PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 3, 1967

THE NON-INVARIANCE OF HYPERBOLICITY IN PARTIAL DIFFERENTIAL EQUATIONS

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Hyperbolicity is shown to be not an absolute invariant in the sense defined by the author. Specifically, an example of a nonhyperbolic system is given with a partial prolongation which is hyperbolic. A large class of systems is found which is closed under modified absolute equivalence and which contains all hyperbolic systems. These ideas are applied to give existence theorems for the initial value problem in several types of nonhyperbolic systems.

Since hyperbolicity is conveniently defined for quasi-linear systems, and as an additional reference, we define in §1 the ideas of partial prolongations and absolute equivalence for such systems. Since these problems and methods are generally local, we usually express them in coordinate notation. Ehresmann's jet notation could have been used to provide an invariant treatment. *We also assume all manifolds and functions are infinitely differentiable*, although it is not difficult to formulate the theorems for less smooth functions using available results in partial differential equations [3].

1. Definitions. Let D^p and D^m be open sets in $R^p = \{(x^1, \cdots, x^p)\}$ and $R^m = \{(z^1, \cdots, x^m)\}$, respectively.

DEFINITION 1. A *system* Σ on $D^p \times D^m$ is a system of functional and quasi-linear partial differential equations with $x^1, \cdots, x^p$ as independent and $z^1, \cdots, z^m$ as dependent variables:

$$f^\alpha(x^1, \cdots, x^p, z^1, \cdots, z^m) = 0 , \qquad \alpha = 1, \cdots, \alpha_1 ,$$
$$L^\beta = A^{\beta i}_\lambda \frac{\partial z^\lambda}{\partial x^i} + B^\beta = 0 , \qquad \beta = 1, \cdots, \beta_1 ,$$

(we use the summation convention), where f^α, $A^{\beta i}_\lambda$ and B^β are (infinitely differentiable) functions on $D^p \times D^m$. It is also required that the equations

$$f^\alpha{}_{,z^\lambda} \frac{\partial z^\lambda}{\partial x^i} + f^\alpha{}_{,x^i} = 0 , \qquad \alpha = 1, \cdots, \alpha_1$$
$$i = 1, \cdots, p ,$$

be a consequence of $L^\beta = 0$. (We use the notation $f^\alpha{}_{,z^\lambda} = \partial f^\alpha / \partial z^\lambda$, etc).

DEFINITION 2. If Σ is the system in Definition 1, its *total prolongation* $P\Sigma$ is the system on $D^p \times D^m \times R^{pm} = \{(x^i, z^\lambda, p^\mu_j) \mid i, j = 1, \cdots, p; \lambda, \mu = 1, \cdots, m\}$ generated by the equations of Σ together with

$$\frac{\partial z^\lambda}{\partial x^i} - p^\lambda_i = 0 , \qquad \frac{\partial p^\lambda_i}{\partial x^j} - \frac{\partial p^\lambda_j}{\partial x^i} = 0 ,$$

$$A^{\beta i}_\lambda \frac{\partial p^\lambda_i}{\partial x^j} + A^{\beta i}_{\lambda}{}_{, z^\mu} p^\mu_j p^\lambda_i + A^{\beta i}_{\lambda}{}_{, x^j} p^\lambda_i$$
$$+ B^\beta{}_{, z^\lambda} p^\lambda_j + B^\beta{}_{, x^j} = 0 ,$$
$$\lambda, \mu = 1, \cdots, m; i, j = 1, \cdots, p; \beta = 1, \cdots, \beta_1 .$$

DEFINITION 3. A *transform* or *change variables* of is a diffeomorphism of the form

$$\bar{z}^\lambda = \bar{\varphi}^\lambda(x^1, \cdots, x^p, z^1, \cdots, z^m)$$
$$x^i = \bar{\psi}^i(x^1, \cdots, x^p)$$

with inverse defined by φ^λ, ψ^i. Then Σ is transformed according to the usual rules:

$$\frac{\partial \bar{z}^\lambda}{\partial \bar{x}^i} = \left(\bar{\varphi}^\lambda{}_{, z^\mu} \frac{\partial z^\mu}{\partial x^k} + \bar{\varphi}^\lambda{}_{, x^k}\right) \psi^k{}_{, \bar{x}^i}.$$

Thus, $D^p \times D^m$ is regarded as a fibre space over D^p.

DEFINITION 4. A system Σ_1 on $D^p \times D^m \times D^n = \{(x^i, z^\lambda, u^\gamma) \mid i = 1, \cdots, p; \lambda = 1, \cdots, m; \gamma = 1, \cdots, n\}$ is a *partial prolongation* of the system Σ in Definition 1 when it is generated by equations equivalent to those in Σ together with equations of the form

$$u^\gamma - C^{\gamma j}_\lambda \frac{\partial z^\lambda}{\partial x^j} - D^\gamma = 0 , \qquad \gamma = 1, \cdots, n ,$$

$$P^\delta = E^{\delta i}_\gamma \frac{\partial u^\gamma}{\partial x^i} + F^{\delta i}_\gamma \frac{\partial z^\lambda}{\partial x^i} + G^\delta = 0 , \qquad \delta = 1, \cdots, \delta_1 ,$$

where $C^{\gamma j}_\lambda, D^\gamma$ are functions on $D^p \times D^m$. It is further required that if $P\Sigma$ is the total prolongation of Σ as in Definition 2, then

$$E^{\delta i}_\gamma \Big(C^{\gamma j}_\lambda \frac{\partial p^\lambda_j}{\partial x^i} + C^{\gamma j}_{\lambda}{}_{, z^\mu} p^\lambda_j p^\mu_i + C^{\gamma j}_{\lambda}{}_{, x^i} p^\lambda_j$$
$$+ D^\gamma{}_{, z^\lambda} p^\lambda_i + D^\gamma{}_{, x^i}\Big) + F^{\delta i}_\lambda p^\lambda_i + G^\delta = 0$$

are to be consequences of the equations in $P\Sigma$. When $C^{\gamma j}_\lambda = 0, \Sigma$ is called an *admissible restriction* of Σ_1.

Observe that $P\Sigma$ is itself a partial prolongation of Σ. If $z^\lambda = \bar{z}^\lambda(x^1, \cdots, x^p)$ is a solution of Σ then $p_i^\lambda = \partial z^\lambda/\partial x^i$ define a solution of $P\Sigma$, and hence $u^\gamma = C_\lambda^{\gamma j}(x, \bar{z})(\partial z^\lambda/\partial x^j) + D^\gamma(x, \bar{z})$ define a solution of Σ_1. The above definitions are somewhat more restrictive than in [2], but they are essentially equivalent. In [2] it was shown that if Σ_1 is a partial prolongation of Σ, then $P\Sigma$ is a partial prolongation of Σ_1.

DEFINITION 5. If $\Sigma_0, \Sigma_1, \cdots, \Sigma_n$ is a finite sequence of systems such that for every $i = 1, \cdots, n$, Σ_i is a partial prolongation or transform of Σ_{i-1} or else Σ_{i-1} is a partial prolongation or transform of Σ_i, then Σ_0 and Σ_n are *absolutely equivalent*.

2. Hyperbolicity.

We now treat D^p as a product space $D^{p-1} \times D^1 = \{(x^1, \cdots, x^{p-1}) \times (x^p)\}$ and discuss systems "hyperbolic in the x^p-direction." Any change of variables must preserve this product structure on D^p.

Notation. If $F(x^i, z^\lambda, \partial z^\mu/\partial x^j)$ is any first-order partial differential function, denote by $F;_j$ the second-order function

$$F;_j = F,_{x^j} + F,_{z^j} \frac{\partial z^\lambda}{\partial x^j} + F,_{(\partial z^\lambda/\partial x^i)} \frac{\partial^2 z^\lambda}{\partial x^i \partial x^j} .$$

(Note that F and $F;_j$ may be regarded as functions on spaces of jets. We always regard $\partial^2 z^\lambda/\partial x^i \partial x^j = \partial^2 z^\lambda/\partial x^j \partial x^i$.)

DEFINITION 6. The system Σ is *involutive in the x^p-direction* if it is generated by equations of the form

$$\begin{aligned} &f^\alpha(x^1, \cdots, x^p, z^1, \cdots, z^m) = 0 , \qquad \alpha = 1, \cdots, \alpha_1 , \\ &M^\lambda = \frac{\partial z^\lambda}{\partial x^p} - \sum_{a=1}^{p=1} A_\mu^{\lambda a} \frac{\partial z^\mu}{\partial x^a} - B^\lambda = 0 , \qquad \lambda = 1, \cdots, m , \\ &L^\beta = \sum_{a=1}^{p-1} C_\mu^{\beta a} \frac{\partial z^\mu}{\partial x^a} + D^\beta = 0 , \qquad \beta = 1, \cdots, \beta_1 . \end{aligned}$$

It is required that

$$\text{(1)} \qquad \begin{aligned} &L^\beta;_p - \sum_{a=1}^{p-1} (R_\gamma^{\beta a} L^\gamma;_a + S_\lambda^{\beta a} M^\lambda;_a) \\ &\qquad - T_\gamma^\beta L^\gamma - U_\lambda^\beta M^\lambda = 0 \qquad (\text{mod}\, f^1, \cdots, f^{\alpha_1}) , \end{aligned}$$

where the $R_\gamma^{\beta a}$, T_γ^β and U_λ^β are functions of x^i, z^λ, M^λ and $M^\lambda;_a$, $a = 1, \cdots, p-1$. (In the following the indices a, b will run over $1, \cdots, p-1$ unless noted otherwise, $i, j = 1, \cdots, p$; $\lambda, \mu = 1, \cdots, m$.)

The equations $M^{\lambda} = 0$, are called the *primary* equations of while equations (1) are the *secondary* equations.

Involutiveness is not in general preserved under partial prolongations, since, in the notation of Definition 4, no P^{δ} need occur. We shall usually be concerned with invariants among systems involutive in the x^p-direction. By a calculation it can be seen that this concept is preserved under a change of coordinates which leaves the product structure on D^p invariant. We introduce the following finer equivalence relation.

DEFINITION 7. Two systems involutive in the x^p-direction are *x^p-equivalent* if they are absolutely equivalent by means of a sequence of systems $\Sigma_0, \Sigma_1, \cdots, \Sigma_n$ where each Σ_i is involutive in the x^p-direction.

DEFINITION 8. Let Σ be involutive in the x^p-direction. Suppose (in the notation of Definition 6) the equations of Σ can be arranged so that the matrices $A^{\lambda a}_{\mu}$ and $R^{\beta a}_{\gamma}$ have the following property: there exist nonsingular real matrices $V^{\lambda}_{\mu}(x, z, \xi_1, \cdots, \xi_{p-1})$ and $W^{\beta}_{\gamma}(x, z, \xi_1, \cdots, \xi_{p-1})$ on $D^p \times D^m \times R^{p-1}$ with inverses $(V^{-1})^{\lambda}_{\mu}$ and $(W^{-1})^{\beta}_{\gamma}$ such that on $D^p \times D^m \times R^{p-1}$,

$$V^{\lambda}_{\mu} A^{\mu a}_{v} \xi_a (V^{-1})^{\nu}_{\omega}$$

and

$$W^{\beta}_{\gamma} R^{\gamma a}_{\delta} \xi_a (W^{-1})^{\delta}_{\varepsilon}$$

are diagonal matrices. Then Σ is *hyperbolic in the x^p-direction.*

By a calculation it can be seen that this concept is preserved under an allowable change of variables.

THEOREM 1. *Let Σ be hyperbolic in the x^p-direction on a neighborhood of (x_0, z_0). Using the notation of Definition 6, let*

$$z^{\lambda} = \varphi^{\lambda}(x^1, \cdots, x^{p-1}), \qquad \lambda = 1, \cdots, m,$$

be defined on a neighborhood of $(x^a_0) \in D^{p-1}$ and satisfy

$$\varphi^{\lambda}(x^a_0) = z^{\lambda}_0, \qquad \lambda = 1, \cdots, m,$$
$$f^{\alpha}(x^1, \cdots, x^{p-1}, x^p_0, \varphi^1, \cdots, \varphi^m) = 0, \qquad \alpha = 1, \cdots, \alpha_1,$$

and

$$C^{\beta a}_{\mu}(x^b, x^p_0, \varphi^{\lambda})\varphi^{\mu}{}_{,x^i} + D^{\beta}(x^b, x^p_0, \varphi^{\lambda}) = 0, \qquad \beta = 1, \cdots, \beta_1.$$

Then there exists a unique solution $z^{\lambda} = F^{\lambda}(x^1, \cdots, x^p)$ of Σ on a neighborhood of (x^i_0) satisfying

$$F^{\lambda}(x^1, \cdots, x^{p-1}, x_0^p) = \varphi^{\lambda}(x^1, \cdots, x^{p-1}), \qquad \lambda = 1, \cdots, m.$$

Proof. Apply standard existence theorems [3] to solve $M^{\lambda} = 0$ with the given initial conditions. This solution is unique. The functions L^{β} satisfy a second system of linear hyperbolic equations with zero initial values. By uniqueness, the L^{β} are zero. Similar arguments show the f^{α} to be zero when evaluated at the solution of $M^{\lambda} = 0$.

EXAMPLE. The system

$$M^1 = \frac{\partial z^1}{\partial x^2} = 0, \qquad M^2 = \frac{\partial z^2}{\partial x^2} - \frac{\partial z^2}{\partial x^1} = 0, \qquad M^3 = \frac{\partial z^3}{\partial x^2} - \frac{\partial z^1}{\partial x^1} = 0$$

is involutive in the x^2-direction, but its matrix $A_{\mu}^{\lambda 1}$, having eigenvalues $0, 0, 1$ and rank 2, is not diagonalizable. The partial prolongation Σ_1 obtained by adding $(\partial z^1/\partial x^1) - u^1 = 0$ and $(\partial u^1/\partial x^2) = 0$ is, however, hyperbolic in the x^2-direction, for this larger system may be written

$$\frac{\partial z^1}{\partial x^2} = 0, \qquad \frac{\partial z^2}{\partial x^2} - \frac{\partial z^2}{\partial x^1} = 0, \qquad \frac{\partial z^3}{\partial x^2} - u^1 = 0,$$
$$\frac{\partial u^1}{\partial x^2} = 0, \qquad \frac{\partial z^1}{\partial x^1} - u^1 = 0.$$

The matrix $A_{\mu}^{\lambda 1}$ is now diagonalized while $R_{\gamma}^{\beta 1}$ is 1-dimensional, hence diagonal.

This example shows hyperbolicity is not invariant under partial prolongations, even if both systems are involutive in the x^p-direction. Yet the initial value problem can be solved for Σ in the example. Given $\varphi^{\lambda}(x^1)$ as initial functions, let $\psi^1 = \partial\varphi^1/\partial x^1$ be an initial function for u^1 in Σ_1. Solving Σ_1 will yield a solution of Σ.

THEOREM 2. *Let Σ be x^p-equivalent to Σ_1 which is hyperbolic in the x^p-direction. Then given the initial conditions of Theorem* 1 *for Σ a solution may be found as in Theorem* 1.

Proof. It suffices to consider one pair at a time in the sequence of systems joining Σ and Σ_1, showing that initial conditions carry over naturally and recalling that any solution of one system induces a solution of the others. Thus, one need consider only single partial prolongations or changes of variables. In each case the result follows from a detailed calculation.

3. Complex systems.

In this section we determine a class of systems which contains the hyperbolic systems and is closed under x^p-equivalence.

DEFINITION 9. A system which is involutive in the x^p-direction is said to be *x^p-complex* on an open set $\mathsf{U} \subset D^p \times D^m$ if, in the notation of Definition 6,

(1) for every choice of variables $x^1, \cdots, x^{p-1}$ and

(2) for every choice of functions H^λ_β on U,

the matrix $A^\lambda_\mu + H^\lambda_\beta C^{\beta 1}_\mu$ always has at least one nonreal eigenvalue at each point of U.

Hyperbolic systems are not x^p-complex since in the notation of Definition 7, with $H^\lambda_\beta = 0$ and $\xi_a = \delta^1_a$, $A^{\lambda 1}_\mu$ is (real) diagonalizable. We shall show that x^p-complex systems form a class which is closed under x^p-equivalence. The complimentary class contains all systems hyperbolic in the x^p-direction.

LEMMA 1. *Let A be an $m \times m$ matrix of functions on an open set $\mathsf{U} \subset D^p \times D^m$. Let*

$$C = \begin{pmatrix} V \\ 0 \end{pmatrix}$$

be an $m \times r$ matrix of functions on U where V is $s \times r$ of rank s. Assume that for every $r \times m$ matrix function H on U, $A + CH$ has at least one nonreal eigenvalue at each point of U. Then $s \leqq m - 2$ and there exists on a neighborhood of each point in U a nonsingular matrix of functions of the form

$$P = \begin{pmatrix} I & 0 \\ 0 & P_1 \end{pmatrix}$$

where I is $s \times s$ identity such that

$$PAP^{-1} = \begin{pmatrix} W & X \\ 0 & A_1 \end{pmatrix}$$

where A_1 is 2×2 with nonreal eigenvalues. The converse is also true.

Proof. The converse follows immediately. The lemma may be proved by induction on $m \geqq 2$. If $m = 2$, then s is $0, 1$ or 2. If $s = 0$, then $C = 0$, hence $A = CH = A = A_1$. When $s = 1$ or 2 the other hypotheses cannot be fulfilled.

Now assume the lemma true for all matrices A of order $< m$. Given any R' of order $s \times s$ and R'' of order $s \times (m - s)$ one may choose H such that

$$A + CH = \begin{pmatrix} R' & R'' \\ A' & A'' \end{pmatrix}.$$

Then if $A + CH$ is to have always a nonreal eigenvalue, certainly $s \leqq m - 2$. On a neighborhood U' of any point in U a nonsingular $(m - s) \times (m - s)$ matrix function P_1 may be found so that

$$P_1 A' = \begin{pmatrix} V' \\ 0 \end{pmatrix} = C'$$

where V' is $t \times (m - s)$ and has rank t. Then

$$\begin{pmatrix} I & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} R' & R'' \\ A' & A'' \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P_1^{-1} \end{pmatrix} = \begin{pmatrix} R', & R'P^{-1} \\ C, & P_1 A'' P_1^{-1} \end{pmatrix}$$

has at least one nonreal eigenvalue for every R', R''. Then so does

$$\begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \begin{pmatrix} R', & R''P_1^{-1} \\ C', & P_1 A'' P_1^{-1} \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}$$

for every choice of K.

In particular, if $R' = KC'$, $R'' = KP_1A'' + KC'P_1 - R'KP_1$, it follows that $P_1A''P_1^{-1} + C'K$ has nonreal eigenvalues for every choice of K. The induction hypothesis applies to $P_1A''P_1^{-1}$ and C', so on a neighborhood of each point in $\mathsf{U}' \subset \mathsf{U}$, there is a matrix P_2 such that

$$\begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix} P_1 A'' P_1^{-1} \begin{pmatrix} I & 0 \\ 0 & P_2^{-1} \end{pmatrix} = \begin{pmatrix} W' & X' \\ 0 & A_2 \end{pmatrix}$$

where A_2 is 2×2 with nonreal eigenvalues, I is $t \times t$ and $t \leqq m - s - 2$. Now take

$$P = \begin{pmatrix} I_2 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I_1 & 0 \\ 0 & P_1 \end{pmatrix}$$

where I_1 is the $s \times s$ and I_2 is the $(s + t) \times (s + t)$ identity matrix.

THEOREM 3. *Let Σ_1 be a partial prolongation of Σ where both are involutive in the x^p-direction. If Σ is x^p-complex on $\mathsf{U} \subset D^p \times D^m$, then Σ_1 is x^p-complex on $\mathsf{U} \times R^n$. If Σ_1 is x^p-complex on $\mathsf{U} \times V$ where U is open in $D^p \times D^m$, then Σ is x^p-complex on U.*

Proof. Let Σ have the form in Definition 6 while Σ_1 contains the additional equations

$$\mathsf{U}^\pi - E_\lambda^{\pi a} \frac{\partial z^\lambda}{\partial x^a} - F^\pi = 0 \,,$$

$$\frac{\partial u^\pi}{\partial x^p} - G_\lambda^{\pi a} \frac{\partial z^\lambda}{\partial x^a} - H_\rho^{\pi a} \frac{\partial u^\rho}{\partial x^a} - I^\pi = 0 \,,$$

$$J_\pi^{\gamma a} \frac{\partial u^\pi}{\partial x^a} + K_\lambda^{\gamma 2} \frac{\partial z^\lambda}{\partial x^a} + N^\gamma = 0 \,,$$

$\pi, \rho = 1, \cdots, n; \gamma = 1, \cdots, \gamma_1$. (Observe that $\partial z^\lambda/\partial x^p$ can be eliminated using equations in Σ.) Definition 4 requires that the equations obtained by replacing $\partial u^\pi/\partial x^i$ by expressions in p_i^λ and $\partial p_i^\lambda/\partial x^j$ must occur in $P\Sigma$. Considering the coefficient of $\partial p_i^\lambda/\partial x^1$ in these expressions one sees that for some functions N_β^γ and M_β^π of x^i, z^λ,

$$J_\pi^{\gamma 1} E_\lambda^{\pi 1} = N_\beta^\gamma C_\lambda^{\beta 1} .$$

and

$$E_\lambda^{\pi 1} A_\rho^{\lambda 1} - H_\rho^{\pi 1} E_\lambda^{\rho 1} = M_\beta^\pi C_\lambda^{\beta 1} .$$

Thus, if $J = (J_\pi^{\gamma 1})$, $E = (E_\lambda^{\pi 1})$, etc.,

$$EJ = CN \quad \text{and} \quad AE - EH = CM .$$

Now assume Σ is x^p-complex, so for every choice of $x^1, \cdots, x^{p-1}$ and M_0, $A + CM_0$ has a nonreal eigenvalue at each point of U. To prove that Σ_1 is x^p-complex we must show that

$$\begin{pmatrix} A & G \\ 0 & H \end{pmatrix} + \begin{pmatrix} C & E & K \\ 0 & 0 & J \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \\ M_5 & M_6 \end{pmatrix} = Z \tag{2}$$

has a nonreal eigenvalue for every choice of $M_1, \cdots, M_6$, where $G = (G_\lambda^{\pi 1})$, $H = (H^{\pi 1})$ etc.

On a neighborhood of $(x_0, z_0) \in \mathsf{U}$ let $\bar{P}$ be an $m \times m$ nonsingular matrix of functions so that

$$\bar{P}C = \begin{pmatrix} V \\ 0 \end{pmatrix} = \bar{C}$$

where V is $s \times r$ and of rank $s \leqq m - 2$. Let $\bar{A} = \bar{P}A\bar{P}^{-1}$. By Lemma 1 there is a nonsingular P on a neighborhood of (x_0, z_0) such that $P\bar{C} = \bar{C}$ and

$$P\bar{A}P^{-1} = \begin{pmatrix} W & X \\ 0 & A_1 \end{pmatrix}$$

where A_1 is 2×2 with nonreal eigenvalues. This P can be chosen so that

$$A_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} .$$

Then the complex vectors

$$q_1 = (0, \cdots, 0, 1, i) , \qquad r_1 = (0, \cdots, 0, 1, -i)$$

satisfy

$$q_1 P\bar{A}P^{-1} = (a - ib)q_1,\ r_1 P\bar{A}P^{-1} = (a + ib)r_1\ ,$$

while

$$q_1 P\bar{C}M_0 P^{-1} = r_1 P\bar{C}M_0 P^{-1} = 0\ .$$

Hence $q = q_1 P\bar{P}$ and $r = r_1 P\bar{P}$ are complex eigenvectors of A which are carried to zero by C.

Since $AE - EH = CM$, qE is an eigenvector of H belonging to the eigenvalue $a - ib$ while rE belongs to $a + ib$. From $EJ = CN$ it follows that $(qE)J = (rE)J = 0$. For every M_6, $H + JM_6$ has these same eigenvectors and values. Letting Q be $n \times n$ nonsingular so that

$$\bar{Q}J = \begin{pmatrix} J' \\ 0 \end{pmatrix}$$

where J' is $t \times U$ with rank $t < n - 2$, there must exist by Lemma 1 a matrix Q_1 on a neighborhood of (x_0, z_0) such that

$$Q_1(\bar{Q}H\bar{Q}^{-1})Q_1^{-1} = \begin{pmatrix} W' & X' \\ 0 & H_1 \end{pmatrix}$$

where H_1 is 2×2 with nonreal eigenvalues. Letting $Q = Q_1\bar{Q}$, it follows that

$$\begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} Z \begin{pmatrix} I & 0 \\ 0 & Q^{-1} \end{pmatrix} = \begin{pmatrix} & A' \quad G' \\ QJM_6 & QHQ^{-1} + QJM_0Q^{-1} \end{pmatrix}.$$

The lower two rows in this matrix are $(0\ H_1)$. It follows that for every choice of $M_1, \cdots, M_6$, the matrix Z has nonreal eigenvalues for any $(x_0, z_0) \in \mathbf{U}$, $(u^1, \cdots, u^n) \in R^n$.

The converse follows at once by choosing $M_2 = \cdots = M_6 = 0$ in Z.

4. Examples. Several nonanalytic systems for which the initial value problem may be solved are x^p-equivalent to systems hyperbolic in the x^p-direction.

THEOREM 4. *The system*

$$\frac{\partial z}{\partial x^p} = f\left(x^1, \cdots, x^p, z, \frac{\partial z}{\partial x^1}, \cdots, \frac{\partial z}{\partial x^{p-1}}\right)$$

is absolutely equivalent to a system hyperbolic in the x^p-direction. (We have not included such general systems is our definitions, but these can be extended in an obvious way).

Proof. The total prolongation is obtained by adding

$$M_i = \partial z/\partial x^i - q_i\,, \qquad L_{ij} = \partial q_i/\partial x^j - \partial q_j/\partial x^i\,,$$
$$N_2 = \partial q_p/\partial x^i - f,_{x^i} - f,_z q_i - f,_{q_a}(\partial q_i/\partial x^a) = 0\,,$$
$$i, j = 1, \cdots, p;\ a = 1, \cdots, p-1\,.$$

Then $q_p - f(x, z, q_1, \cdots, q_{p-1}) = 0$ is the 0-order equation; $M_p = 0$, $L_{ap} = 0$ and $N_p = 0$ are the primary equations, and $M_a = 0, L_{ab} = 0$, $N_a = 0$ generate the secondary equations. Since

$$L_{ab;p} = L_{pa;b} - L_{pb;a}$$
$$M_{a;p} = M_{p;a} - L_{ap}\,,$$
$$N_{a;p} = N_{p;a} + M_a(f,_{x^p,z} + f,_{z,z^q}p - f,_{q_b,z}L_{bp}) - f,_z L_{ap}$$
$$- L,_{q_a,a}L_{bp} - f,_{q_b,q_c}L_{cp}\frac{\partial q_b}{\partial x^a}$$
$$- f, q_b L_{pb;a}$$

the secondary equations have matrix $R_\gamma^{\beta a} = 0$, in the notation of Definition 6. The primary equations may be arranged in the form

$$M_p = 0\,,$$
$$L_{ap} + N_a = \frac{\partial q_a}{\partial x^p} - f,_{q_b}\frac{\partial q_a}{\partial x^b} - f,_{x^a} - f,_{z_q}q_a = 0\,,$$
$$N_p = 0\,,$$

with relevant matrix $A_\mu^{\lambda b} = \delta_\mu^\lambda f,_{q_b}$. For any $\xi_1, \cdots, \xi_{p-1}$, $A_\mu^{\lambda b}\xi_b$ is diagonal.

The example in § 2 can be generalized in the following

THEOREM 5. *If the system Σ on $D^2 \times D^m$*

$$M^\lambda = \frac{\partial z^\lambda}{\partial x^2} - A_\mu^\lambda\frac{\partial z^\mu}{\partial x^1} - B^\lambda = 0\,, \qquad \lambda = 1, \cdots, m\,,$$
$$L^\beta = C_\lambda^\beta\frac{\partial z^\lambda}{\partial x^1} + D^\beta = 0\,, \qquad \beta = 1, \cdots, \beta_1\,,$$

is involutive in the x^2-direction and satisfies

(1) *A_μ^λ, B^λ are constants,*

(2) *A_μ^λ has only real eigenvalues and elementary divisors of degree at most 2,*

(3) *the auxiliary system is hyperbolic, then Σ is x^2-equivalent to a system hyperbolic in the x^2-direction.*

Proof. Let $P = (v^\lambda_\mu)$ be a nonsingular constant matrix with inverse $P^{-1} = (w^\lambda_\mu)$ so that $P^{-1}AP = \operatorname{diag}(J_1, \cdots, J_r, \lambda^{r+1}, \cdots, \lambda^n) = J$ where

$$J_k = \begin{pmatrix} \lambda^k & 1 \\ 0 & \lambda^k \end{pmatrix} .$$

Then $AP = PJ$, so

$$A^\mu_\lambda v^{2k-1}_\mu = \lambda^1 v^{2k-1}_\lambda , \qquad k = 1, \cdots, r .$$

Consider the partial prolongation Σ_1 on $D^2 \times D^m \times R^k$ with new variables $u^1, \cdots, u^k$ obtained by adding to Σ the equations

$$I^\theta = \frac{\partial u^\theta}{\partial x^2} - \lambda^\theta \frac{\partial u^\theta}{\partial x^1} = 0 \quad \text{(no summation)}$$
$$K^\theta = u^\theta - v^{2\theta-1}_\lambda \frac{\partial z^\lambda}{\partial x^1} , \qquad \theta = 1, \cdots, k .$$

It is not difficult to check that Σ_1 is a partial prolongation. Since $K^\theta; \hat{2} = I^\theta + \lambda^\theta K^\theta; \hat{1} - v^{2\theta-1}_\lambda M^\lambda; \hat{1}$, (no summation on θ), Σ_1 is involutive in the x^2-direction and its secondary system is hyperbolic.

The primary equations of Σ_1 may be written in the form

$$M^\lambda - j^\mu_{2\theta-1} w^\lambda_\mu K^\theta = 0 , \quad \text{(summing on } \theta)$$
$$I^\theta = 0$$

where $J = (j^\lambda_\mu)$. That is,

$$\frac{\partial z^\lambda}{\partial x^2} - A^\lambda_\mu \frac{\partial z^\mu}{\partial x^1} + w^\lambda_\mu j^\mu_{2\theta-1} v^{2\nu-1} \frac{\partial z^\nu}{\partial x^1}$$
$$- j^\mu_{2\theta-1} w^\lambda_\mu u^\theta - B^\lambda = 0 .$$

Since

$$\begin{aligned} & w^\mu_\lambda (A^\nu_\mu - j^\omega_{\omega\theta-1} w^\nu_2 v^{2\theta-1}_\mu) u^\pi_\nu \\ & = j^\pi_\lambda - \delta^{2\theta-1}_\lambda j^\omega_{2\theta-1} \delta^\pi_\omega = j^\pi_\lambda - \delta^{2\theta-1}_\lambda j^\pi_{2\theta-1} \\ & = \begin{cases} j^\pi_\lambda \text{ if } \lambda \neq 2\varphi - 1 , & 1 \leqq \varphi \leqq r \\ 0 \text{ if } \lambda = 2\varphi - 1 , & 1 \leqq \varphi \leqq r , \end{cases} \end{aligned}$$

the relevant matrix for Σ_1 is diagonalized.

COROLLARY. *For a system satisfying the conditions of Theorem 5 the initial value problem is well posed.*

The author would like to thank J. Jans for helpful discussions of the algebraic problems in this paper.

References

1. R. Hermann, *E. Cartan's Geometric Theory of Partial Differential Equations*, Advances in Mathematics, **1** (1965), 265-317.
2. H. H. Johnson, *Absolute equivalence of exterior differential systems*, (to appear in Illinois J. Math. **10** (1966), 407-411
3. P. Lax, *Partial differential equations*, Notes, NYU (1950-51).
4. L. Mansfield, *A generalization of the Cartan-Kahler theorem*, Thesis, 1965, University of Washington.

Received July 11, 1966.

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