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A PHRAGMEN-LINDELÖF THEOREM FOR FUNCTION ALGEBRAS

I. GLICKSBERG

Let A be a function algebra, considered as a closed subalgebra of $C(\mathfrak{M})$, where $\mathfrak{M}$ is the space of multiplicative linear functionals on A. Let ∂ denote the Šilov boundary of A. We shall call $\mathfrak{M}\backslash\partial$ the "interior of $\mathfrak{M}$" and say a function g on this "interior" is A-holomorphic if each φ in $\mathfrak{M}\backslash\partial$ has a neighborhood on which g is uniformly approximable by elements of A.

What we shall observe here is that results of the Phragmén-Lindelöf type apply to certain A-holomorphic functions.

These results follow easily from the type of argument used in an earlier paper [1] in which function-algebra analogues of some classical results of function theory were obtained; the present note is essentially an addendum to [1] (where "A-holomorphic" [3] was "locally approximable"). Other results of the Phragmén-Lindelöf type have been obtained by Quigley [2].

Our analogue of the usual Phragmén-Lindelöf result replaces the point at infinity by a peak set lying in the Šilov boundary.

THEOREM 1. *Suppose $f \in A$ peaks on $F \subset \partial$, and g is an A-holomorphic function defined and continuous on $\mathfrak{M}\backslash F$. Suppose g is bounded on $\partial\backslash F$ and for some α, $0 < \alpha < 1$, and $k > 0$*

$$g \exp\left(\frac{-k}{|1-f|^{\alpha}}\right) \tag{1}$$

is bounded on the interior of $\mathfrak{M}$. Then g is bounded on $\mathfrak{M}\backslash F$ by its bound on $\partial\backslash F$.

Thus an unbounded A-holomorphic function continuous on $\mathfrak{M}\backslash F$ cannot increase too slowly as we approach F. Actually g need only be defined on $\mathfrak{M}\backslash\partial$ (and A-holomorphic) if we replace $\partial\backslash F$ by a deleted neighborhood of it in $\mathfrak{M}$.

THEOREM 2. *With f, F and α as above, let g be an A-holomorphic function which is bounded on the intersection V of a neighborhood of $\partial\backslash F$ with the interior of $\mathfrak{M}$, and suppose* (1) *holds. Then g is bounded by its bound on V.*

Both of these results are easy consequences of the local maximum modulus principle [4] and classical arguments. A little more is needed for the following extension of the Phragmén-Lindelöf corollary concerning a bounded analytic function on a sector having a limit as $z \to \infty$ along the bounding rays.

THEOREM 3. *Suppose g is a bounded function on $\mathfrak{M}$ which is A-holomorphic, has its restriction to ∂ continuous, and in fact is continuous at each point of $\partial \setminus \bigcup_{n=1}^{\infty} K_n$, where K_n is a zero set of A lying in the*[1] *Choquet boundary. Then g is continuous on $\mathfrak{M}$.*

Thus we cannot have too small a set of discontinuities for an A-holomorphic function which has a continuous restriction to the Šilov boundary and also is continuous at a fairly large set of points in ∂.

As a mixture of Theorems 1 and 3 we obtain

COROLLARY 4. *Suppose g is a (not necessarily bounded) function on $\mathfrak{M}$ which is A-holomorphic, has its restriction to ∂ continuous and is continuous at each point of $\partial \setminus \bigcup_{n=1}^{\infty} K_n$, where K_n is a zero set of A lying in the Choquet boundary. Suppose $f \in A$ peaks on K_1 while* (1) *is bounded on the interior of $\mathfrak{M}$. Then g is continuous on $\mathfrak{M}$.*

Proofs. Our proof of Theorem 1 is simply an imitation of a classical argument [5]. To begin let $\alpha < \beta < 1$; noting that

$$|\arg(1 - f)| \leqq \pi/2 ,$$

we have an element $(1 - f)^\beta$ in A (where we apply the principal branch of z^β to $1 - f$, so $|\arg(1 - f)^\beta| \leqq \beta\pi/2 < \pi/2$). Now fix β and $\varepsilon > 0$. For $\operatorname{Re} z \geqq 0$ and $z = re^{i\theta} \neq 0$ ($|\theta| \leqq \pi/2$)

$$(2) \qquad \begin{aligned} \left|\exp\left(-\frac{\varepsilon}{z^\beta}\right)\exp\frac{k}{|z|^\alpha}\right| &= \exp(-\varepsilon r^{-\beta}\cos\beta\theta + kr^{-\alpha}) \\ &= \exp\{-r^{-\beta}(\varepsilon\cos\beta\theta - kr^{\beta-\alpha})\} \\ &\leqq \exp(-cr^{-\beta}) \end{aligned}$$

for some $c > 0$ if r is sufficiently small, and this of course implies (2) is bounded on $\operatorname{Re} z \geqq 0$. Thus

[1] The Choquet boundary consists of all points in the Šilov boundary having unique representing measures. In the metric case it coincides with the set of peak points.

$$\exp\left(-\frac{\varepsilon}{(1-f)^{\beta}}\right)\exp\left(\frac{k}{|1-f|^{\alpha}}\right) \tag{3}$$

is bounded on $\mathfrak{M}\backslash F$, whence

$$g\exp\left(-\frac{\varepsilon}{(1-f)^{\beta}}\right) \tag{4}$$

is bounded on $\mathfrak{M}\backslash F$ as the product of (1) and (3). But the exponential in (4), and thus (4) itself, is A-holomorphic and we can argue that by [1, Th, 4.8], (4) is bounded on $\mathfrak{M}\backslash F$ by its bound over $\partial\backslash F$, hence by $\sup|g(\partial\backslash F)|$ since the exponential is of modulus $\leqq 1$. So for any φ in $\mathfrak{M}\backslash F$ we have

$$\left|g(\varphi)\exp\left(-\frac{\varepsilon}{(1-f(\varphi))^{\beta}}\right)\right| \leqq \sup|g(\partial)\backslash F)|\ , \tag{5}$$

and letting $\varepsilon \rightarrow 0$ yields the desired result.

Actually, once we have seen (4) is a bounded A-holomorphic function we should appeal directly to Rossi's local maximum modulus principle [4] to obtain (5). Indeed, extend (4) to all of $\mathfrak{M}$ by setting it equal to zero on F; since (3) tends to zero as we approach F (by (2)) we obtain a continuous function h on $\mathfrak{M}$. Now let B be the closed subalgebra of $C(\mathfrak{M})$ generated by h and A. To obtain (5) we need only see $\partial_B \subset \partial$ since then

$$|h(\varphi)| \leqq \sup|h(\partial_B)| \leqq \sup|h(\partial)| = \sup|h(\partial\backslash F)|\ ,$$

because h vanishes on F, and this is (5).

We now argue exactly as in [1, 3.2]: if $\varphi \in \partial_B \cap (\mathfrak{M}\backslash\partial)$ we choose a neighborhood U_φ of φ in $\mathfrak{M}\backslash\partial$ on which h (and thus any element of B) is uniformly approximable by elements of A. Since $\varphi \in \partial_B$ we must have a φ' in U_φ and an h' in B with

$$|h'(\varphi')| > \sup|h'(\text{bndry } U_\varphi)| \tag{6}$$

and thus this holds for some approximating element h'' in A. But that violates the local maximum modulus principle, so $\partial_B \cap (\mathfrak{M}\backslash\partial) = \varnothing$, and $\partial_B \subset \partial$.

This argument yields a simple proof of Theorem 2. In that result, as is now apparent, we need only show the function

$$h = g\exp\left(\frac{-\varepsilon}{(1-f)^{\beta}}\right)$$

on $\mathfrak{M}\backslash\partial$ is bounded by its bound on V.

Now choose a deleted neighborhood W of F on which

$$|h| < \sup |h(V)| + \eta$$

(where $\eta > 0$), which is possible since $h \to 0$ as we approach F, exactly as before. Removing $(V \cup W)^-$ from the interior $\mathfrak{M}\backslash\partial$ we obtain an open subset U of $\mathfrak{M}\backslash\partial$ with $U^- \cap \partial = \emptyset$ so that bndry $U \subset (V \cup W)^-$. With B now the closed subalgebra of $C(U^-)$ generated by A and h we see that $\partial_B \subset$ bndry U by just the above application of local maximum modulus. Hence $\partial_B \subset (V \cup W)^-$, so that

$$\sup |h(\mathfrak{M}\backslash\partial)| \leqq \sup |h(V \cup W)| \leqq \sup |h(V)| + \eta ;$$

since $\eta > 0$ is arbitrary, this shows h is bounded by its bound over V, as desired.

We can now proceed to the proof of Theorem 3, which involves some modifications in the arguments of [1, §4]. Let B_0 denote the uniformly closed algebra of bounded functions on $\mathfrak{M}$ generated by g and A; trivially $\mathfrak{M}$ can be viewed as a subset of $\mathfrak{M}_{B_0}$ and we let X denote the closure of $\mathfrak{M}$ in $\mathfrak{M}_{B_0}$. X is a boundary for B_0, so $B = B_0^\wedge | X$ is a closed subalgebra of $C(X)$.

Since g and the elements of A are continuous when restricted to either ∂ or $\mathfrak{M}\backslash\partial$, the natural injection of each of these spaces into X is continuous, and of course one-to-one. In particular then the compact space ∂ is imbedded homeomorphically in X. But in fact the same is true of $\mathfrak{M}\backslash\partial$ since the map ρ: $X \to \mathfrak{M}$ dual to $A \to B$ clearly provides inverses for the injections $\partial \to X$, $\mathfrak{M}\backslash\partial \to X$. (Note that $\hat{f}(x) = f(\rho(x))$ for $f \in A$, $x \in X$.)

Now each of the sets $\mathfrak{M}\backslash\partial$ and $\partial\backslash(\mathfrak{M}\backslash\partial)^-$ is imbedded as an open subset of X. To see this note that each φ_0 in $\mathfrak{M}\backslash\partial$ has a compact neighborhood in $\mathfrak{M}$ disjoint from ∂ of the form

$$U = \{\varphi \in \mathfrak{M}: |f_i(\varphi) - f_i(\varphi_0)| \leqq \varepsilon, \qquad i = 1, \cdots, n\} ;$$

since $X = (\mathfrak{M}\backslash U)^- \cup U^- = (\mathfrak{M}\backslash U)^- \cup U$, $x \in X\backslash U$ implies $x \in (\mathfrak{M}\backslash U)^-$, and so $|\hat{f}_i(x) - f_i(\varphi_0)| \geqq \varepsilon$ for some i, whence

$$\begin{aligned} W_{\varphi_0} &= \{\varphi \in \mathfrak{M}: |f_i(\varphi) - f_i(\varphi_0)| < \varepsilon/2, i = 1, \cdots, n\} \\ &= \{x \in X: |\hat{f}_i(x) - f_i(\varphi_0)| < \varepsilon/2, i = 1, \cdots, n\} \end{aligned}$$

is a neighborhood of φ_0 in X lying wholly within $\mathfrak{M}\backslash\partial$, so $\mathfrak{M}\backslash\partial$ is open in X as asserted. The same argument, starting from a compact neighborhood in ∂ disjoint from $(\mathfrak{M}\backslash\partial)^-$, yields a neighborhood W_{φ_0} of $\varphi_0 \in \partial\backslash(\mathfrak{M}\backslash\partial)^-$ in X lying wholly in $\partial\backslash(\mathfrak{M}\backslash\partial)^-$, so this set is also open in X. Moreover, the existence of W_{φ_0} shows ρ is one-to-one over $\mathfrak{M}\backslash\partial$ and $\partial\backslash(\mathfrak{M}\backslash\partial)^-$. For $\hat{f}_i(x) = \hat{f}_i(\rho(x))$, $x \in X, f_i \in A$, so $\rho(x) \in \mathfrak{M}\backslash\partial$ implies $x \in W_{\rho(x)} \subset \mathfrak{M}\backslash\partial$; similarly $\rho(x) \in \partial\backslash(\mathfrak{M}\backslash\partial)^-$ implies $x \in \partial\backslash(\mathfrak{M}\backslash\partial)^-$. So

$$\rho^{-1}(\mathfrak{M}\backslash\partial) = \mathfrak{M}\backslash\partial, \rho^{-1}(\partial\backslash(\mathfrak{M}\backslash\partial)^-) = \partial\backslash(\mathfrak{M}\backslash\partial)^- ,$$

and thus ρ is clearly one-to-one over these sets.

Since $\mathfrak{M}\backslash\partial$ is open in X local maximum modulus applies to show $\partial_B \cap (\mathfrak{M}\backslash\partial) = \emptyset$ exactly as in [1, 3.2] or in our proof of Theorem 1: for any $\varphi \in \partial_B \cap (\mathfrak{M}\backslash\partial)$ has a neighborhood U_φ in $\mathfrak{M}\backslash\partial$ on which g (and so any element of B) is uniformly approximable by elements of A; since $\mathfrak{M}\backslash\partial$ is open in X, U_φ is open in X and thus we find φ' in U_φ and h' in B satisfying (6) since $\varphi \in \partial_B$, and this contradicts local maximum modulus exactly as in the proof of Theorem 1. Thus

$$\partial_B \cap (\mathfrak{M}\backslash\partial) = \emptyset ,$$

and since $\rho^{-1}(\mathfrak{M}\backslash\partial) = \mathfrak{M}\backslash\partial$, we conclude that $\rho(\partial_B) \subset \partial$.

To complete our proof we need only see ρ is one-to-one on X: for then ρ is a homeomorphism of X with $\mathfrak{M}$ (since $\rho(X) \subset \mathfrak{M}$ and $\rho(X) \supset (\mathfrak{M}\backslash\partial) \cup (\partial\backslash(\mathfrak{M}\backslash\partial)^-)$), and continuity of $g \circ \rho = \hat{g}$ on X implies that of g on $\mathfrak{M} = \rho(X)$.

We have already seen $\rho^{-1}(x) = \{x\}$ for x in $(\mathfrak{M}\backslash\partial) \cup (\partial\backslash(\mathfrak{M}\backslash\partial)^-)$, and for x in $\partial\backslash\bigcup K_n$ the assumed continuity of g at x implies $\rho^{-1}(x) = \{x\}$: for each h in B_0 is continuous at x, and so if $\rho(y) = x$ and the net $\{\varphi_\delta\}$ in $\mathfrak{M}$ converges to y in X then $\rho(\varphi_\delta) = \varphi_\delta \to \rho(y) = x$ in $\mathfrak{M}$, whence $\hat{h}(y) = \lim \hat{h}(\varphi_\delta) = \hat{h}(x)$ for all h in B_0, and $y = x$. Thus we need only see $\rho(y) = x$ for x in K_n implies $y = x$, and since we know this holds for x in $\partial\backslash(\mathfrak{M}\backslash\partial)^-$, we can assume $x \in (\mathfrak{M}\backslash\partial)^-$ as well.

So suppose $\rho(y) = x \in K_n \cap (\mathfrak{M}\backslash\partial)^-$. Since K_n lies in the Choquet boundary of A, only the unit point mass δ_x at x, among all probability measures on ∂, can represent x on A. Thus if we knew $\partial_B = \partial$ then any probability measure μ on $\partial_B = \partial$ representing y on B would necessarily represent $\rho(y) = x$ on A, whence $\mu = \delta_x$ and $y = x$.

So we need only see $\partial_B\backslash\partial = \emptyset$ (since clearly $\partial \subset \partial_B$). As we saw, $\rho(\partial_B) \subset \partial$, and ρ is one-to-one over $(\partial\backslash\bigcup K_n) \cup (\partial\backslash(\mathfrak{M}\backslash\partial)^-)$ so that

$$\rho(\partial_B\backslash\partial) \subset (\mathfrak{M}\backslash\partial)^- \cap (\bigcup K_n) .$$

So by category if $\partial_B\backslash\partial \neq \phi$ one of the closed sets

$$E_n = \rho^{-1}[K_n \cap (\mathfrak{M}\backslash\partial)^-] \cap (\partial_B\backslash\partial)$$

in the locally compact space $\partial_B\backslash\partial$ has nonvoid interior in $\partial_B\backslash\partial$, hence in ∂_B. But $K_n = g_n^{-1}(0)$, $g_n \in A$ so that $y \in E_n$ lies in $\hat{g}_n^{-1}(0) = (g_n \circ \rho)^{-1}(0)$. In fact y lies in the topological boundary in X of $\hat{g}_n^{-1}(0)$. For

$$\rho(y) \in (\mathfrak{M}\backslash\partial)^-, y \notin \partial ,$$

and thus y has a neighborhood in X disjoint from ∂, whence y lies in the closure in X of $\mathfrak{M}\backslash\partial$ (since $(\mathfrak{M}\backslash\partial) \cup \partial$ is dense in X). But $\hat{g}_n^{-1}(0) \cap (\mathfrak{M}\backslash\partial) = g_n^{-1}(0) \cap (\mathfrak{M}\backslash\partial) = \phi$, so that y lies in the topological boundary of $\hat{g}_n^{-1}(0)$ as asserted.

Thus we have seen that E_n has nonvoid interior in ∂_B and lies in the topological boundary of $\hat{g}_n^{-1}(0)$ in X, which contradicts [1, 2.2]. Our assumption that $\partial_B \backslash \partial$ is nonvoid must therefore be false, and $\partial_B = \partial$ as desired, completing our proof.

Corollary 4 follows directly from the preceding. Indeed if we set

$$h = \begin{cases} g \exp\left(\dfrac{-\varepsilon}{(1-f)^{\beta}}\right) & \text{on } \mathfrak{M}\backslash K_1 \\ 0 & \text{on } K_1, \qquad \alpha < \beta < 1 , \end{cases}$$

then $h \mid \partial$ is continuous and Theorem 3 implies $h \in C(\mathfrak{M})$. So h is bounded by its bound over ∂, exactly as in the proof of Theorem 1, and so we see the same is true of g. Hence by Theorem 3, $g \in C(\mathfrak{M})$.

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UNIVERSITY OF WASHINGTON