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A SUM OF A CERTAIN DIVISOR FUNCTION FOR ARITHMETICAL SEMI-GROUPS

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Let $\{b_n\}$ denote the set of elements of a free ordered arithmetical semi-group with multiplication and a known counting function. Using the corresponding terminology of arithmetic let $b_n = d\delta$ and let $\tau'(b_n)$ denote the number of divisors d of b_n where both d are δ and square free. Then it is shown here that $T(x)$ defined by

$$T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n) \sim Ax \log x$$

where A is a constant depending on b_u.

A more explicit definition of the semi-group is as follows. Suppose there is an infinite sequence $\{p\}$ of real numbers, which we will call generalised primes, such that

$$1 < p_1 < p_2 < \cdots .$$

Form the set $\{b\}$ of all p-products, i.e., products $p_1^{v_1} p_2^{v_2} \cdots$, where $v_1, v_2, \cdots$ are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalised integers and suppose that no two generalised integers are equal if their v's are different. Then assume $\{b\}$ may be arranged as an increasing sequence:

$$1 = b_1 < b_2 < \cdots < b_n < \cdots .$$

We say $d \mid b_n$ if $d \in \{b\}$ and there exists $\delta \in \{b\}$ such that $d\delta = b_n$; d and δ are then called complementary divisors of b_n. Let $\tau'(b_n)$ be the number of divisors d of b_n where both d and its complementary divisor are square free. In fact

$$\tau'(b_n) = \sum_{\substack{d\delta = b_n \\ d \text{ square free} \\ \delta \text{ square free}}} 1 . \tag{1.1}$$

This means that $\tau'(b_n) = 0$ unless b_n is of the form $\prod_{ij} p_i p_j^2$. Let x be any positive number and b_u any generalised integer. The sum to be evaluated, $T(x)$ is defined by

$$T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n) \tag{1.2}$$

where (b_n, b_u) denotes the greatest common divisor of b_n and b_u. In order to evaluate this sum a further assumption on the number of generalised integers less than or equal to x is required. Let $[x]$ denote the number of generalised integers $\leqq x$.

Assume

$$[x] = x + R(x), R(x) = 0\ (x^{\alpha}) \text{ and } 0 < \alpha < 1\ . \tag{1.3}$$

Using (1.3) it will be shown that when b_u is square free

$$T(x) = Ax \log x + 0\left(x \exp. \left\{\frac{(\log b_u)^{1-\alpha}}{\log\log b_u}\right\}\right) \tag{1.4}$$

where

$$A = \prod_{p \mid b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4}\ .$$

This sum is similar to that found by Gordon and Rogers in [2]. Also using the methods of [2] exactly analagous results for arithmetical semi-groups can be found to those shown by Gordon and Rogers. The only extra difficult result required is the prime number theorem for generalised integers. This is proved in [6] and is

$$\pi(x) = \frac{x}{\log x} + 0\left(\frac{x}{\log^2 x}\right)\ . \tag{1.5}$$

2. Supplementary definitions and results. Define the Möbius function $\mu(b_n)$ for the semi-group as follows: $\mu(b_n) = 0$ if b_n has a square factor $\mu(b_n) = (-1)^k$, where k denotes the number of prime divisors of b_n and b_n has no square factor; $\mu(1) = 1$. Let $\phi(x, b_u)$ denote the number of generalised integers $\leqq x$ which are prime to b_u. Then it is proved in [3] that

$$\sum_{d \mid b_n} \mu(d) = \begin{cases} 0 \text{ when } b_n \neq 1 \\ 1 \text{ when } b_n = 1\ , \end{cases} \tag{2.1}$$

and in [4] that

$$\phi(x, b_u) = \sum_{d \mid b_u} \mu(d)\left[\frac{x}{d}\right]\ . \tag{2.2}$$

Hence using assumption (1.3) we have

$$\begin{aligned} \phi(x, b_u) &= x \sum_{d \mid b_u} \frac{\mu(d)}{d} + 0\left(x^{\alpha} \sum_{d \mid b_u} \frac{|\mu(d)|}{d^{\alpha}}\right)\ . \\ &= xf(b_u) + 0(x^{\alpha} f_{\alpha}(b_u)) \text{ say}\ . \end{aligned} \tag{2.3}$$

Then as is shown in [3], and in any case as the functions are multiplicative

$$(2.4)\qquad \begin{aligned} f(b_u) &= \sum_{d \mid b_u} \frac{\mu(d)}{d} = \prod_{p \mid b_u}\left(1 - \frac{1}{p}\right), \\ f_\alpha(b_u) &= \sum_{d \mid b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p \mid b_u}\left(1 + \frac{1}{p^\alpha}\right). \end{aligned}$$

Define $\zeta(s) = \sum_{n=1}^{\infty} b_n^{-s} (s > 1)$. Then it is proved in [1], using an assumption equivalent to (1.3) that

$$\zeta(s) = \prod_{r=1}^{\infty} (1 - p_r^{-s})^{-1} .$$

Hence

$$\frac{1}{\zeta(s)} = \prod_{r=1}^{\infty} (1 - p_r^{-s}) = \sum_{n=1}^{\infty} \mu_n b_n^{-s} .$$

Abel's transformation, in the following form, will be used to give some necessary estimates. Suppose $\{b_n\}$ and $\{a_n\}$ are given with $b_1 \leqq b_2 \leqq \cdots, b_n \to \infty$. Let $A(x) = \sum_{b_n \leqq x} a_n$. Suppose $\psi(x)$ has a continuous derivative $\psi'(x)$ for all x involved. Then

$$\sum_{b_n \leqq x} a_n \psi(b_n) = A(x)\psi(x) - \int_{b_1}^{x} A(u)\psi'(u)du .$$

Using (1.3) and this transformation, we obtain the following results.

$$(2.5)\qquad \sum_{b_n \leqq x} \frac{1}{b_n^\beta} = \frac{x^{1-b}}{1-\beta} + \gamma_\beta + 0(x^{\alpha-\beta}), \begin{cases} \beta \neq 1 \\ \beta \neq \alpha , \end{cases}$$

and γ_β is a constant equal to $\zeta(\beta)$ when $\beta > 1$.

$$(2.6)\qquad \sum_{b_n \leqq x} \frac{1}{b_n^\alpha} = \frac{x^{1-\alpha}}{1-\alpha} + 0(\log x) .$$

$$(2.7)\qquad \sum_{b_n > x} \frac{1}{b_n^\beta} = \zeta(\beta) - \sum_{b_n \leqq x} \frac{1}{b_n^\beta} = 0(x^{1-\beta}) \quad \text{for } \beta > 1 .$$

Again using (1.5) and Abel's transformation we obtain

$$(2.8)\qquad \sum_{p \leqq x} \frac{1}{p^\alpha} = \frac{x^{1-\alpha}}{(1-\alpha)\log x} + 0\left(\frac{x^{1-\alpha}}{\log^2 x}\right)$$

$$(2.9)\qquad \sum_{p \leqq x} \log p = x + 0(x/\log x) .$$

Define

$$(2.10)\qquad \lambda(b_u) = \sum_{\substack{b_n = 1 \\ (b_n, b_u) = 1}}^{\infty} \frac{\mu(b_n)}{b_n^2} = \prod_{\substack{p \\ p \nmid b_u}} \left(1 - \frac{1}{p^2}\right).$$

Then from (2.7) we have

(2.11) $$\sum_{\substack{b_n \leqq x \\ (b_n, b_u)=1}} \frac{\mu(b_n)}{b_n^2} = \lambda(b_u) + 0(x^{-1}) .$$

3. The Q function.

Let

$$q_u(b_n) = \begin{cases} 1 \text{ if } b_n \text{ is square free and } (b_n, b_u) = 1 \\ 0 \text{ otherwise .} \end{cases}$$

$$Q_u(x) = \sum_{b_n \leqq x} q_u(b_n) .$$

$$e(b_n) = \begin{cases} 1 \text{ if } b_n = 1 \\ 0 \text{ if } b_n \neq 1 . \end{cases}$$

Then from (2.1)

$$q_u(b_n) = e((b_n, b_u)) \sum_{d^2\delta = b_n} \mu(d) .$$

This gives

$$\begin{aligned} Q_u(x) &= \sum_{b_n \leqq x} e((b_n, b_u)) \sum_{d^2\delta = b_n} \mu(d) \\ &= \sum_{\substack{d^2\delta \leqq x \\ (d, b_u) = (\delta, b_u) = 1}} \mu(d) \\ &= \sum_{\substack{d \leqq \sqrt{x} \\ (d, b_u)=1}} \mu(d)\phi\left(\frac{x}{d^2}, b_u\right) \\ &= \sum_{\substack{d \leqq \sqrt{x} \\ (d, b_u)=1}} \mu(d)\left\{\frac{x}{d^2}f(b_u) + 0\left(\frac{x^\alpha}{d^{2\alpha}}f(b_u)\right)\right\} \end{aligned}$$

from (2.3)

$$= xf(b_u)\{\lambda(b_u) + 0(x^{-1/2})\} + 0\left(x^\alpha f_\alpha(b_u)\left\{\frac{x^{(1-2\alpha)/2}}{1 - 2\alpha} + \gamma_{2\alpha}\right\}\right)$$

from (2.11) and (2.5). Hence

(3.1) $$Q_u(x) = xf(b_u)\lambda(b_u) + 0(x^{1/2}f_\alpha(b_u)) + 0(x^\alpha f_\alpha(b_u)) .$$

4. The evaluation of the sum of the divisor function $T(x)$.

Replacing in (1.2) the value for $\tau'(b_n)$ defined in (1.1), we have the result that $T(x)$ is the number of elements in the class satisfying $d\delta = b_n$, $\mu^2(d) = \mu^2(\delta) = 1$, where $b_n \leqq x$, $(b_n, b_u) = 1$. This is the same class as that for which $d\delta \leqq x$, $(d, b_u) = (\delta, b_u) = 1$ and $\mu^2(d) = \mu^2(\delta) = 1$. Rearranging the order of summation we have that $T(x)$ is the number of elements in the class satisfying $\delta \leqq x/d$, $(\delta, b_u) = 1$, δ square free, where $d \leqq x$, $(d, b_u) = 1$ and d is square free. Hence

$$T(x) = \sum_{d \leqq x} q_u(d) \sum_{\delta \leqq x/d} q_u(\delta)$$
$$= \sum_{d \leqq x} q_u(d) \left\{ \frac{x}{d} f(b_u)\lambda(b_u) + 0\left(\frac{x^{1/2}}{d^{1/2}} f_\alpha(b_u)\right) + 0\left(\frac{x^\alpha}{d^\alpha} f_\alpha(b_u)\right)\right\}$$

from (3.1)

$$= xf(b_u)\lambda(b_u) \sum_{d \leqq x} \frac{q_u(d)}{d} + 0\left(x^{1/2} f_\alpha(b_u) \sum_{d \leqq x} \frac{1}{d^{1/2}}\right) + 0\left(x^\alpha f_\alpha(b_u) \sum_{d \leqq x} \frac{1}{d^\alpha}\right)$$
$$= xf(b_u)\lambda(b_u) \sum_{d \leqq x} \frac{q_u(d)}{d} + 0(xf_\alpha(b_u))$$

from (2.5) and (2.6).

Now from (3.1) and Abel's transformation we have

$$\sum_{d \leqq x} \frac{q_u(d)}{d} = f(b_u)\lambda(b_u) \log x + 0(f_\alpha(b_u)) + 0(x^{-1/2} f_\alpha(b_u)) + 0(x^{\alpha-1} f_\alpha(b_u)) .$$

Substituting this result in the expression for $T(x)$ we obtain

$$T(x) = f^2(b_u)\lambda^2(b_u) x \log x + 0(xf_\alpha(b_u)) . \tag{4.1}$$

From the definition in (2.4) we have

$$f_\alpha(b_u) = \sum_{d \mid b_u} \frac{|\mu(d)|}{d^\alpha} \leqq \sum_{d \mid b_u} \frac{1}{d^\alpha} \leqq \sum_{d \mid b_u} 1 = 0(b_u^\delta) \tag{4.2}$$

where δ is any positive real number. This is proved in [5, Th. 5] and is true for all b_u. However, when b_u is square free we can obtain a better value for $f_\alpha(b_u)$ by using the prime number theorem. Suppose b_u is square free and let $b_u = p_{u1} p_{u2} \cdots p_{uk} \geqq p_1 p_2 \cdots p_k$. Then

$$\log b_u \geqq \sum_{p \leqq p_k} \log p = p_k + 0(p_k / \log p_k) \tag{4.3}$$

from (2.9). Hence

$$f_\alpha(b_u) = \sum_{d \mid b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p \mid b_u} \left(1 + \frac{1}{p^\alpha}\right) \leqq \prod_{p \leqq p_k} \left(1 + \frac{1}{p^\alpha}\right)$$
$$\leqq \prod_{p \leqq (1+o(1)) \log b_u} \left(1 + \frac{1}{p^\alpha}\right)$$

from (4.3), and so

$$\log f_\alpha(b_u) \leqq \sum_{p \leqq (1+o(1)) \log b_u} \frac{1}{p^\alpha}(1 + o(1)) .$$

Then from (2.8)

$$f_\alpha(b_u) = 0\left(\exp.\left\{\frac{(\log b_u)^{1-\alpha}}{\log\log b_u}\right\}\right) \tag{4.4}$$

for b_u square free. Now from (2.4) and (2.10)

$$\begin{aligned} f^2(b_u)\lambda^2(b_u) &= \prod_{p \mid b_u}\left(1 - \frac{1}{p}\right)^2 \prod_{\substack{p \\ p \nmid b_u}}\left(1 - \frac{1}{p^2}\right)^2 \\ &= \prod_{p \mid b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4} \\ &= A \text{ (say)} . \end{aligned}$$

Hence from (4.1), (4.2) and (4.4) we have

$$T(x) = A\,x \log x + 0(xb_u^\delta) \tag{4.5}$$

for all b_u and all positive real numbers δ and

$$T(x) = Ax \log x + 0\left(x \exp\left\{\frac{(\log b_u)^{1-\alpha}}{\log\log b_u}\right\}\right) \tag{4.6}$$

for all square free b_u.

This is the result given for $T(x)$ in (1.4). Since

$$\prod_p \left(1 - \frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)} ,$$

the value for A may also be written

$$A = \frac{1}{\zeta^2(2)} \prod_{p \mid b_u} \frac{p^2}{(p+1)^2} .$$

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