *then the sequence of polynomials* $\{F_n^{(s)}(w)\}_0^{\infty}$ *converges to an entire function* $F^{(s)}(w)$ *as* $n \to \infty$.

Proof. Since $F_n^{(s)}(w) = w^n\phi_n^{(s)}(1/w)$ and

$$\phi_{n+1}^{(s)}(x) = (x - a_n^{(s)})\phi_n^{(s)}(x) - b_n^{(s)}\phi_{n-1}^{(s)}(x) ,$$

we have

$$F_{n+1}^{(s)}(w) = (1 - a_n^{(s)}w)F_n^{(s)}(w) - b_n^{(s)}w^2F_{n-1}^{(s)}(w) .$$

Next we let

$$G_n^{(s)}(w) = \max \{| F_n^{(s)}(w) |, | F_{n-1}^{(s)}(w) |\} .$$

Then

$$| F_{n+1}^{(s)}(w) | \leqq \{1 + | a_n^{(s)} | | w |\} | F_n^{(s)}(w) | + b_n^{(s)} | w^2 | | F_{n-1}^{(s)}(w) | ,$$

so

$$| F_{n+1}^{(s)}(w) | \leqq \{1 + | a_n^{(s)} | | w | + b_n^{(s)} | w^2 |\}G_n^{(s)}(w) .$$

Since we also have

$$G_{n+1}^{(s)}(w) \leqq \max \{| F_{n+1}^{(s)}(w) |, G_n^{(s)}(w)\} ,$$

this gives

$$G_{n+1}^{(s)}(w) \leqq (1 + | a_n^{(s)} | | w | + b_n^{(s)} | w^2 |)G_n^{(s)}(w) .$$

Now by definition $F_0^{(s)}(w) = 1$, so $G_0^{(s)}(w) = 1$. Then by iterating the above inequality, we obtain

$$G_{n+1}^{(s)}(w) \leqq \prod_{i=1}^{n} \{1 + | a_i^{(s)} | | w | + b_i^{(s)} | w^2 |\}$$

and thus

$$| F_{n+1}^{(s)}(w) | \leqq \prod_{i=1}^{n} \{1 + | a_i^{(s)} | | w | + b_i^{(s)} | w^2 |\} .$$

We next note that the infinite product

$$E^{(s)}(w) = \prod_{1}^{\infty} \{1 + | a_i^{(s)} | | w | + b_i^{(s)} | w^2 |\}$$

converges uniformly for values of w in a bounded domain because of the hypothesis $\sum_1^\infty b_i < \infty$ and $\sum_0^\infty | a_i | < \infty$. Using this we can show that the sequence $\{F_n^{(s)}(w)\}_0^\infty$ is a Cauchy sequence in any bounded domain. We proceed as follows:

$$| F_{n+1}^{(s)}(w) - F_n^{(s)}(w) | = | -a_n^{(s)}wF_n^{(s)}(w) - b_n^{(s)}w^2F_{n-1}^{(s)}(w) | ,$$

so

$$| F_{n+1}^{(s)}(w) - F_n^{(s)}(w) | \leqq | w | E^{(s)}(w)(| a_n^{(s)} | + b_n^{(s)} | w |) ,$$

and thus for each $N \geqq 1$

$$| F_{n+N}^{(s)}(w) - F_n^{(s)}(w) | \leqq \sum_{i=0}^{N-1} | F_{n+N-i}^{(s)}(w) - F_{n+N-1-i}^{(s)}(w) |$$
$$\leqq | w | E^{(s)}(w) \left\{ \sum_{i=n}^{n+N-1} (| a_i^{(s)} | + b_i^{(s)} | w |) \right\} .$$

Now let $\varepsilon > 0$ be given and suppose w is in the domain $D = \{w: | w | \leqq M\}$. Since $\sum_1^\infty (| a_i | + b_i)$ converges, we can choose n so large that for each $N \geqq 1$ we have

$$\left\{ \sum_n^{n+N} (| a_i | + b_i M) \right\} \cdot M \cdot \max_{w \in D} E^{(s)}(w) < \varepsilon .$$

Thus we see that in the uniform norm $\{F_n^{(s)}(w)\}$ is a Cauchy sequence in the domain D, and hence converges uniformly to a function analytic in D. Since this holds for all bounded domains, the resulting function is entire and the theorem is proven.

Next we give some necessary conditions for the convergence of $F_n^{(s)}(w)$ to an entire function $F^{(s)}(w)$. We first note a fact about the structure of $F_n^{(s)}(w)$. Namely; for $s \geqq 0$ and $n \geqq 2$

(4–A)
$$F_n^{(s)}(w) = 1 - \left\{ \sum_0^{n-1} a_i^{(s)} \right\} \cdot w + \left\{ \sum_{0 \leqq i < j \leqq n-1} a_i^{(s)} \cdot a_j^{(s)} - \sum_1^{n-1} b_i^{(s)} \right\} w^2 + 0(w^3) .$$

The proof of (4–A) is a simple induction argument and is omitted here.

THEOREM 4.8. *The following conditions are each necessary for the convergence of the sequence $\{F_n^{(s)}(w)\}$ to an entire function.*

(i) $\sum_0^\infty a_i$ *converges.*

(ii) $\sum_0^\infty a_i^2 < \infty$.

(iii) $\sum_1^\infty b_i < \infty$.

Proof. Suppose the polynomials $F_n^{(s)}(w)$ converge to the entire function $F^{(s)}(w) = e_1^{(s)} + e_2^{(s)} w + e_3^{(s)} w^2 + \cdots$. Then by (4–A) we have $e_1^{(s)} = 1, e_2^{(s)} = \sum_{i=0}^\infty a_i^{(s)}$ and

$$e_3^{(s)} = \lim_{n \to \infty} \left\{ \sum_{0 \leq i < j \leqq n} a_i^{(s)} a_j^{(s)} - \sum_{i=1}^n b_i^{(s)} \right\} .$$

Now for each n

$$\left(\sum_0^n a_i^{(s)} \right)^2 = \sum_0^n a_i^{(s)2} + 2 \sum_{0 \leqq i < j \leqq n} a_i^{(s)} a_j^{(s)} ;$$

therefore,

$$\sum_{0 \leqq i < j \leqq n} a_i^{(s)} a_j^{(s)} = \frac{\left(\sum_0^n a_i^{(s)}\right)^2 - \sum_0^n (a_i^{(s)})^2}{2},$$

and thus

$$e_3 = \lim_{n\to\infty} \left\{ \frac{\left(\sum_0^n a_i^{(s)}\right)^2 - \sum_0^n (a_i^{(s)})^2}{2} - \sum_{i=1}^n b_i^{(s)} \right\}.$$

Next since $\sum_0^\infty a_i^{(s)} = e_2$, the above can be written

$$\begin{aligned} 2e_3 &= \lim_{n\to\infty} \left\{ e_2^2 - \sum_0^n [(a_i^{(s)})^2 + 2b_i^{(s)}] \right\} \\ &= e_2^2 - \lim_{n\to\infty} \sum_{i=0}^n [(a_i^{(s)})^2 + 2b_i^{(s)}]. \end{aligned}$$

Hence we see that $\sum_{i=0}^\infty ((a_i^{(s)})^2 + 2b_i^{(s)}) = e_2^2 - 2e_3$, and since each b_i is positive, we can conclude $\sum_{i=0}^\infty b_i < \infty$ and $\sum_{i=0}^\infty a_i^2 < \infty$.

In the special case of positive a_i's we can combine Theorem 4.7 and 4.8 to yield necessary and sufficient conditions for convergence of $F_n^{(s)}$ to an entire function.

COROLLARY 4.9. *If a_i and b_i are positive for each i then the following are necessary and sufficient conditions for the polynomials $F_n^{(s)}(w)$ to converge to an entire function.*

(i) $\sum_0^\infty a_i < \infty$.

(ii) $\sum_1^\infty b_i < \infty$.

Proof. Since each a_i is positive, conditions (i) and (ii) which are necessary by Theorem 4.8 are also sufficient by Theorem 4.7.

5. **Constructing $\psi^{(s)}(x)$ for $n \geqq 2$.** In this section we allow the point set $\{a_0, a_1, \cdots\}$ to have any finite number of limit points. We are again able to construct $\psi^{(s)}(x)$ by using Theorem 3.2 and Lemma 3.4. First, however, we must obtain a Mittag-Leffler type expansion for the function $K^{(s)}(x)$ of Theorem 3.2 in the case where $2 \leqq n < \infty$. The following preliminary work will aid in obtaining this expansion.

We first recall that from Theorems 3.2 and 3.3 we know that $K^{(s)}(x)$ is meromorphic in $C - \{\alpha_1, \cdots, \alpha_n\}$ and is analytic outside some circle $|x| = R$. Also, $K^{(s)}(x)$ is the limit of rational functions whose poles and zeros are real, simple, and interlaced; so the same properties hold for it. Hence we can find real numbers β_0 and β_n such that $\beta_0 < \beta_n$ and $K^{(s)}(x)$ is analytic in $C - [\beta_0, \beta_n]$. We also choose β_i, $i = 1, 2, \cdots, n-1$ in such a way that $\alpha_i < \beta_i < \alpha_{i+1}$ and $K^{(s)}(x)$

is analytic at β_i. This is possible because $K^{(s)}(x)$ has only a countable number of singularities in $C - \{\alpha_1, \cdots, \alpha_n\}$. Next we label the poles and residues of $K^{(s)}(x)$ as follows:

(a) $\{\alpha_{i,k}^{(s)}\}_{k=1}^{\infty}$ is the set of poles of the function $K^{(s)}(x)$ in (β_{i-1}, β_i) ordered so that $|\alpha_{i,k}^{(s)} - \alpha_i| \geqq |\alpha_{i\,k+1}^{(s)} - \alpha_i|$ $i = 1, 2, \cdots, n$; $k = 1, 2, \cdots$.

(b) $A_{i,k}^{(s)}$ is the residue of $K^{(s)}$ at the pole $\alpha_{i,k}^{(s)}$.

We label the poles and residues of the rational functions $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ in a similar manner:

(c) $\{\alpha_{i,k,m}^{(s)}\}_{k=1}^{m(i)}$ is the set of poles of $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ in (β_{i-1}, β_i) ordered so that $|\alpha_{i,k,m}^{(s)} - \alpha_i| \geqq |\alpha_{i,k+1,m}^{(s)} - \alpha_i|$ $i = 1, 2, \cdots, n$; $k = 1, 2, \cdots, m(i)$; $m = 1, 2, \cdots$; $m(1) + m(2) + \cdots + m(n) = m$,

(d) $A_{i,k,m}^{(s)}$ is the residue of $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ at $\alpha_{i,k,m}^{(s)}$.

Now, from the interlacing of the zeros and poles of $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ we know that the residues $A_{i,k,m}^{(s)}$ are positive. Also by Hurwitz's theorem as $m \to \infty$ $A_{i,k,m}^{(s)} \to A_{i,k}^{(s)}$ and $\alpha_{i,k,m}^{(s)} \to \alpha_{i,k}^{(s)}$. This leads us to represent the function $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ as a sum of functions. Namely, let $f^{(s)}(i, m; x) = \sum_{k=1}^{m(i)} A_{i,k,m}^{(s)}/(x - \alpha_{i,k,m}^{(s)})$. Then

$$\text{(5-A)} \qquad \begin{aligned} \frac{\phi_{m-1}^{(s+1)}(x)}{\phi_m^{(s)}(x)} &= \sum_{i=1}^{n} \sum_{k=1}^{m(i)} \frac{A_{i\,k\,m}^{(s)}}{x - \alpha_{i,k,m}^{(s)}} \\ &= \sum_{i=1}^{n} f^{(s)}(i, m; x) , \end{aligned}$$

where $f^{(s)}(i, m; x)$ is a rational function whose poles are real, simple, and have positive residue. But any rational function whose poles are real and simple with positive residue also has real simple zeros which interlace with the poles. Thus $\phi_{m-1}^{(s+1)}(x)/\phi_m^{(s)}(x)$ is the sum of n rational functions $f(i, m; x)$ each having real, simple poles and zeros which are interlaced on (β_{i-1}, β_i) $i = 1, 2, \cdots, n$. We next need a lemma concerning the residues $A_{i\,k,m}^{(s)}$ and $A_{i\,k}^{(s)}$.

LEMMA 5.1. *For each* $s \geqq 0$ *we have*

(i) $\lim_{m\to} \sum_{i=1}^{n} \sum_{k=1}^{m(i)} A_{i,k,m}^{(s)} = 1$,

(ii) $\sum_{i=1}^{n} \sum_{k=1}^{\infty} A_{i\,k}^{(s)} \leqq 1$.

Proof. Since

$$\lim_{m\to} \frac{1}{w} \frac{\phi_{m-1}^{(s+1)}(1/w)}{\phi_m^{(s)}(1/w)} = (1/w)K_*^{(s)}(w)$$

and since this convergence is uniform about $w = 0$, we see from Theorem 3.3 that

$$\text{(5-B)} \qquad \lim_{m\to} \left\{\frac{1}{w} \frac{\phi_{m-1}^{(s+1)}(1/w)}{\phi_m^{(s)}(1/w)}\right\}_{w=0} = 1 .$$

But by (5-A)

$$\left\{\frac{1}{w}\,\frac{\phi_{m-1}^{(s+1)}(1/w)}{\phi_m^{(s)}(1/w)}\right\} = \sum_{i=1}^{n}\sum_{k=1}^{m(i)} \frac{A_{i,k,m}^{(s)}}{w\{(1/w) - \alpha_{i,k,m}^{(s)}\}} = \sum_{i=1}^{n}\sum_{k=1}^{m(i)} \frac{A_{i\,k.m}^{(s)}}{1 - w\alpha_{i,k,m}^{(s)}}\,.$$

So by setting $w = 0$ we obtain

$$\left\{\frac{1}{w}\,\frac{\phi_{m-1}^{(s+1)}(1/w)}{\phi_m^{(s)}(1/w)}\right\}_{w=0} = \sum_{i=1}^{n}\sum_{k=1}^{m(i)} A_{i,k,m}^{(s)}\,,$$

and (i) of the lemma follows from (5-B).

Next, as noted above, $A_{i,k,m}^{(s)} \to A_{i,k}^{(s)}$ as $m \to \infty$. Therefore, for each integer K,

$$\sum_{i=1}^{n}\sum_{k=1}^{K} A_{i,k}^{(s)} = \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{K} A_{i,k,m}^{(s)} \leqq \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{m(i)} A_{i,k,m}^{(s)} = 1\,.$$

Hence the monotonically increasing sequence $\{\sum_{i=1}^{n}\sum_{k=1}^{K} A_{i.k}^{(s)}\}_{K=n}^{\infty}$ is uniformly bounded. Thus $\sum_{i=1}^{n}\sum_{k=1}^{\infty} A_{i,k}^{(s)} \leqq 1$ and (ii) holds.

Since much of the remainder of this section deals with subsequences of subsequences, we shall attempt to simplify our notation by some temporary conventions. First we assume that s is a fixed nonnegative integer and then we suppress it from our notation. Next we change our multiple subscripts to arguments. Thus, in particular, we now write $A_{i,k}^{(s)} = A(i, k)$, $A_{i,k,m}^{(s)} = A(i, k, m)$, $\alpha_{i,k}^{(s)} = \alpha(i, k)$ and $\alpha_{i,k,m}^{(s)} = \alpha(i, k, m)$.

We now consider the sequence $\{f(1, m; x)\}$ of rational functions and proceed as follows to obtain a convergent subsequence: For $n \geqq 1$, let $D(1, n)$ be a domain defined by

$$D(1, n) = \{x\colon |x - \alpha_1| \geqq 1/n\}\,.$$

Then from our labeling of the poles of $K(x)$ we know that only finitely many of the points $\alpha(1, k)$, $(k = 1, 2, \cdots)$, lie in $D(1, n)$. Let these points be $\alpha(1, 1), \alpha(1, 2), \cdots, \alpha(1, N_n)$. Now as $m \to \infty$, $\alpha(1, k, m) \to \alpha(1, k)$; so there exists an integer M_n such that $m \geqq M_n$ implies

$$|\alpha(1, k, m) - \alpha(1, k)| \leqq 1/n\,, \qquad k = 1, 2, \cdots, N_n\,.$$

Therefore, if we restrict x to the domain

$$D^*(1, n) = \{x\colon |x - \alpha_1| \geqq 1/n,\ |x - \alpha(1, k)| \geqq 2/n,\ k = 1, \cdots, N_n\}$$

and choose $m \geqq M_n$, we have

$$\left| \sum_{k=1}^{m_1} \frac{A(1, k, m)}{x - \alpha(1, k, m)} \right| \leqq \sum_{k=1}^{m(1)} \frac{A(1, k, m)}{|x - \alpha(1, k, m)|}$$
$$\leqq n\left(\sum_{k=1}^{m(1)} A(1, k, m)\right)$$
$$\leqq n .$$

This means that the sequence $\{f(1, m; x)\}_{M_n}^{\infty}$ is uniformly bounded in $D^*(1, n)$ and hence contains a subsequence which converges uniformly. Moreover, each function in this sequence is analytic in $D^*(1, n)$, so the limit function is also analytic in $D^*(1, n)$. We now repeat this process for each n, starting with $n = 1$ and at each stage using the convergent subsequence obtained at the previous stage. Then we apply the Cantor diagonal process and obtain a sequence which converges uniformly on every compact subset of $C - \{\alpha_1; \alpha(1, 1), \alpha(1, 2), \cdots\}$. Denote this final subsequence by $\{f(1, m'; x)\}$ and let the limit function be $f_1(x)$. Then

$$\lim_{m' \to \infty} (f(1, m'; x)) = f_1(x) ,$$

and the convergence is uniform on compact subsets of

$$C - \{\alpha_1; \alpha(1, 1), \alpha(1, 2), \cdots\} .$$

Therefore, $f_1(x)$ is analytic in $C - \{\alpha_1; \alpha(1, 1), \alpha(1, 2), \cdots\}$.

We next consider the sequences $\{f(2, m'; x)\}$, and by repeating the above process we obtain a new subsequence which converges uniformly on the compact subsets of $C - \{\alpha_2; \alpha(2, 1), \alpha(2, 2), \cdots\}$. Let $f_2(x)$ be the limit of this sequence. Then $f_2(x)$ is analytic on $C - \{\alpha_2; \alpha(2, 1), \alpha(2, 2), \cdots\}$, and

$$f_2(x) = \lim_{m'' \to \infty} f(2, m''; x)$$

where m'' is a subsequence of m'. Continuing in this manner with $\{f(3, m''; x)\}$, etc., we eventually obtain a subsequence $\{m^*\}$ and functions $f_1(x), \cdots, f_n(x)$ such that

$$\lim_{m^* \to \infty} f(i, m^*; x) = f_i(x) , \qquad i = 1, 2, \cdots, n ,$$

where $f_i(x)$ is analytic in $C - \{\alpha_i; \alpha(i, 1), \alpha(i, 2), \cdots\}$. Next for each m we have

$$f(1, m; x) + \cdots + f(n, m; x) = \sum_{i=1}^{n} \sum_{k=1}^{m(i)} \frac{A(i, k, m)}{x - \alpha(i, k, m)} .$$

and since

$$K(x) = \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{k=1}^{m(i)} \frac{A(i, k, m)}{x - \alpha(i, k, m)}$$
$$= \lim_{m^*\to\infty} \sum_{i=1}^{n} \sum_{k=1}^{m(i)} \frac{A(i, k, m^*)}{x - \alpha(i, k, m^*)} ,$$

we see that

$$K(x) = \lim_{m^*\to\infty} \{f(1, m^*; x) + \cdots + f(n, m^*; x)\}$$
$$= f_1(x) + \cdots + f_n(x) .$$

This, in turn, means that for each $i, (1 \leqq i \leqq n), f_i(x)$ has a simple pole at the points $\alpha(i, 1), \alpha(i, 2), \cdots$ because $K(x)$ has simple poles at these points, while $f_i(x), (i \neq j)$, is analytic here. We summarize the above results as follows:

THEOREM 5.2. *For each* $s \geqq 0$, *if* $\{a_0, a_1, \cdots\}$ *has derived set* $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ *where each* α_i *is finite and if* $\{b_i\}_1^\infty$ *satisfies* $b_i > 0$ *for each* i *and* $b_i \to 0$, *then the continued fraction*

$$K^{(s)}(x) = \frac{1 \mid}{\mid x - a_s} - \frac{b_{s+1} \mid}{\mid x - a_{s+1}} - \frac{b_{s+2} \mid}{\mid x - a_{s+2}} - \cdots$$

can be represented in the form

$$K^{(s)}(x) = f_1^{(s)}(x) + f_2^{(s)}(x) + \cdots + f_n^{(s)}(x)$$

where each $f_i^{(s)}(x)$ *is meromorphic in* $C - \{\alpha_i\}$ *and analytic in* $C - [\beta_{i-1}, \beta_i]$, *for* β_i's *satisfying*

$$\beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{n-1} < \alpha_n < \beta_n .$$

We now turn to the problem of expanding each function $f_i^{(s)}(x)$ in a Mittag-Leffler type series. We have the following result:

THEOREM 5.3. *For each* $i \in \{1, 2, \cdots, n\}$ *we have*

$$f_i(x) = -B_i + \frac{A_i}{x - \alpha_i} + \sum_{k=1}^{\infty} \frac{A(i, k)}{x - \alpha(i, k)}$$

where

(a) $\sum_{k=1}^{\infty} A(i, k) < \infty$

(b) $\sum_{i=1}^{n} B_i = 0.$

Proof. We first recall that each $f(i, m; x)$ is a rational function with its zeros and poles interlaced. We also note that $K(x)$ has residue $A(i, k)$ at the pole $\alpha(i, k), i = 1, \cdots, n; k = 1, 2, \cdots$. Therefore, since $K(x) = f_1(x) + \cdots + f_n(x)$, it follows that $f_i(x)$ has residue $A(i, k)$ at the pole $\alpha(i, k)$ because for $j \neq i$ $f_j(x)$ is analytic at $\alpha(i, k)$, $k = 1, 2, \cdots$. Next $f_i(x)$ is meromorphic in $C - \{\alpha_i\}$, so setting $x = \alpha_i + 1/y$, $f_i(\alpha_i + 1/y)$ is meromorphic in the finite y plane with poles at the points $y = 1/(\alpha(i, k) - \alpha_i)$, $k = 1, 2, \cdots$, and residue $-A(i, k)/(\alpha(i, k) - \alpha_i)^2$ at $1/(\alpha(i, k) - \alpha_i)$. We now apply Montel's theorem (Theorem 4.1 above) to the meromorphic function $f_i(\alpha_i + 1/y)$. This is possible because

$$f_i(\alpha_i + 1/y) = \lim_{m^* \to \infty} f(i, m^*; \alpha_i + 1/y)$$

and $f(i, m, z)$ is a rational function whose zeros and poles interlace. This gives

$$\begin{aligned} f_i(\alpha_i + 1/y) &= -B_i + A_i y \\ &\quad + \sum_{k=1}^{\infty} \frac{-A(i, k)}{(\alpha(i, k) - \alpha_i)^2} \left\{ \frac{1}{y - 1/(\alpha(i, k) - \alpha_i)} + \frac{\alpha(i, k) - \alpha_i}{1} \right\} \end{aligned}$$

where

$$\sum_{k=1}^{\infty} \frac{(\alpha(i, k) - \alpha_i)^2}{1} \frac{A(i, k)}{(\alpha(i, k) - \alpha_i)^2} = \sum_{k=1}^{\infty} A(i, k) < \infty .$$

Changing from the y-plane to the x-plane, $y = 1/(x - \alpha_i)$; so

$$\begin{aligned} f_i(x) &= -B_i + \frac{A_i}{x - \alpha_i} \\ &\quad + \sum_{k=1}^{\infty} \frac{-A(i, k)}{(\alpha(i, k) - \alpha_i)^2} \left\{ \frac{1}{\dfrac{1}{x - \alpha_i} + \dfrac{1}{\alpha(i, k) - \alpha_i}} + \frac{1}{\dfrac{1}{\alpha(i, k) - \alpha_i}} \right\} \\ &= -B_i + \frac{A_i}{x - \alpha_i} + \sum_{k=1}^{\infty} \frac{A(i, k)}{x - \alpha(i, k)} . \end{aligned}$$

This gives the desired representation, and we need only show $\sum_{i=1}^{n} B_i = 0$ to complete the proof. Now

$$K(x) = f_1(x) + f_2(x) + \cdots + f_n(x) ,$$

so

$$K(x) = -\sum_{i=1}^{n} B_i + \sum_{i=1}^{n} \frac{A_i}{x - \alpha_i} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{A(i, k)}{x - \alpha(i, k)} .$$

Hence

$$K_*(w) = -\sum_{i=1}^{n} B_i + \sum_{i=1}^{n} \frac{A_i w}{1 - A_i w} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{A(i, k) w}{1 - \alpha(i, k) w} ,$$

so $K_*(0) = -\sum_{i=1}^{n} B_i$. But by Theorem 3.3, $K_*(0) = 0$, and thus

$\sum_{i=1}^{n} B_i = 0$ and the proof is complete.

We are now ready for the main theorem of this section.

THEOREM 5.4. *Let the real sequences* $\{a_i\}_0^\infty$ *and* $\{b_i\}_1^\infty$ *satisfy the following conditions*:

(1) *The derived set of* $\{a_0, a_1, a_2, \cdots\}$ *is* $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, *where each* α_i *is finite and* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

(b) $b_i > 0$ *for each* i *and* $\lim_{i\to\infty} b_i = 0$.

Then if $\psi^{(s)}(x)$ *and* $K^{(s)}(x)$ *are defined by* (5) *and* (6) *of* § 2, *we have*

(i) $K^{(s)}(x)$ *is meromorphic in* $C - \{\alpha_1, \cdots, \alpha_n\}$ *and has a representation of the form*

$$K^{(s)}(x) = \sum_{i=1}^{n} \frac{A_i^{(s)}}{x - \alpha_i} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{A_{i,k}^{(s)}}{x - \alpha_{i,k}^{(s)}} ,$$

where $A_i^{(s)} \geqq 0$ *and* $A_{i,k}^{(s)} < 0$, $i = 1, \cdots, n$; $k = 1, 2, \cdots$ *and* $\sum_{k=1}^{\infty} A_{i,k}^{(s)} < \infty$, $i = 1, 2, \cdots, n$.

(ii) $\psi^{(s)}(x)$ *is a unique jump function and* $\mathscr{S}(\psi^{(s)})$ *is contained in the closure of the set of poles of* $K^{(s)}(x)$.

(iii) $\psi^{(s)}(\alpha_{i,k}^{(s)} + 0) - \psi^{(s}(\alpha_{i,k}^{(s)} - 0) = A_{i,k}^{(s)}$ $i = 1, \cdots, n; k = 1, 2, \cdots$.

(iv) $\psi^{(s)}(\alpha_i + 0) - \psi^{(s)}(\alpha_i - 0) = A_i^{(s)}$, $i = 1, 2, \cdots, n$.

Proof. (i) is just Theorems 3.2, 5.2, and 5.3. The uniqueness of $\psi^{(s)}(x)$ follows from condition (2) and Carleman's theorem (Theorem 4.3 above). Thus to complete the proof we show that a jump function $\psi^{(s)}(x)$ with jumps defined by conditions (iii) and (iv) is actually a distribution function for the polynomials $\{\phi_n^{(s)}\}_0^\infty$ defined by (4) of § 2. Now by Lemma 3.4 we have for $0 \leqq p \leqq n$ and some $R > 0$

$$\frac{1}{2\pi i} \int_{|x|=R} x^p \phi_n^{(s)}(x) K^{(s)}(x) dx = \left\{\prod_{i=1}^{n} b_{i+s}\right\} \delta_{n,p} .$$

By using the representation for $K^{(s)}(x)$ given by part (i), this can be written

$$\text{(5-C)} \quad \begin{aligned} &\frac{1}{2\pi i} \int_{|x|=R} x^p \phi_n^{(s)}(x) \left\{\sum_{i=1}^{n} \frac{A_i^{(s)}}{x - \alpha_i} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{A_{i,k}^{(s)}}{x - \alpha_{i,k}^{(s)}}\right\} dx \\ &= \left\{\prod_{i=1}^{n} b_{i+s}\right\} \delta_{n,p} . \end{aligned}$$

Now since R was chosen so that $(1/w)K_*^{(s+i)}(w)$ was analytic in $\{w: |w| \leqq 1/R\}$, $i = 1, \cdots, n$, it follows that $K^{(s)}(x)$ is analytic outside and on $|x| = R$; so the series converges uniformly here and we can integrate term-by-term. This gives

$$\frac{1}{2\pi i}\int_{|x|=R} x^p\phi_n^{(s)}(x)\sum_{i=1}^{n}\frac{A_i^{(s)}}{x-\alpha_i}+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{A_{i,k}^{(s)}}{x-\alpha_{i,k}^{(s)}}\{dx\}$$

$$=\sum_{i=1}^{n}(\alpha_i)^p\phi_n^{(s)}(\alpha_i)A_i^{(s)}+\sum_{i=1}^{n}\sum_{k=1}^{\infty}(\alpha_{i,k}^{(s)})^p\phi_n^{(s)}(\alpha_{i,k}^{(s)})A_{i,k}^{(s)}$$

$$=\int_{-\infty}^{+\infty}x^p\phi^{(s)}(x)d\psi^{(s)}(x)\ .$$

Therefore, if we define $\psi^{(s)}(x)$ to be a jump function with jumps given by (iii) and (iv), then

$$\int_{-\infty}^{+\infty}x^p\phi_n^{(s)}(x)d\psi^{(s)}=\left\{\prod_{i=1}^{n}b_{i+s}\right\}\delta_{n,p}\ .$$

Thus $\psi^{(s)}$ is a distribution function for $\phi_n^{(s)}(x)$ and the proof is complete.

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