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# ON SOME HYPONORMAL OPERATORS

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**Let $H$ be a Hilbert space and $T$ a hyponormal operator ($T^*T - TT^* \geqq 0$). The first result is: if $(T^*)^p T^q$ is a completely continuous operator then $T$ is normal.**

**Secondly, part we introduce the class of operators on a Banach space which satisfy the condition**

$$\| x \| = 1 \qquad \| Tx \|^2 \leqq \| T^2 x \|$$

**and we prove the following:**

1. $\gamma_T = \lim \| T^n \|^{1/n} = \| T \|$;
2. **if $T$ is defined on Hilbert space and is completely continuous then $T$ is normal.**

In what follows for this section we suppose that $T$ is a hyponormal operator on Hilbert space $H$.

THEOREM 1.1. *If $T$ is completely continuous then $T$ is normal.*

This is known ([1], [2], [3]).

The main result of this section is as follows.

THEOREM 1.2. *If $T^{*p}T^q$ is completely continuous where $p$ and $q$ are positive integers then $T$ is normal.*

LEMMA. Let $\| T \| = 1$. Then in the Hilbert space $H$ there exists a sequence $\{x_n\}$, $\| x_n \| = 1$ such that

$$\| T^* x_n \| \to 1 \tag{1}$$

$$\| T^m x_n \| \to 1 \qquad m = 1, 2, 3, \cdots, \tag{2}$$

$$\| T^* T x_n - x_n \| \to 0 \tag{3}$$

$$\| T T^* x_n - x_n \| \to 0 \tag{4}$$

$$\| T^* T^m x_n - T^{m-1} x_n \| \to 0 \qquad m = 1, 2, 3, \cdots. \tag{5}$$

*Proof.* We observe that (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3). Thus it remains to prove (1), (2), and (5).

By definition there exists a sequence $\{x_n\}$, $\| x_n \| = 1$ such that

$$\| T^* x_n \| \to \| T^* \| = \| T \| = 1 .$$

It is known [3] that for $x$, $\| x \| = 1$

$$\| Tx \|^2 \leqq \| T^2 x \| .$$

Since

$$\| T^* x_n \|^2 \leqq \| T x_n \|^2 \leqq \| T^2 x_n \| \leqq 1$$

we have

$$\lim \| T^2 x_n \| = 1 .$$

If

$$\| T^{k-1} x_n \| \to 1$$
$$\| T^k x_n \| \to 1$$

then

$$\lim \| T^{k+1} x_n \| = 1 .$$

Now

$$\left\| T^2 \frac{T^{k-1} x_n}{\| T^{k-1} x_n \|} \right\| \geqq \left\| T \frac{T^{k-1} x_n}{\| T^{k-1} x_n \|} \right\|^2$$

we have

$$\| T^{k+1} x_n \| \to 1 .$$

By induction we have the relation (2).

For (5) we put

$$y_n(m) = T^* T^m x - T^{m-1} x_n$$

and

$$\delta_n(m) = \| y_n(m) \|^2 .$$

We have

$$\begin{aligned} \delta_n(m) &= \| T^* T^m x_n \|^2 - 2 \| T^m x_n \|^2 + \| T^{m-1} x_n \|^2 \\ &\leqq \| T^m x_n \|^2 - 2 \| T^m x_n \|^2 + \| T^{m-1} x_n \|^2 \\ &= \| T^{m-1} x_n \|^2 - \| T^m x_n \|^2 . \end{aligned}$$

By (2) we obtain that $\delta_n(m) \to 0$ for every $m$. This proves the lemma.

*Proof of the Theorem* 1.2. Let $p$ and $q$ the integers such that $T^{*p} T^q$ is a completely continuous operator. By the lemma

$$T^* T^q x_n - T^{q-1} x_n \to 0$$

($\{x_n\}$ is the sequence of lemma). It is clear that $\{T^{*p-1} T^{q-1} x_n\}$ admits a subsequence which is convergent. Also, by the lemma and this result we obtain a subsequence of $\{T^{*p-2}T^{q-2}x_n\}$ which is convergent. The process can be repeated and we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent.

Let $x_0 = \lim x_{n_k}$. Thus

$$T^*Tx_0 = x_0$$
$$TT^*x_0 = 0 .$$

The closed subspace $M_T = \{x, TT_x^* = x\}$ is a nonzero subspace. By the Lemma 2 of [2] $T$ has a approximate proper value

$$Ty_n - \lambda y_n \to 0 .$$

The above arguments show that every sequence of approximate eigenvectors $\{y_n\}$ of $T$ belonging to $\bar{\lambda}$ with $|\bar{\lambda}| = 1$ contains a convergent subsequence so that $\bar{\lambda}$ is an eigenvalue of $T^*$, hence $\lambda$ is of $T$.

Let $M$ be the smallest closed linear subspace which contains every proper subspace of $T$ and $N = M^{\perp}$. It is known that $N$ is invariant for $T^*$ and thus $T^{*p}T^q$ is completely continuous on $N$. It is known that $T_N$ is hyponormal. This shows that $N = \{0\}$ and $M = H$. The theorem is proved.

II. In this section we introduce a class of operators on any Banach space $B$.

DEFINITION 2.1. The operator $T$ is said to be of class $N$ if

$$x \in B, \|x\| = 1 \qquad \|Tx\|^2 \leqq \|T^2x\| .$$

LEMMA 2.1. *Every hyponormal operator is of class N.*

*Proof.*

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leqq \|T^*Tx\| \leqq \|T^2x\| .$$

It is clear by this lemma that these operators are extension of a class of hyponormal operators.

LEMMA 2.2. *If $T$ is of class $N$ and*

$$(1)\quad \|T\| = 1 , \qquad (2)\quad \|x_n\| \to 1 , \qquad (3)\quad \|Tx_n\| \to 1 .$$

*Then* $\|T^m x_n\| \to 1$ $(m = 1, 2, 3, \cdots,)$.

*Proof.* This is easy consequence of the inequality

$$\|T^m x_n\| = \|T^2 \cdot T^{m-2}x_n\| \geqq \frac{\|T^{m-1}x_n\|}{\|T^{m-2}x_n\|} .$$

THEOREM 2.1. *If $T$ is of class $N$*

$$\| T \| = \lim \| T^n \|^{1/n} = \delta_T .$$

*Proof.* For every $n$, Lemma 2.2. leads to relation $\| T^n \| = \| T \|^n$ which gives Theorem 2.1.

COROLLARY 2.1. *A generalised nilpotent operator $T$ of the class $N$ is necessarily zero.*

LEMMA 2.3. *If $T$ is of class $N$ on a Hilbert space $H$ and $\| T \| = 1$ then*

$$M_{T^*} = \{x,\ TT^* = x\}$$

*is invariant under $T$.*

*Proof.* Let $x \in M_{T^*}$, $\| x \| = 1$. Then

$$\begin{aligned} \| T^*Tx - x \|^2 &= \| T^*Tx \|^2 - 2 \| Tx \| + 1 \\ &= \| T^*Tx \|^2 - 2 \| TTT^*x \|^2 + 1 \\ &= \| T^*Tx \|^2 - 2 \left\| T^2 \frac{T^*x}{\| T^*x \|} \right\|^2 \cdot \| T^*x \|^2 + 1 \\ &\leqq \| T^*Tx \|^2 - 2 \| TT^*x \|^4 \frac{1}{\| T^*x \|^2} + 1 \\ &\leqq \| T^*Tx \|^2 - \frac{2}{\| T^*x \|^2} + 1 \leqq 0 . \end{aligned}$$

Thus $\| T^*Tx - x \| = 0$. It is clear that

$$Tx = TT^*(Tx) = T(T^*Tx)$$

which shows that $Tx \in M_{T^*}$.

We observe that $T/M$ is an isometric operator.

THEOREM 2.2 *If $T$ is of the class $N$ on a Hilbert space and $T^*$ is completely continuous for some $k \geqq 1$ then $T$ is normal.*

*Proof.* (for $\| T \| = 1$) From the completely continuous property of $T^k$ it is clear that the subspace

$$M_{T^*} = \{x,\ TT^* = x\}$$

is not $\{0\}$. Also $M_{T^*}$ is finite dimensional because it is invariant under $T^k$ which is isometric and completely continuous and $M_{T^*}$ reduces $T$. We consider the subspace $M_{T^*}^{\perp}$ and continue in this way and obtain that $T$ is normal.

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## References

1. T. Andô, *On hyponormal operators*, Proc. Amer. Math. Soc. **14** (1963), 290-291.
2. S. Berberian, *A note on hyponormal operators*, Pacif. J. Math. **12** (1962), 1172-1176.
3. J. Stampfli, *Hyponormal operators*, Pacif. J. Math. **12** (1962), 1453-58.

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