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MULTIPLY TRANSITIVE GROUPS OF TRANSFORMATIONS

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A group G of homeomorphisms of a topological space X onto itself is called n-transitive if any set of n points in X can be mapped onto any other set of n points by some member of G. In this paper, we investigate the transitivity of G when X is euclidean m-space E^m or real projective m-space Π^m, and G properly contains the group A_m of affine transformations or the group P_m of projective transformations, respectively. We show that $G \supset A_1$ implies that G is at least 3-transitive, $G \supset P_1$ implies that G is at least 4-transitive, and, for a fairly wide class of groups, G is n-transitive for every n. For higher dimensional spaces, our information is considerably more meager. We show that $G \supset A_m$ or $G \supset P_m$ implies that G is at least 3-transitive, and that if some member of G leaves fixed the points of some open set, then G is n-transitive for every n.

2. **Multiple transitivity.** Let X be a topological space and $H(X)$ the group of all homeomorphisms of X onto itself. The identity of $H(X)$ will be denoted by e. For each $h \in H(X)$, we set $K(h) = \{x \in X: h(x) = x\}$, and observe that

$$K(h_1h_2) \supset K(h_1) \cap K(h_2) , \qquad K(h_1h_2h_1^{-1}) = h_1(K(h_2)) .$$

For any subgroup G of $H(X)$ and any $x \in X$, we call $G(x) = \{g(x): g \in G\}$ an orbit of G and note that orbits are either coincident or disjoint. When n is a positive integer, we define G to be n-transitive if, for any subsets $\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\}$ of n distinct points in X, we can find $g \in G$ such that $g(x_i) = y_i$ $(i = 1, \cdots, n)$. If g is unique, we call G strictly n-transitive. If G is n-transitive for every n, we will call G ω-transitive. When X is a connected, locally euclidean manifold of dimension $m \geqq 2$, then $H(X)$ is clearly ω-transitive, but $H(E^1)$ is only 2-transitive, and $H(\Pi^1)$ is only 3-transitive under the above definition. To remedy this, we will modify the definition in these two cases by requiring that as i increases from 1 to n, x_i should move in the positive sense of orientation, and y_i should move in either the positive or negative sense. Thus $H(X)$ is also ω-transitive when $X = E^1$ or Π^1. The group $H^+(X)$ of orientation-preserving homeomorphisms of X evidently sends any positively oriented n-tuple into any other positively oriented n-tuple for every n. We will say that a subgroup G of $H^+(X)$ is n-transitive relative to $H^+(X)$ if G sends any positively oriented n-tuple into any other positively oriented n-tuple.

LEMMA 1. *Let X be a topological space and G a subgroup of $H(X)$. Suppose that, for each subset L of n points in X and each $x \in X - L$, the orbit $G_0(x)$ of the group $G_0 = \{g \in G: L \subset K(g)\}$ has a nonempty interior in X. Then $G_0(x)$ contains a connected component of $X - L$.*

Proof. Let $U \subset G_0(x)$ be an open subset of X, and $y \in G_0(x)$ be arbitrary. Then we can find $g_1, g_2 \in G_0$ with the properties $g_1(x) \in U$ and $g_2(x) = y$. Thus $y = g_2(x) \in g_2 g_1^{-1}(U) \subset G_0(x)$, and y lies in the interior of $G_0(x)$, so that $G_0(x)$ is open. The orbits of G_0 are either coincident or disjoint, and no two of them can intersect the same connected component of $X - L$ unless they coincide. Since $e \in G_0$, we have $x \in G_0(x)$, and the orbits $G_0(x)$ cover $X - L$. Hence, each of them contains a connected component.

LEMMA 2. *With the same hypotheses as in Lemma 1, suppose X is a connected, locally euclidean manifold of dimension $m \geqq 2$, and G is n-transitive for some n. Then G is $(n + 1)$-transitive.*

Proof. To show that G is $(n + 1)$-transitive, it is evidently sufficient to show that, for any points $x_1, \cdots, x_{n+1}, y_{n+1} \in X$, there is a $g \in G$ satisfying $g(x_i) = x_i$ $(i = 1, \cdots, n)$ and $g(x_{n+1}) = y_{n+1}$. Since $X - \{x_1, \cdots, x_n\}$ is connected, this is precisely the conclusion of Lemma 1.

LEMMA 3. *With the same hypotheses as in Lemma 1, suppose $X = E^1$, G is n-transitive for some $n \geqq 2$, and the condition "$x \in X - L$" is replaced by "x lies to the right of L". Then G is $(n + 1)$-transitive. If $G \subset H^+(E^1)$ is n-transitive $(n \geqq 0)$ relative to $H^+(E^1)$, then G is $(n + 1)$-transitive relative to $H^+(E^1)$.*

Proof. Let $x_1 < \cdots < x_{n+1}$ and either (i) $y_1 < \cdots < y_{n+1}$ or (ii) $y_1 > \cdots > y_{n+1}$ be given. In case (i), we choose $g_1 \in G$ so that $g_1(x_i) = y_i$ $(i = 1, \cdots, n)$. Since g_1 is order-preserving, we have $g_1(x_{n+1}) > y_n$, and the same argument as in the proof of Lemma 1 shows that the orbit $G_0(g_1(x_{n+1}))$ is the open interval (y_n, ∞), where $G_0 = \{g \in G: \{y_1, \cdots, y_n\} \subset K(g)\}$. Thus we can find $g_2 \in G_0$ satisfying $g_2(g_1(x_{n+1})) = y_{n+1}$, so that $g_2 g_1(x_i) = y_i$ $(i = 1, \cdots, n + 1)$. This also suffices to prove the last statement in the Lemma. In case (ii), we choose $g_3 \in G$ so that $g_3(x_i) = y_i$ $(i = 2, \cdots, n + 1)$. From $n \geqq 2$ we infer that g_3 is order-reversing, whence $g_3(x_1) > y_2$, and we can find $g_4 \in G$ satisfying $y_i \in K(g_4)$ $(i = 2, \cdots, n + 1)$ and $g_4(g_3(x_1)) = y_1$. Thus $g_4 g_3(x_i) = y_i$ $(i = 1, \cdots, n + 1)$.

If, in the hypothesis of Lemma 3, "x lies to the right of L" is replaced by "x lies to the left of L", then an argument similar to the preceding one yields the same conclusions.

3. Extensions of finite sets. Let L be a finite subset of an arbitrary subset M of a topological space X, and G a subgroup of $H(X)$. We set $M_0 = M$ and, for $i \geqq 0$,

$$M_{i+1} = \bigcup \{g(M_i) \cup g^{-1}(M_i) : g \in G \text{ and } g(L) \subset M_i\} .$$

Since $e \in G$ and $L \subset M_0$, we have $M_0 \subset M_1$ and, in general, $M_i \subset M_{i+1}$. Thus $\{M_i\}$ is an increasing family of sets, and we shall call its union N the extension of M with respect to L and G. We observe that if $g \in G$ and $g(L) \subset N$, then $g(N) = N$. For $g(L)$ is finite and so is contained in some M_k, whence $g(M_i) \subset M_{i+1}$ and $g^{-1}(M_i) \subset M_{i+1}$ for each $i \geqq k$. Hence, $g(N) \subset N$, $g^{-1}(N) \subset N$, and $g(N) = N$.

LEMMA 4. *Suppose X is a Hausdorff space, L has n points, G is n-transitive and has the property that, for any net $\{g_k\}$ in G and any $g \in G$, $\lim_k g_k(x) = g(x)$ for all $x \in L$ implies*

$$\lim_k g_k(x) = g(x) , \qquad \lim_k g_k^{-1}(x) = g^{-1}(x) , \qquad x \in X .$$

Then $g(L) \subset \bar{N}$ implies $g(\bar{N}) = \bar{N}$, where $\bar{N}$ is the closure of N.

Proof. If $L = \{x^1, \cdots, x^n\}$ and $g(L) \subset \bar{N}$, then we can find a net $\{(x_k^1, \cdots, x_k^n)\}$ of n-tuples in N such that $\lim_k x_k^i = g(x^i)$ $(i = 1, \cdots, n)$. The n-transitivity of G implies that there are elements $g_k \in G$ satisfying $g_k(x^i) = x_k^i$ for each i and k. Thus

$$\lim_k g_k(x^i) = \lim_k x_k^i = g(x^i) , \qquad i = 1, \cdots, n$$

implies

$$\lim_k g_k(x) = g(x) , \qquad \lim_k g_k^{-1}(x) = g^{-1}(x) , \qquad x \in X .$$

From the remark preceding the lemma, $g_k(L) \subset N$ implies $g_k(x)$, $g_k^{-1}(x) \in N$ for $x \in N$, whence $g(x), g^{-1}(x) \in \bar{N}$ for $x \in N$. Consequently, $g(N) \subset \bar{N}$, $g(\bar{N}) \subset \bar{N}$, $g^{-1}(N) \subset \bar{N}$, $g^{-1}(\bar{N}) \subset \bar{N}$, and $g(\bar{N}) = \bar{N}$.

LEMMA 5. *Let X be m-dimensional euclidean space E^m, G the group A_m of affine transformations defined on E^m, L consist of $m + 1$ points which do not lie on any $(m - 1)$-dimensional hyperplane, and $M \supset L$ consist of $m + 2$ points. Then N is dense in E^m.*

Proof. We recall that the elements a of A_m have the form

$a(x) = t + Tx$, where $t \in E^m$, and T is a nonsingular linear transformation of E^m onto itself. Moreover, A_m is strictly $(m+1)$-transitive on $(m+1)$-tuples which do not lie on any $(m-1)$-dimensional hyperplane. We first consider the case $m = 1$. The hypothesis of Lemma 4 is clearly satisfied with $n = 2$. Let $L = \{x_1, x_2\}$ and $M = \{x_1, x_2, x_3\}$. Evidently we can arrange the indices so that either (i) $x_1 < x_2$, $x_1 < x_3$ or (ii) $x_1 > x_2$, $x_1 > x_3$. We will complete the proof for case (i); case (ii) is handled in exactly the same way. Choose $a_1 \in A_1$ so that $a_1(x_1) = x_1$ and $a_1(x_2) = x_3$. Then $a_1(L) \subset N$, and the remark preceding Lemma 4 implies that $a_1(N) = N$. Indeed, $a_1^k(N) = N$ for any integer k, where a_1^k is the k-th iterate of a_1. Now a_1 is order-preserving and has just one fixed point at x_1, so that $\{a_1^k(x_2): -\infty < k < +\infty\}$ has x_1 and $+\infty$ as limit points. In other words, N contains a sequence which converges to x_1 from the right and another which converges to $+\infty$. If $\bar{N} \neq E^1$, then $E^1 - \bar{N}$ is the union of disjoint open intervals. Let $I = (\lambda, \mu)$ be one of these, where we allow $\lambda = -\infty$ or $\mu = +\infty$. If $\lambda \neq -\infty$, we can find $a_2 \in A_1$ satisfying $a_2(x_1) = \lambda$ and $\lambda < a_2(x_2) \in N$, whence a_2 is order-preserving, $a_2(L) \subset \bar{N}$, $a_2(\bar{N}) = \bar{N}$, and $a_2^{-1}(I) \subset E^1 - \bar{N}$. But $a_2^{-1}(\lambda)$ is the left endpoint of $a_2^{-1}(I)$, while $a_2^{-1}(\lambda) = x_1$ has a sequence in $\bar{N}$ converging to it from the right, so that part of this sequence must lie in $a_2^{-1}(I)$, which is impossible. If $\lambda = -\infty$, then $\mu \leqq x_1$, and we choose $a_3 \in A_1$ so that $a_3(x_2) = x_2$, $x_1 < a_3(x_1) \in N$, and $a_3(x_1) < x_2$. Thus a_3 is order-preserving, $a_3(L) \subset N$, $a_3(N) = N$, and $a_3(I) \subset E^1 - \bar{N}$. But $a_3(\mu) > \mu$, and $a_3(\mu)$ is the right endpoint of $a_3(I)$, whence $\mu \in a_3(I)$, which is impossible. Therefore, $\bar{N} = E^1$.

We now proceed by induction on m. Suppose the lemma has been proved in all dimensions less than a certain m,

$$L = \{x_1, \cdots, x_{m+1}\} \subset \{x_0, x_1, \cdots, x_{m+1}\} = M \subset E^m ,$$

and L does not lie on any $(m-1)$-dimensional hyperplane. We can arrange the indices in L so that either (i) x_0 lies on the $(m-1)$-dimensional hyperplane X determined by $x_2, \cdots, x_{m+1}$, or (ii) x_0 and x_1 lie on the same side of X. To see this, we set up a coordinate system in E^m in which the points of L are the origin and unit points on the coordinate axes. If each point of L lay on the side opposite x_0 of the $(m-1)$-dimensional hyperplane through the remaining points of L, then all the coordinates of x_0 would be negative, while x_0 lay on the side opposite the origin of the hyperplane through the unit points, which is impossible. In case (ii), choose $a_0 \in A_m$ so that $a_0(x_1) = x_0$ and $a_0(x_i) = x_i$ $(i = 2, \cdots, m+1)$. We will show that $x_1, a_0(x_1)$, and $a_0^2(x_1)$ are collinear. Since $K(a_0) = X$, we can refer $a_0(x) = t_0 + T_0x$ to a coordinate system in E^m relative to which $x_1 = (0, \cdots, 0, 1)$, X is the set of points with last coordinate 0, $t_0 = (0, \cdots, 0)$, and T_0 has the form

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \alpha_1 \\ 0 & 1 & 0 & \cdots & \alpha_2 \\ 0 & 0 & 1 & \cdots & \alpha_3 \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_m \end{pmatrix}, \qquad \alpha_m > 0 .$$

Thus we have

$$\begin{aligned} a_0(x_1) &= (\alpha_1, \cdots, \alpha_{m-1}, \alpha_m) , \\ a_0^2(x_1) &= (\alpha_1(1 + \alpha_m), \cdots, \alpha_{m-1}(1 + \alpha_m), \alpha_m^2) , \\ a_0(x_1) - x_1 &= (\alpha_1, \cdots, \alpha_{m-1}, \alpha_m - 1) , \\ a_0^2(x_1) - a_0(x_1) &= (\alpha_1\alpha_m, \cdots, \alpha_{m-1}\alpha_m, (\alpha_m - 1)\alpha_m) \\ &= \alpha_m(a_0(x_1) - x_1) , \end{aligned}$$

whence x_1, $a_0(x_1) = x_0$, and $a_0^2(x_1) = a_0(x_0) = y_0$ are collinear, and $y_0 \neq x_0, x_1$. We will show next that there is a subset L' of M which contains x_0, x_1, and $m - 1$ of the remaining m points of L, but which does not lie on any $(m - 1)$-dimensional hyperplane. If $L' = \{x_0, x_1, \cdots, x_m\}$ will not work, then let k be the least integer such that $2 \leqq k \leqq m$ and $\{x_0, x_1, \cdots, x_k\}$ lies on some $(k - 1)$-dimensional hyperplane, and set $L' = M - \{x_k\}$. Now if L' lay on an $(m - 1)$-dimensional hyperplane X_{m-1}, then the unique $(k - 1)$-dimensional hyperplane through $\{x_0, x_1, \cdots, x_{k-1}\}$ must contain x_k and lie in X_{m-1}, so that $M \subset X_{m-1}$, which is impossible. Hence, $L' = M - \{x_k\}$ satisfies our condition. Let x_j be a fixed element of $L' - \{x_0, x_1\}$, Y be the $(m - 1)$-dimensional hyperplane through $L'' = L' - \{x_j\}$, $M'' = L'' \cup \{y_0\}$, and $a_1 \in A_m$ map L onto L'. Since $\{y_0, x_0, x_1\}$ is collinear, and $x_0, x_1 \in L''$, we have $M'' \subset Y$. Now L'' contains m points, M'' contains $m + 1$ points, and the group B of elements in A_m which fix x_j and map Y onto itself acts on Y exactly like A_{m-1}. By our induction hypothesis, the extension N'' of M'' with respect to L'' and B is dense in Y. We will show that $\bigcup M_i'' = N'' \subset N = \bigcup M_i$ by showing inductively that $M_i'' \subset N$. First, $a_0(L) \subset M$ implies $y_0 = a_0(x_0) \in a_0(M) \subset N$, so that $M_0'' = M'' \subset N$. Suppose now that $M_i'' \subset N$ for some i, and $b(L'') \subset M_i''$ for some $b \in B$. Then $a_1(L) = L' \subset M$ implies $a_1(N) = N$, and

$$ba_1(L) = b(L') = \{x_j\} \cup b(L'') \subset \{x_j\} \cup M_i'' \subset N$$

implies $ba_1(N) = N$. Thus $b(N) = b(a_1(N)) = N$, $b(M_i'') \cup b^{-1}(M_i'') \subset N$, and $M_{i+1}'' \subset N$, so that $N'' \subset N$. Suppose $\{y_1, \cdots, y_{m-1}\}$ is a subset of N'' which does not lie in any $(m - 3)$-dimensional hyperplane. Since L'' does not lie on any $(m - 2)$-dimensional hyperplane, we can find an $x_i \in L''$ such that $\{x_i, y_1, \cdots, y_{m-1}\}$ does not lie on any $(m-2)$-dimensional hyperplane. Then $\{x_i, x_j, y_1, \cdots, y_{m-1}\}$ does not lie on any $(m-1)$-dimensional hyperplane, and we can find an $a_2 \in A_m$ which maps L' onto $\{x_i, x_j, y_1, \cdots, y_{m-1}\}$ in such a way that $a_2(x_i) = x_j$ and $a_2(x_j) = x_i$. From $a_2a_1(L) = a_2(L') \subset N$, we infer that $a_2a_1(N) = N$ and $a_2(N) = a_2(a_1(N)) = N$, so that $a_2(N'') \subset N$. Now $a_2(N'')$ is a dense subset of $a_2(Y)$, and $a_2(Y)$ is an $(m-1)$-dimensional hyperplane through $\{x_j\}$ and $\{y_1, \cdots, y_{m-1}\}$. The union of such hyperplanes as $\{y_1, \cdots, y_{m-1}\}$ ranges over N'' is clearly dense in E^m, whence N is dense in E^m, and our main induction step is complete for case (ii). For case (i), the preceding argument becomes considerably simpler. We set

$$L'' = \{x_2, \cdots, x_{m+1}\} , \qquad M'' = \{x_0, x_2, \cdots, x_{m+1}\} ,$$

and let B be the set of elements in A_m which fix x_1 and map X onto itself. Then $N'' \subset N$, and N'' is dense in X. The last part of the argument with $L' = L$, $Y = X$, and $x_j = x_1$ shows that N is dense in E^m in this case as well.

LEMMA 6. *The conclusion of Lemma* 5 *remains valid if, in the hypothesis, we set* $m = 1$ *and replace* A_1 *with the group* A_1^+ *of order-preserving elements in* A_1.

Proof. We observe that all of the elements in A_1 which appear in the proof of Lemma 5 are order-preserving. The only other lemma used in that proof was Lemma 4 which assumes that G is 2-transitive. Although A_1^+ is only 2-transitive relative to $H^+(E^1)$, the net $\{g_k\}$ can still be found, if we recall that any pair of points which lies sufficiently close to a positively oriented pair is also positively oriented.

LEMMA 7. *Let* X *be a topological space,* L *consist of* n *points,* $L \subset M$, $f \in H(X)$, G *and* G' *be subgroups of* $H(X)$, *and* G' *have the property that if* $g' \in G'$ *and* $K(g')$ *contains* n *points, then* $g' = e$. *Suppose that, for every* $g \in G$, *there is a* $g' \in G'$ *such that* $fg(x) = g'f(x)$ *for all* $x \in M$. *Then* $fg(x) = g'f(x)$ *for all* x *in the extension* N *of* M *with respect to* L *and* G.

Proof. We will prove the result inductively for the sets $M = M_0, M_1, M_2, \cdots$. Suppose that, for every $g \in G$, there is a $g' \in G'$ such that $fg(x) = g'f(x)$ for all $x \in M_i$, and $g_1(L) \subset M_i$, where $g_1 \in G$. If $y \in L$, then $g_1(y) \in M_i$ and

$$(1) \qquad fg(g_1(y)) = g'f(g_1(y)) , \qquad y \in L .$$

We know that there are elements $g_1', g_2' \in G'$ satisfying

$$(2) \qquad fg_1(y) = g_1'f(y)\ , \qquad fgg_1(y) = g_2'f(y)\ , \qquad y \in M_i\ .$$

Combining (1) and (2) and recalling that $L \subset M_i$, we obtain

$$g_2'f(y) = fgg_1(y) = g'fg_1(y) = g'g_1'f(y)\ , \qquad y \in L\ .$$

Thus $f(y) \in K(g_2'^{-1}g'g_1')$, $f(L) \subset K(g_2'^{-1}g'g_1')$, and $f(L)$ contains n points, so that $g_2'^{-1}g'g_1' = e$ and $g_2' = g'g_1'$. From (2) we have

$$fgg_1(y) = g_2'f(y) = g'g_1'f(y) = g'fg_1(y)\ , \qquad y \in M_i\ ,$$

that is, $fg(x) = g'f(x)$ for all $x \in g_1(M_i)$. To see that $fg(x) = g'f(x)$ for all $x \in g_1^{-1}(M_i)$, we observe that $L \subset M_i$ implies

$$(3) \qquad fgg_1^{-1}(y) = g'fg_1^{-1}(y)\ , \qquad y \in g_1(L)\ .$$

We can also find elements $g_3', g_4' \in G'$ satisfying

$$(4) \qquad fg_1^{-1}(y) = g_3'f(y)\ , \qquad fgg_1^{-1}(y) = g_4'f(y)\ , \qquad y \in M_i\ .$$

From (3), (4), and $g_1(L) \subset M_i$ we obtain

$$g_4'f(y) = fgg_1^{-1}(y) = g'fg_1^{-1}(y) = g'g_3'f(y)\ , \qquad y \in g_1(L)\ .$$

Thus $fg_1(L) \subset K(g_4'^{-1}g'g_3')$ and $g_4' = g'g_3'$. Finally, from (4) we have

$$fgg_1^{-1}(y) = g_4'f(y) = g'g_3'f(y) = g'fg_1^{-1}(y)\ , \qquad y \in M_i\ ,$$

in other words, $fg(x) = g'f(x)$ for all $x \in g_1^{-1}(M_i)$. Therefore, $fg(x) = g'f(x)$ for all $x \in M_{i+1}$, and the induction step is complete.

LEMMA 8. *With the same hypotheses as in Lemma* 7, *suppose* $G = G'$ *and* $f(x) = x$ *for all* $x \in M$. *Then* $f(x) = x$ *for all* $x \in N$.

Proof. Again we proceed by induction on the sets M_i. Suppose $f(x) = x$ for all $x \in M_i$, and $g_1(L) \subset M_i$, where $g_1 \in G$. Then we can find $g_1' \in G$ such that

$$fg_1(x) = g_1'f(x) = g_1'(x)\ , \qquad x \in M_i\ .$$

Since $L, g_1(L) \subset M_i$, we have

$$g_1(y) = fg_1(y) = g_1'(y)\ , \qquad y \in L\ ,$$

whence $L \subset K(g_1^{-1}g_1')$ and $g_1 = g_1'$. Thus $fg_1(x) = g_1(x)$ for all $x \in M_i$, that is, $f(z) = z$ for all $z \in g_1(M_i)$. Similarly, there is a $g_2' \in G$ satisfying

$$fg_1^{-1}(x) = g_2'f(x) = g_2'(x)\ , \qquad x \in M_i\ ,$$

$$g_1^{-1}(y) = fg_1^{-1}(y) = g_2'(y)\ , \qquad y \in g_1(L)\ ,$$

so that $g_1^{-1} = g_2'$ and $fg_1^{-1}(x) = g_1^{-1}(x)$ for all $x \in M_i$. Therefore, $f(z) = z$ for all $z \in M_{i+1}$, and the induction step is complete.

THEOREM 1. *Suppose* $X = E^1$, L *consists of two points,* M *of three points,* $f \in H^+(E^1)$, *and, for every* $a \in A_1^+$, *there is an* $a' \in A_1^+$ *such that* $fa(x) = a'f(x)$ *for all* $x \in M$. *Then* $f \in A_1^+$.

Proof. The hypotheses of Lemma 7 are evidently satisfied when $n = 2$ and $G = G' = A_1^+$, whence $fa(x) = a'f(x)$ for all $x \in N$. By Lemma 6, N is dense in E^1, and the continuity of a, a', and f implies that $fa = a'f$, that is, $fA_1^+f^{-1} \subset A_1^+$. If we choose $a_1 \in A_1^+$ so that $a_1(0) = f(0)$, $a_1(1) = f(1)$, and set $f_1 = a_1^{-1}f$, then $0, 1 \in K(f_1)$ and $f_1A_1^+f_1^{-1} \subset A_1^+$. In particular, if we define $a_2(x) = 1 + x$ for $x \in E^1$, then $a_3 = f_1a_2f_1^{-1} \in A_1^+$. Now $K(a_3) = f_1(K(a_2)) = f_1(\emptyset) = \emptyset$, so that a_3 is also a translation, and $a_3(0) = 1$ implies $a_3 = a_2$. Thus $2 = a_3(1) = f_1a_2f_1^{-1}(1) = f_1(2)$, and $0, 1, 2 \in K(f_1)$. Setting $M = \{0, 1, 2\}$ in Lemmas 6 and 8, we conclude that $f_1 = e$ and $f = a_1 \in A_1^+$.

4. 3-transitive groups containing A_m and P_m. We are now ready to investigate the transitivity of groups of homeomorphisms of euclidean m-space E^m or real projective m-space Π^m which contain the affine group A_m or the projective group P_m, respectively, as a proper subgroup. The groups which we will consider are all obtained by adjoining some homeomorphism to A_m or P_m and generating the smallest group containing them. Any larger group will obviously have at least as high a degree of transitivity. In the case $m = 1$, we will obtain slightly sharper results by adjoining an element of $H^+(E^1)$ or $H^+(\Pi^1)$ to A_1^+ or P_1^+, respectively, and considering transitivity relative to $H^+(E^1)$ or $H^+(\Pi^1)$. Then if an orientation-reversing element of A_1 or P_1 is added, the resulting group will clearly have the same degree of transitivity relative to $H(E^1)$ or $H(\Pi^1)$, respectively.

THEOREM 2. *If* $f \in H^+(E^1) - A_1$, *then the group* G *generated by* f *and* A_1^+ *is* 3-*transitive relative to* $H^+(E^1)$.

Proof. Given any three points $x_1 < x_2 < x_3$ in E^1, let $L = \{x_1, x_2\}$ and $M = \{x_1, x_2, x_3\}$. For each $a \in A_1^+$, we can find $a' \in A_1^+$ satisfying $a'(f(x_i)) = fa(x_i)$ $(i = 1, 2)$. If $a(x) = \alpha + \beta x$ and $a'(x) = \alpha' + \beta' x$, then α' and β' must satisfy the equations

$$\alpha' + \beta' f(x_1) = f(\alpha + \beta x_1) ,$$
$$\alpha' + \beta' f(x_2) = f(\alpha + \beta x_2) ,$$

so that α' and β' are continuous functions of α and β. We can identify

A_1^+ with the set of pairs (α, β) of real numbers, where $\beta > 0$. If we give A_1^+ the euclidean topology of a half-plane and hold $x \in E^1$ fixed, then the mapping $a \to a(x)$ or $(\alpha, \beta) \to \alpha + \beta x$ from A_1^+ into E^1 is evidently continuous. Since f and f^{-1} are continuous, so also is the mapping $a \to \varphi(a) = f^{-1}a'^{-1}fa(x_3)$ from A_1^+ into E^1. From Theorem 1, we know that there is at least one $a_0 \in A_1^+$ such that $a_0'f(x_3) \neq fa_0(x_3)$, for otherwise $f \in A_1^+$, contrary to our hypothesis. Thus $\varphi(a_0) \neq x_3$ while $\varphi(e) = x_3$. From the connectedness of A_1^+ we infer that $\varphi(A_1^+)$ is a nondegenerate interval and so contains an open set. Moreover, $f^{-1}a'^{-1}fa \in G$ and $x_1, x_2 \in K(f^{-1}a'^{-1}fa)$. By Lemma 3, G is 3-transitive relative to $H^+(E^1)$.

THEOREM 3. *If* $m \geqq 2$ *and* $f \in H(E^m) - A_m$, *then the group* G *generated by* f *and* A_m *is* 3-*transitive.*

Proof. We know that A_m maps any noncollinear triple onto any other noncollinear triple. If we can show that G maps every collinear triple onto some noncollinear triple, then we will have established that G is 3-transitive. Let M be a collinear triple, $L \subset M$ consist of two points, X be the line through M, and suppose that, for every $a \in A_m$, $fa(M)$ is a collinear triple. The group B of all those elements in A_m which map X onto itself behaves exactly like A_1 on X. By Lemma 5, the extension N of M with respect to L and B is dense in X. We will show by induction on the sets M_i that, for every $a \in A_m$, $fa(N)$ is a collinear set. Suppose $fa(M_i)$ is a collinear set for each $a \in A_m$, and $b(L) \subset M_i$ for some $b \in B$. Then $fa(b(M_i)) = fab(M_i)$ and $fa(b^{-1}(M_i)) = fab^{-1}(M_i)$ are each collinear, and

$$(5) \qquad \begin{aligned} &fa(M_i) \cap fa(b(M_i)) \supset fa(b(L)) \; , \\ &fa(M_i) \cap fa(b^{-1}(M_i)) \supset fa(L) \; . \end{aligned}$$

Since $fa(b(L))$ and $fa(L)$ each contain two points, the sets $fa(M_i)$, $fa(b(M_i))$, and $fa(b^{-1}(M_i))$ all lie on the same line, so that $fa(M_{i+1})$ is collinear, and the induction step is complete. From $\bar{N} = X$ we infer that $fa(X)$ is collinear for each $a \in A_m$. If Y is any line in E^m, then we can choose $a_0 \in A_m$ such that $a_0(X) = Y$, whence $f(Y) = fa_0(X)$ is also collinear. Since Y is closed, connected, and separated by each of its points, the same must also be true of $f(Y)$ so that $f(Y)$ is a line. Let Y_1, Y_2 be parallel lines and Z a line which meets them both. Then $Y_1 \cap Y_2 = \emptyset$, and any line which meets Z and Y_1 in distinct points mush also meet Y_2. Since f preserves these incidence relations, we conclude that $f(Y_1)$ and $f(Y_2)$ are parallel. Let L' consist of the origin and the m unit points in a coordinate system for E^m, and let M' be the set of 2^m vertices of the unit cube determined by L'.

Then $fa(M')$ is the set of vertices of a parallelotope for each $a \in A_m$, and we can find $a' \in A_m$ satisfying $fa(x) = a'f(x)$ for all $x \in M'$. If we select $a_1 \in A_m$ so that $a_1(x) = f(x)$ for all $x \in M'$ and set $f_1 = a_1^{-1}f$, then $M' \subset K(f_1)$ and

$$f_1a(x) = a_1^{-1}fa(x) = a_1^{-1}a'f(x) = a_1^{-1}a'a_1f_1(x) , \qquad x \in M' .$$

We infer from Lemmas 5 and 8 that $f_1 = e$ and $f = a_1$, which contradicts the hypothesis of our theorem. Hence, $fa(M)$ is not collinear for some $a \in A_m$.

The conclusion of Theorem 3 seems especially weak in view of the fact that A_m itself is $(m + 1)$-transitive on subsets which do not lie on any $(m - 1)$-dimensional hyperplane. The difficulty in extending our method to higher transitivity comes from (5). If we knew, for example, that $fa(b(L))$ and $fa(L)$ each contained three points, it would not follow that these triples were noncollinear, and we could not conclude that $fa(M_i)$, $fa(b(M_i))$, and $fa(b^{-1}(M_i))$ were coplanar.

LEMMA 9. *Suppose the group F generated by A_1^+ and $f \in H^+(E^1)$ is n-transitive relative to $H^+(E^1)$. If we extend f to an element $\bar{f}$ of $H^+(\Pi^1)$ by making $\bar{f}$ fix the point at infinity, then the group G generated by P_1^+ and $\bar{f}$ is $(n + 1)$-transitive relative to $H^+(\Pi^1)$.*

Proof. An element $p \in P_1^+ = P_1 \cap H^+(\Pi^1)$ has the form $p(x) = (\alpha x + \beta)/(\gamma x + \delta)$, where $\alpha\delta - \beta\gamma > 0$. We can identify A_1^+ with the subgroup of P_1^+ which leaves fixed the point ∞ at infinity. Suppose that $\{x_1, \cdots, x_{n+1}\}$ and $\{y_1, \cdots, y_{n+1}\}$ are given such that, as i increases from 1 to $n + 1$, x_i and y_i each move in the positive sense of orientation. Choose $p_0, p_1 \in P_1^+$ so that $p_0(x_1) = \infty$ and $p_1(y_1) = \infty$. Then $\{p_0(x_2), \cdots, p_0(x_{n+1})\}, \{p_1(y_2), \cdots, p_1(y_{n+1})\} \subset \Pi^1 - \{\infty\}$, and the points in each set increase with i. Thus we can find $g_0 \in F$ satisfying $g_0(p_0(x_i)) = p_1(y_i)$ $(i = 2, \cdots, n + 1)$, and $g_1 = p_1^{-1}\bar{g}_0p_0 \in G$ must satisfy $g_1(x_i) = y_i$ $(i = 1, \cdots, n + 1)$.

THEOREM 4. *If $f \in H^+(\Pi^1) - P_1^+$, then the group G generated by f and P_1^+ is 4-transitive relative to $H^+(\Pi^1)$.*

Proof. Let $f(\infty) = x_0$, and choose $p_0 \in P_1^+$ so that $p_0(x_0) = \infty$. Then $p_0f(\infty) = \infty$, and the restriction f_0 of p_0f to $\Pi^1 - \{\infty\} = E^1$ belongs to $H^+(E^1)$. Theorem 2 says that the group F generated by f_0 and the set A_1^+ of those elements of P_1^+ which fix ∞ is 3-transitive relative to $H^+(E^1)$, and Lemma 9 gives the desired result.

THEOREM 5. *If $m \geqq 2$ and $f \in H(\Pi^m) - P_m$, then the group G generated by f and P_m is 3-transitive.*

Proof. Since P_m maps any noncollinear triple onto any other noncollinear triple, our result will be proved if we can show that, for any collinear triple M, there is a $p \in P_m$ such that $fp(M)$ is noncollinear. Suppose that, for some collinear triple $M = \{x_1, x_2, x_3\}$ and every $p \in P_m$, $fp(M)$ is collinear. Let X be a projective line in Π^m, $p_0 \in P_m$ map M into X, and Q be the subgroup of P_m which maps X onto itself. We know that Q acts like P_1 on X and is, therefore, 3-transitive without regard to orientation. Let $x \in X - \{p_0(x_1), p_0(x_2)\}$ be arbitrary, and choose $q \in Q$ so that $\{p_0(x_1), p_0(x_2)\} \subset K(q)$ and $q(p_0(x_3)) = x$. Then $fq(p_0(M))$ and $f(p_0(M))$ are each collinear and have two points in common, so that $f(x)$ lies on the projective line Y through $f(p_0(M))$, and $f(X) \subset Y$. Since f is a homeomorphism, and X, Y are topological circles, we must have $f(X) = Y$. If Z denotes the $(m-1)$-dimensional projective hyperplane at infinity, then any projective line which meets Z in two points must lie in Z. Moreover, $f(Z)$ must have the same property, for f preserves incidence relations. Hence, $f(Z)$ is a projective hyperplane, and $f(Z)$ has dimension $m-1$. If we choose $p_1 \in P_m$ so that $p_1(Z) = f(Z)$ and set $f_1 = p_1^{-1}f$, then $f_1(Z) = Z$, and the restriction f_1^* of f_1 to $\Pi^m - Z = E^m$ maps lines onto lines. Following the argument in the proof of Theorem 3, we infer that f_1^* is affine, $f_1 \in P_m$, and $f \in P_m$, which contradicts the hypothesis of our theorem. Therefore, $fp(M)$ is noncollinear for some $p \in P_m$.

5. **ω-transitive groups.** So far, we have not exhibited any f such that the group generated by f and A_m is ω-transitive. This we will now do. As before, the results for the case $m = 1$ are much stronger than those for $m > 1$, and this seems to be due to the fact that a nondegenerate connected subset of E^1 has a nonempty interior. The conditions which we shall impose on f all have to do with its fixed point set and require, at the very least, that this should have a nonempty interior.

THEOREM 6. *Suppose $f \in H^+(E^1)$, $f \neq e$, and $K(f)$ contains a half-line. Then the group G generated by f and the set B of all translations in A_1^+ is ω-transitive relative to $H^+(E^1)$.*

Proof. Let $x_1 < \cdots < x_{n+1}$ be arbitrary points of E^1, and suppose $(-\infty, x_0]$ is a connected component of $K(f)$. The case $[x_0, +\infty) \subset K(f)$ is handled in the same way. Choose $b_0 \in B$ so that $b_0(x_0) = x_{n+1}$. If we set $f_0 = b_0 f b_0^{-1}$, then $K(f_0) = b_0(K(f))$ has $(-\infty, x_{n+1}]$ as a connected component. The elements of B have the form $b(x) = \beta + x$, and if we give to B the topology induced by the euclidean topology for β, then the mapping $\varphi(b) = bf_0b^{-1}(x_{n+1})$ from B into E^1 becomes continuous. Now $\varphi(e) = f_0(x_{n+1}) = x_{n+1}$, and we can find a connected neighborhood

$U \subset B$ of e so that $b \in U$ implies $b(x_{n+1}) \in (x_n, +\infty)$. Since x_{n+1} is a boundary point of $K(f_0)$, we can find $b \in U$ with the property that $b^{-1}(x_{n+1}) \in E^1 - K(f_0)$, whence $\varphi(b) \neq x_{n+1}$. From the connectedness of U we infer that $\varphi(U)$ is a nondegenerate interval which must have a nonempty interior. If we set $G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\}$, then $b \in U$ implies

$$K(bf_0b^{-1}) \supset b((-\infty, x_{n+1}]) = (-\infty, b(x_{n+1})] \supset (-\infty, x_n] ,$$

so that $bf_0b^{-1} \in G_0$ and $\varphi(U) \subset G_0(x_{n+1})$. Lemma 3 tells us that if we know G to be n-transitive relative to $H^+(E^1)$, then G is $(n+1)$-transitive. Since G is clearly 0-transitive, a simple induction argument shows that G is ω-transitive.

Clearly the group G_2 generated by f and any conjugate hBh^{-1} of B, where $h \in H(E^1)$, is also ω-transitive relative to $H^+(E^1)$. For the fixed point set of $f_1 = h^{-1}fh$ is homeomorphic to that of f, so that the group G_1 generated by f_1 and B is ω-transitive by Theorem 6, and $G_2 = hG_1h^{-1}$. Similar remarks apply to the other theorems in this section. We also observe that some groups generated by $f \in H^+(E^1) - A_1^+$ and B are not even 2-transitive. Choose $b_0 \in B$ and $f \in H^+(E^1) - A_1^+$ so that $b_0(x) = \beta_0 + x$, where $\beta_0 \neq 0$, and f has period β_0 in the sense that $f(\beta_0 + x) = \beta_0 + f(x)$, or $b_0fb_0^{-1} = f$. Now f and each element of B commutes with b_0, so every element of the group G generated by f and B commutes with b_0. If any such element maps x into y, then it maps $x + \beta_0$ into $y + \beta_0$, and G is not 2-transitive.

THEOREM 7. *Suppose* $\{f_1, f_2, \cdots\} \subset H^+(E^1)$, *and, for every compact subset* C *of* E^1, *there is an* f_m *satisfying* $E^1 \neq K(f_m) \supset C$. *Then the group* G *generated by* $\{f_1, f_2, \cdots\}$ *and* B *is* ω*-transitive relative to* $H^+(E^1)$.

Proof. Let $x_1 < \cdots < x_{n+1}$ be arbitrary points in E^1, and f_m have the property that $E^1 \neq K(f_m) \supset [x_1 - 1, x_{n+1}]$. If $K(f_m)$ contains a half-line, then our result follows from Theorem 6. We will assume, therefore, that the connected component $[y_0, y_1]$ of $K(f_m)$ which contains $[x_1 - 1, x_{n+1}]$ is bounded. Choose $b_0 \in B$ so that $b_0(y_1) = x_{n+1}$, set $g_0 = b_0f_mb_0^{-1}$, and let $\varphi(b) = bg_0b^{-1}(x_{n+1})$ for each $b \in B$. Then $K(g_0)$ has $[y_2, x_{n+1}]$ as a connected component, where $y_2 = b_0(y_0) \leqq x_1 - 1$. As in the proof of Theorem 6, φ is continuous, $\varphi(e) = x_{n+1}$, and we can find a connected neighborhood $U \subset B$ of e such that $b \in U$ implies $b(x_{n+1}) \in (x_n, +\infty)$ and $b(y_2) \in (-\infty, x_1)$. Again there is a $b \in U$ such that $\varphi(b) \neq x_{n+1}$, and if we define G_0 as before, then $b \in U$ implies

$$K(bg_0b^{-1}) \supset b([y_2, x_{n+1}]) \supset [x_1, x_n] ,$$

so that $bg_0b^{-1} \in G_0$ and $\varphi(U) \subset G_0(x_{n+1})$. The rest of the proof follows that of Theorem 6.

THEOREM 8. *Suppose* $f, g \in H^+(E^1)$, $E^1 \neq K(f)$ *has a nonempty interior, and* $K(g) = \{y_0\}$. *Then the group* G *generated by* f, g *and* B *is* ω*-transitive relative to* $H^+(E^1)$.

Proof. Choose $y_1, y_2 \in E^1$ and $b_0 \in B$ so that $[y_1, y_2] \subset K(f)$ and $b_0(y_0) = y_1$. If we set $g_0 = b_0 g b_0^{-1}$, then $K(g_0) = \{y_1\}$, and if we define $g_1 = g_0$ in case $g_0(y_2) > y_2$ and $g_1 = g_0^{-1}$ in case $g_0^{-1}(y_2) > y_2$, then $g_1^m(y_2) \longrightarrow +\infty$ as $m \longrightarrow +\infty$. Finally, let $b_m(x) = \beta_m + x$ and

$$f_m = b_m^{-1} g_1^m f g_1^{-m} b_m \ .$$

Then

$$\begin{aligned} K(f_m) &= b_m^{-1} g_1^m(K(f)) \supset b_m^{-1}([y_1, g_1^m(y_2)]) \\ &= [-\beta_m + y_1, -\beta_m + g_1^m(y_2)] \ . \end{aligned}$$

If we choose $\beta_m = g_1^m(y_2)/2$, then any compact subset of E^1 will eventually lie in some $K(f_m)$, and our result follows from Theorem 7.

COROLLARY. *With the same hypotheses as in Theorem* 8, *the group generated by* f *and* A_1^+ *is* ω*-transitive relative to* $H^+(E^1)$.

THEOREM 9. *Suppose* $\{f_1, f_2, \cdots\} \subset H^+(\Pi^1)$, *and there is a point* $y_0 \in \Pi^1$ *such that, for every neighborhood* U *of* y_0, *we can find an* f_m *satisfying* $\Pi^1 \neq K(f_m) \supset \Pi^1 - U$. *Then the group* G *generated by* $\{f_1, f_2, \cdots\}$ *and* Q *is* ω*-transitive relative to* $H^+(\Pi^1)$, *where* Q *is the group of "rotations"* $q \in P_1^+$ *of the form* $q(x) = (\alpha x - \beta)/(\beta x + \alpha)$ *with* α, β *real and not both* 0.

Proof. The name "rotation" for an element of Q is suggested by the fact that Q is strictly 1-transitive, so that e is the only one of its elements with fixed points. We can identify Q with the set of ordered pairs (α, β), excluding $(0, 0)$, but we must also identify (α, β) with $(\lambda\alpha, \lambda\beta)$ for each real $\lambda \neq 0$. Thus Q is topologically equivalent to Π^1, that is, a circle. The action of Q on Π^1 is, therefore, the same as that of the group of real numbers modulo 2π on itself by means of left translation. We will show, first of all, that the group G_1 of those elements in G which fix ∞ is ω-transitive relative to $H^+(E^1)$. Let $x_1 < \cdots < x_{n+1} \in E^1 \subset \Pi^1$ be arbitrary, $q_0 \in Q$ map y_0 into $x_{n+1} + 1$, and f_m have the property that

$$\Pi^1 \neq K(f_m) \supset \Pi^1 - q_0^{-1}((x_{n+1}, x_{n+1} + 2)) \ .$$

Setting $f = q_0 f_m q_0^{-1}$, we have $\Pi^1 \neq K(f) \supset \Pi^1 - (x_{n+1}, x_{n+1} + 2)$. Let y_1 be the right-hand endpoint of the connected component D of $K(f)$ which contains $\Pi^1 - (x_{n+1}, x_{n+1} + 2)$, where Π^1 is oriented so as to agree with the ordering of E^1. If we choose $q_1 \in Q$ so that $q_1(y_1) = x_{n+1}$ and set $g_1 = q_1 f q_1^{-1}$, then $q_1(D)$ is a connected component of $K(g_1)$ which contains $\Pi^1 - (x_{n+1}, x_{n+1} + 2)$. We define $\varphi(q) = q g_1 q^{-1}(x_{n+1})$ for each $q \in Q$, and observe that φ is continuous, $\varphi(e) = x_{n+1}$, and there is a connected neighborhood $V \subset Q$ of e such that $q \in V$ implies $q((x_{n+1}, x_{n+1}+2)) \subset (x_n, +\infty)$. As before, $\varphi(V)$ has a nonempty interior, and $q \in V$ implies

$$K(qg_1q^{-1}) \supset q(\Pi^1 - (x_{n+1}, x_{n+1} + 2)) \supset \Pi^1 - (x_n, +\infty) ,$$

so that $qg_1q^{-1} \in G_1$. If we set $G_0 = \{g \in G_1 : \{x_1, \cdots, x_n\} \subset K(g)\}$, then $G_0(x_{n+1})$ has a nonempty interior, and Lemma 3 implies that G_1 is ω-transitive relative to $H^+(E^1)$. To show that G is ω-transitive relative to $H^+(\Pi^1)$, we can apply the argument in the proof of Lemma 9 with P_1^+ replaced by Q, for only the 1-transitivity of P_1^+ was used in that case.

THEOREM 10. *Suppose $f, g \in H^+(\Pi^1)$, $\Pi^1 \neq K(f)$ has a nonempty interior, and $K(g) = \{y_0\}$. Then the group G generated by f, g and Q is ω-transitive relative to $H^+(\Pi^1)$.*

Proof. Choose $y_1 < y_2$ in $E^1 \subset \Pi^1$ and $q_0, q_1 \in Q$ so that $[y_1, y_2] \subset K(f)$, $q_0(y_0) = \infty$, and $q_1(y_1) = \infty$. Then $g_0 = q_0 g q_0^{-1}$ has only one fixed point at ∞, and $f_0 = q_1 f q_1^{-1}$ leaves fixed the points of $[-\infty, y_3]$, where $y_3 = q_1(y_2)$ and, for the sake of our interval notation, we identify $-\infty$ and $+\infty$ with ∞. Now $\{g_0^k(y_3): -\infty < k < +\infty\}$ has $+\infty$ as a limit point, and, for every neighborhood U of ∞, we can find an integer k satisfying

$$\Pi^1 - U \subset [-\infty, g_0^k(y_3)] \subset K(g_0^k f_0 g_0^{-k}) .$$

Our result now follows from Theorem 9.

COROLLARY. *With the same hypotheses as in Theorem 10, the group generated by f and P_1^+ is ω-transitive relative to $H^+(\Pi_1)$.*

THEOREM 11. *Suppose X is a locally compact, locally connected metric space which can not be separated by any finite set,*

$$\{f_1, f_2, \cdots\} \subset H(X) ,$$

and $y_0 \in X$ has the property that $\{X - K(f_k)\}$ is a base for the neighborhoods of y_0. Let $R \subset H(X)$ be a 1-transitive group of *isometries of* X, *and* $R_0 = \{r \in R: r(y_0) = y_0\}$. *Suppose there is a continuous mapping* σ *from* $[0, 1]$ *into* R *with the topology of uniform convergence on compact sets such that* $\sigma(0) \in R_0$, $\sigma(1) \in R - R_0$, *and, for each* $y \in X$, $R_0(y)$ *is the sphere containing* y *with center at* y_0. *Then the group* G *generated by* $\{f_1, f_2, \cdots\}$ *and* R *is* ω-*transitive.*

Proof. Let $x_1, \cdots, x_{n+1} \in X$ be given, and

$$G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\} .$$

If we can show that $G_0(x_{n+1})$ has a nonempty interior, then our result will follow by induction from Lemma 1. Since G is 1-transitive, we may assume that $x_{n+1} = y_0$. For let $g_0 \in G$ map x_{n+1} into y_0, and

$$G'_0 = \{g' \in G: \{g_0(x_1), \cdots, g_0(x_n)\} \subset K(g')\} .$$

Then $g \in G_0$ implies $g_0 g g_0^{-1} \in G'_0$, and $g' \in G'_0$ implies $g_0^{-1} g' g_0 \in G_0$, whence $g_0^{-1} G'_0 g_0 = G_0$. If we know that $G'_0(y_0)$ has a nonempty interior, then

$$G_0(x_{n+1}) = g_0^{-1} G'_0 g_0(x_{n+1}) = g_0^{-1}(G'_0(y_0))$$

also has a nonempty interior. Hence, we can assume that $x_{n+1} = y_0$. If we set $\sigma(t) = r_t$ for $t \in [0, 1]$, then $\alpha = \rho(r_1(y_0), y_0) > 0$, where ρ is the metric for X. Let β be the shortest distance from y_0 to $\{x_1, \cdots, x_n\}$, U_ε the open ball with center y_0 and radius $\varepsilon = \min(\alpha, \beta/2)$, and f_k such that $y_0 \in X - K(f_k) \subset U_\varepsilon$. Since $\varepsilon \leqq \alpha$, and $\rho(r_t(y_0), y_0)$ is a continuous function of t, we can find $\delta \in [0, 1]$ satisfying $\rho(r_t(y_0), y_0) \leqq \varepsilon$ for $t \in [0, \delta]$ and $\rho(r_\delta(y_0), y_0) = \varepsilon$. This also implies that $\rho(y_0, r_t^{-1}(y_0)) \leqq \varepsilon$ for $t \in [0, \delta]$. If we set

$$G_1 = \{s r_t^{-1} f_k r_t s^{-1}: t \in [0, \delta], s \in R_0\} ,$$

then $G_1 \subset G_0$. For

$$\begin{aligned} K(s r_t^{-1} f_k r_t s^{-1}) &= s r_t^{-1}(K(f_k)) \supset X - s r_t^{-1}(U_\varepsilon) \supset X - s(U_{2\varepsilon}) \\ &= X - U_{2\varepsilon} \supset \{x_1, \cdots, x_n\} . \end{aligned}$$

Moreover,

$$r_t^{-1} f_k r_t(y_0) \in r_t^{-1} f_k(\bar{U}_\varepsilon) \subset r_t^{-1}(\bar{U}_\varepsilon) \subset \bar{U}_{2\varepsilon} ,$$

and if we hold t fixed and let s vary, then

$$s r_t^{-1} f_k r_t s^{-1}(y_0) = s(r_t^{-1} f_k r_t(y_0))$$

is a sphere with center y_0 and radius

$$\theta(t) = \rho(y_0, r_t^{-1} f_k r_t(y_0)) , \qquad t \in [0, \delta] .$$

Since $r_\delta(y_0)$ lies on the boundary of $\bar{U}_\varepsilon$, we have $r_\delta^{-1} f_k r_\delta(y_0) = y_0$, and since $r_0(y_0) = y_0 \in X - K(f_k)$, we have $r_0^{-1} f_k r_0(y_0) \neq y_0$. Thus $\theta(0) \neq 0$, and $\theta(\delta) = 0$. Now the local compactness and local connectedness of X implies that the mapping $h \rightarrow h^{-1}$ is continuous, and $(h, x) \rightarrow h(x)$ is jointly continuous in the topology of uniform convergence on compact sets [1], so that $\theta: [0, \delta] \rightarrow E^1$ is continuous, and $\theta([0, \delta])$ is a nondegenerate interval. Hence, $G_1(y_0)$ contains all spheres with center y_0 and radius less than some positive number, and $G_1(y_0) \subset G_0(y_0)$ has a nonempty interior.

COROLLARY 1. *With the same hypotheses as in Theorem* 11, *suppose that we have* $f, g \in H(X)$ *with the property that* $\{g^k(X - K(f)) : k \geqq 0\}$ *is a base for the neighborhoods of* y_0. *Then the group generated by* f, g, *and* R *is* ω*-transitive.*

Proof. We set $f_k = g^k f g^{-k}$ and apply Theorem 11.

COROLLARY 2. *Suppose* $X = E^m$ $(m \geqq 2)$, R *is the group of rigid motions of* E^m, $y_0 \in E^m$, *and* $\{f_1, f_2, \cdots\}$ *is as in the hypothesis of Theorem* 11. *Then* G *is* ω*-transitive.*

Proof. For the mapping σ, we set $r_t(x) = tx_0 + x$, where $x_0 \neq 0$ is a fixed point of E^m.

COROLLARY 3. *Suppose* $X = \Pi^m$ $(m \geqq 2)$, R *is the set of elements in* P_m *which can be represented by* $(m + 1)$*-th order unitary matrices,* $y_0 \in \Pi^m$, *and* $\{f_1, f_2, \cdots\}$ *is as in the hypothesis of Theorem* 11. *Then* G *is* ω*-transitive.*

Proof. If we regard Π^m as the unit sphere in E^{m+1} with antipodal points identified and the metric induced by E^{m+1}, then the elements of R are isometries of Π^m. For the mapping σ, we choose a one-parameter subgroup of rotations about some axis which does not pass through y_0.

LEMMA 10. *Let* X *be a topological space,* G *a subgroup of* $H(X)$, φ *a homeomorphism from* E^1 *onto a closed subset* Y *of* X, *and* $F = \{g \in G : g(Y) = Y\}$. *Suppose* $\varphi^{-1} F \varphi$ *contains* A_1, *and there is a* $g_0 \in G$ *with the properties* $K(g_0) \supset \varphi([0, 1])$ *and* $g_0(Y) - Y \neq \varnothing$. *Then for any interval* $I = [\alpha, \beta]$ *in* E^1 *and any* $y \in Y - \varphi(I)$, *we can find a* $g \in G$ *such that* $K(g) \supset \varphi(I)$ *and* $g(y) \in X - Y$.

Proof. Let $G_0 = \{g \in G : \varphi(I) \subset K(g)\}$ and $Y_0 = \{y \in Y : G_0(y) - Y \neq \varnothing\}$. Clearly Y_0 is open in Y. If $a \in A_1$ and $a(I) \supset I$, then we will show that $a\varphi^{-1}(Y_0) \subset \varphi^{-1}(Y_0)$. We first choose $f \in F$ so that $\varphi^{-1} f \varphi = a$. For each

$t \in \varphi^{-1}(Y_0)$, there is a $g \in G_0$ satisfying $g\varphi(t) \in X - Y$. Then

$$K(fgf^{-1}) = f(K(g)) \supset f\varphi(I) = \varphi a(I) \supset \varphi(I)$$

implies that $fgf^{-1} \in G_0$. From

$$fgf^{-1}(\varphi a(t)) = fgf^{-1}(f\varphi(t)) = fg\varphi(t) \in f(X - Y) = X - Y$$

we infer that $a(t) \in \varphi^{-1}(Y_0)$ and $a\varphi^{-1}(Y_0) \subset \varphi^{-1}(Y_0)$. Since we can always find an $a \in A_1$ such that $a(I) \supset I$, and a maps any point in $E^1 - I$ into any other point further away from I, it follows that if $\varphi^{-1}(Y_0) \neq \emptyset$, then $\varphi^{-1}(Y_0)$ is the union of two half-lines, that is, $E^1 - \varphi^{-1}(Y_0) = [\gamma, \delta] \supset [\alpha, \beta] = I$. We will show that $\varphi^{-1}(Y_0) \neq \emptyset$ and $[\alpha, \beta] = [\gamma, \delta]$ by deriving a contradiction from the assumption $\gamma < \alpha$. The case $\delta > \beta$ is handled in a similar manner. Let C be the connected component of $\varphi^{-1}(K(g_0))$ which contains $[0, 1]$. Then C is a closed interval with at least one endpoint ε, and we can find an $a_0 \in A_1$ so that $a_0(C) \supset I$ and $a_0(\varepsilon) = \alpha$. If $f_0 \in H(X)$ is an extension of $\varphi a_0 \varphi^{-1}$, and $g_1 = f_0 g_0 f_0^{-1}$, then

$$K(g_1) = f_0(K(g_0)) \supset \varphi a_0 \varphi^{-1}(K(g_0)) \supset \varphi a_0(C) \supset \varphi(I)$$

implies that $g_1 \in G_0$ and $a_0(C)$ is a connected component of $\varphi^{-1}(K(g_1))$. Choose $y_0 \in Y$ so that $g_0(y_0) \in X - Y$. From $g_1(f_0(y_0)) = f_0 g_0(y_0) \in X - Y$ we infer that $Y_0 \neq \emptyset$ and $\varphi^{-1}(Y_0) \neq \emptyset$. Evidently $t \in [\gamma, \alpha]$ implies $g_1\varphi(t) \in Y$, and we can find $t_0 \in (\gamma, \alpha)$ so close to α that $t_0 \neq \varphi^{-1}g_1\varphi(t_0) \in (\gamma, \alpha)$. We may assume, in fact, that $\varphi^{-1}g_1\varphi(t_0) < t_0$; for if $\varphi^{-1}g_1\varphi(t_0) > t_0$, then we would work with g_1^{-1}. Choose $a_1 \in A_1^+$ so that $a_1(I) \supset I$, $a_1(t_0) = \gamma$, and let $f_1 \in H(X)$ be an extension of $\varphi a_1 \varphi^{-1}$. As we have already seen, $g_2 = f_1 g_1 f_1^{-1} \in G_0$. Now

$$\begin{aligned} \varphi^{-1}g_2\varphi(\gamma) &= \varphi^{-1}f_1 g_1 f_1^{-1}\varphi(\gamma) = a_1\varphi^{-1}g_1\varphi a_1^{-1}(\gamma) \\ &= a_1\varphi^{-1}g_1\varphi(t_0) < a_1(t_0) = \gamma \end{aligned}$$

implies that $g_2\varphi(\gamma) \in Y_0$, and we can find $g_3 \in G_0$ satisfying $g_3(g_2\varphi(\gamma)) \in X - Y$. Since $g_3 g_2 \in G_0$, we conclude that $\varphi(\gamma) \in Y_0$ which contradicts our hypothesis. Hence, $\gamma = \alpha$, and our result is proved.

THEOREM 12. *Let X be a topological space which can not be separated by any finite subset, R a subgroup of $H(X), f \in H(X)$, φ a homeomorphism from E^1 onto a closed subset Y of X, and $S = \{g \in R: g(Y) = Y\}$. Suppose $\varphi^{-1}S\varphi \supset A_1$, $S_0 = \{g \in G: Y \subset K(g)\}$ is 1-transitive on $X - Y, K(f) \supset \varphi([0, 1])$, and $f(Y) - Y \neq \emptyset$. Then the group G generated by f and R is ω-transitive.*

Proof. We proceed by induction on the transitivity and assume that G is n-transitive for some $n \geqq 0$. If $x_1, \cdots, x_{n+1} \in X$ are given,

$G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\}$, and we can show that $G_0(x_{n+1})$ is an open subset of X, then Lemma 1 will imply that G is $(n+1)$-transitive, and our induction step will be complete. By hypothesis, there is a $g_0 \in G$ which maps $\{x_2, \cdots, x_n\}$ into $\varphi((0, 1))$ and x_{n+1} into $\varphi(1)$. We consider three cases for the position of $g_0(x_1)$. In case (i), $g_0(x_1) \in Y$ and $\varphi^{-1}g_0(x_1) < 1$. Then we can find an interval $I = [\alpha, \beta]$ which contains $\varphi^{-1}g_0(x_1), \cdots, \varphi^{-1}g_0(x_n)$ but not $\varphi^{-1}g_0(x_{n+1})$, and Lemma 10 gives us a $g_1 \in G$ with the properties $\varphi(I) \subset K(g_1)$ and $g_1(g_0(x_{n+1})) \in X - Y$. Since $S_0(g_1g_0(x_{n+1})) = X - Y$ is open in X, it follows that

$$g_0^{-1}g_1^{-1}S_0g_1g_0(x_{n+1}) = g_0^{-1}g_1^{-1}(X - Y)$$

is open in X. From $g \in S_0$ we infer that

$$K(g_0^{-1}g_1^{-1}gg_1g_0) \supset g_0^{-1}g_1^{-1}(Y) \supset g_0^{-1}g_1^{-1}\varphi(I) = g_0^{-1}\varphi(I) \supset \{x_1, \cdots, x_n\} ,$$

whence $g_0^{-1}g_1^{-1}S_0g_1g_0 \subset G_0$, and our induction step is complete in case (i). In case (ii), $g_0(x_1) \in X - Y$. Now Lemma 10 gives us a $g_2 \in G$ with the properties $K(g_2) \supset \{g_0(x_2), \cdots, g_0(x_{n+1})\}$ and $g_2\varphi(0) \in X - Y$. We can also find $g_3 \in S_0$ satisfying $g_3(g_2\varphi(0)) = g_0(x_1)$. Setting $g_4 = g_2^{-1}g_3^{-1}g_0$, we have

$$g_4(x_i) = g_2^{-1}g_3^{-1}g_0(x_i) = g_0(x_i) , \qquad 2 \leqq i \leqq n+1 ,$$
$$g_4(x_1) = g_2^{-1}g_3^{-1}g_0(x_1) = \varphi(0) .$$

Thus case (ii) can be reduced to case (i) with g_0 replaced by g_4. In case (iii), $g_0(x_1) \in Y$ and $\varphi^{-1}g_0(x_1) > 1$. Again Lemma 10 gives us a $g_5 \in G$ such that $K(g_5) \supset \{g_0(x_2), \cdots, g_0(x_{n+1})\}$ and $g_5(g_0(x_1)) \in X - Y$. Setting $g_6 = g_5g_0$, we have

$$g_6(x_i) = g_5g_0(x_i) = g_0(x_i) , \qquad 2 \leqq i \leqq n+1 ,$$
$$g_6(x_1) = g_5g_0(x_1) \in X - Y .$$

Thus case (iii) can be reduced to case (ii) with g_0 replaced by g_6, and all the cases relating to the position of $g_0(x_1)$ have been disposed of.

THEOREM 13. *The conclusion of Theorem* 12 *remains valid if we replace* E^1 *by* Π^1, *that is, a circle, and* A_1 *by* P_1.

Proof. The proof of Theorem 12 up to the definition of g_0 can be carried over unchanged. This time, however, we choose g_0 so as to map $\{x_1, \cdots, x_n\}$ into $\varphi((0, 1))$ and consider two cases for the position of $g_0(x_{n+1})$. In case (i), $g_0(x_{n+1}) \in X - Y$. As we have already seen in the proof of Theorem 12, this implies that $G_0(x_{n+1})$ is open in X, and our induction step is complete in this case. In case (ii), $g_0(x_{n+1}) \in Y$. By hypothesis, these is some point $y_0 \in Y - \varphi([0, 1])$

satisfying $f(y_0) \in X - Y$. We choose $p_1 \in P_1$ and a neighborhood U of $\varphi^{-1}g_0(x_{n+1})$ so that $U \subset \Pi^1 - \{\varphi^{-1}g_0(x_1), \cdots, \varphi^{-1}g_0(x_n)\}$, $p_1(\varphi^{-1}(y_0)) = \varphi^{-1}g_0(x_{n+1})$, and $p_1(\Pi^1 - [0, 1]) \subset U$. Let $g_1 \in S$ be an extension of $\varphi p_1 \varphi^{-1}$, and $g_2 = g_1 f g_1^{-1}$. Then

$$\begin{aligned} K(g_2) &= g_1(K(f)) \supset g_1\varphi([0, 1]) \\ &= \varphi p_1([0, 1]) \supset \varphi(\Pi^1 - U) \supset \{g_0(x_1), \cdots, g_0(x_n)\} , \\ g_2(g_0(x_{n+1})) &= g_1 f g_1^{-1}(g_0(x_{n+1})) \\ &= g_1 f \varphi p_1^{-1} \varphi^{-1} g_0(x_{n+1}) = g_1 f \varphi \varphi^{-1}(y_0) \in g_1(X - Y) = X - Y . \end{aligned}$$

If we set $g_3 = g_2 g_0$, then

$$\begin{aligned} g_3(x_i) &= g_2 g_0(x_i) = g_0(x_i) , \qquad 1 \leqq i \leqq n , \\ g_3(x_{n+1}) &= g_2 g_0(x_{n+1}) \in X - Y , \end{aligned}$$

and case (ii) can be reduced to case (i) with g_0 replaced by g_3. Thus all the cases relating to the positions of $g_0(x_{n+1})$ have been disposed of.

COROLLARY. *Suppose R is a subgroup of $H(X)$, $f \in H(X)$, $X \neq K(f)$ has a nonempty interior, and either* (i) *$X = E^m$ and $R = A_m$, or* (ii) *$X = \Pi^m$ and $R = P_m$. Then the group G generated by f and R is ω-transitive.*

Proof. The case $m = 1$ has already been verified in Theorems 8 and 10, so we will assume that $m \geqq 2$. We first consider case (i) and choose points $x_0 \in \operatorname{int} K(f)$ and $x_1 \in E^m - K(f)$. If $f(x_1)$ does not lie on the line Y through x_0 and x_1, then our result follows from Theorem 12, since $K(f) \cap Y$ contains a nondegenerate interval. If $f(x_1) \in Y$, then we choose a rotation $a_1 \in A_m$ about the point x_1 through such a small positive angle that $K(f) \cap a_1^{-1}(Y)$ contains a nondegenerate interval I. Setting $f_1 = a_1 f a_1^{-1}$, we have

$$\begin{aligned} K(f_1) \cap Y &= a_1(K(f)) \cap Y = a_1(K(f) \cap a_1^{-1}(Y)) \supset a_1(I) , \\ f_1(x_1) &= a_1 f a_1^{-1}(x_1) = a_1 f(x_1) \in X - Y , \end{aligned}$$

and our result again follows from Theorem 12 with f replaced by f_1. Case (ii) is handled in exactly the same way, for we can identify E^m with the finite part of Π^m, and a_1 can be extended to an element of P_m.

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