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KERNEL REPRESENTATIONS OF OPERATORS AND THEIR ADJOINTS

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If S is a locally compact and Hausdorff space and A is a continuous linear operator from $C_0(S)$ into the space $C(T)$ with the supremum norm topology then the Riesz Representation Theorem yields the formula $[Af](x) = \int_S f(y)\lambda(x, dy)$, where for each $x \in T$ $\lambda(x, \cdot)$ is a complex-valued regular Borel measure on S. More generally a study is made of kernel functions λ such that $\int_S f(y)\lambda(\cdot, dy) \in C(T)$ for f of compact support on S. It is shown that $\lambda(\cdot, E)$ is measurable for each Borel set E and that $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ is a regular measure on S yielding the adjoint formula $A^*\nu = \mu$. Necessary and sufficient conditions are given on λ so that $A^{**}(C(S)) \subset C(T)$ and that A^{**} be continuous from $C(S)_\beta$ to $C(T)_\beta$ when S is paracompact. Furthermore, kernel representations of β-continuous operators are studied with applications to semi-groups of operators in $C_0(S)$ and $C(S)_\beta$ when S is locally compact.

We point out that as a consequence of our work the condition (1.7) in the paper by Foguel [7] follows from (1.6) when the space is locally compact and Hausdorff. Further the regularity of the above measure yields the more specific vector-valued measure representation of A, $\mu(E) = \lambda(\cdot, E)$ in the sense of [5, Th. 2, p. 492].

DEFINITION AND NOTATION. If X is a locally compact Hausdorff space we denote by $C(X)$, $C_0(X)$ and $C_c(X)^+$ the collection of all bounded continuous complex-valued functions on X, those vanishing at infinity, and those nonnegative functions of compact support, respectively. The σ-algebra of Borel sets is the σ-algebra generated by the open subsets of X. We denote by $M(X)$ the space of bounded regular Borel measures on X with variation norm and by $B(X)$ the space of bounded Borel measurable functions on X. Let $M(X)^+$ denote the nonnegative measures in $M(X)$. We give $B(X)$, $C_0(X)$ and $C(X)$ the supremum norm topology and $\|f\| = \sup\{|f(x)| : x \in X\}$.

We wish to consider two further topologies on the space $C(X)$. We denote by $C(X)_\beta$ the space $C(X)$ with the locally convex topology defined by the collection of seminorms $P_\phi(f) = \|\phi f\|$, $\phi \in C_0(X)$. Buck [1] has shown that $C(X)_\beta$ has adjoint or dual space $M(X)$. We denote by $C(X)_{\beta'}$ the space $C(X)$ with the locally convex topology whose base of neighborhoods at the origin consists of all convex, balanced, absorbent sets V such that for each $r > 0$ there is a β neighborhood of the origin, W, such that $W \cap B_r \subset V$ where $B_r = \{f \in C(X) : \|f\| \leqq r\}$. In a recently submitted paper Dorroh [4] introduces this topology and shows that $C(X)_{\beta'}$ has dual $M(X)$ and that $\beta = \beta'$ for X a paracompact space. Further results on $C(X)_\beta$ and $C(X)_{\beta'}$ have been recently obtained by Collins and Dorroh in [2]. A set $H \subset M(X)$ is β-equicontinuous (β'-equicontinuous) if there is a $\beta(\beta')$ neighborhood of 0, W, such that $\left|\int_X f d\mu\right| \leqq 1$ for all $f \in W$ and $\mu \in H$. The β-equicontinuous sets of $M(X)$ have been characterized by Conway [3] who has shown that H is β-equicontinuous if and only if H is uniformly bounded and for each $\varepsilon > 0$ there is a compact set $K \subset X$ such that the variation of μ on $X - K$ is less than ε for all $\mu \in H$. Since β' is a finer topology than β any β-equicontinuous set is β'-equicontinuous and these are the same when X is paracompact.

Suppose S and T are locally compact and Hausdorff. Let Δ denote the collection of open sets in S and $\sigma(\Delta)$ the collection of Borel sets. We consider complex-valued functions λ defined on $T \times \sigma(\Delta)$ such that $\lambda(x) = \lambda(x, \cdot) \in M(S)$. For brevity we will denote this by $\lambda: T \to M(S)$. We denote the norm of the measure $\lambda(x)$ by $\|\lambda(x)\|$ and set $\|\lambda\| = \sup\{\|\lambda(x)\| : x \in T\}$. If $f \in B(S)$ we write $\lambda(f)$ for the function defined by $\lambda(f)(x) = \int_S f(y)\lambda(x, dy)$ and $\lambda(\cdot, E)$ is the function whose value at x is $\lambda(x, E)$ for $E \in \sigma(\Delta)$. We let $|\lambda|(x, E)$ be the variation of the measure $\lambda(x, \cdot)$ on the set E. We will say that the kernel λ satisfies condition $E(E')$ if $\{\lambda(x) : x \in K\}$ is β-equicontinuous (β'-continuous) for each compact set $K \subset T$.

Finally we take our topology from [8] and topological vector space terminology from [9]. We make use of the Riesz Representation theorem throughout and in particular its corollary:

$$|\mu|(U) = \sup\left\{\left|\int f d\mu\right| : f \in C_c(S),\ \|f\| \leqq 1,\ \text{support}\,(f) \subset U\right\}$$

for each open set U.

We prove the following theorems.

THEOREM 1. (1) *If* $\lambda: T \to M(S)^+$ *and* $\lambda(f)$ *is lower semi-continuous for each* $f \in C_c(S)^+$ *then* $\lambda(\cdot, E)$ *is Borel measurable for each* $E \in \sigma(\Delta)$.

(2) *If* $\lambda: T \to M(S)$ *and* $\lambda(f) \in C(T)$ *for all* $f \in C_c(S)$ *then* $\lambda(\cdot, E)$ *and* $|\lambda|(\cdot, E)$ *are measurable for each* $E \in \sigma(\Delta)$.

(3) *If* λ *satisfies* (1) *or* (2) *and* $\|\lambda\| < \infty$ *then* $\lambda(f) \in B(T)$ *for* $f \in B(S)$.

THEOREM 2. *If* λ *satisfies* (3) *of Theorem* 1 *then for each* $\nu \in M(T)$

the formula $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ *defines a regular Borel measure on* S *such that* $|\mu|(E) \leqq \int_T |\lambda|(x, E)|\nu|(dx)$ *and for* $f \in B(S)$ *we have* $\int f d\mu = \int \lambda(f) d\nu$.

THEOREM 3. *Suppose* A *is a continuous linear operator from the space* X *to the space* Y *where* X *denotes* $C_0(S)$, $C(S)_\beta$ *or* $C(S)_{\beta'}$ *and* Y *denotes* $C(T)$, $C(T)_\beta$ *or* $C(S)_{\beta'}$. *Then there is a unique mapping* $\lambda: T \to M(S)$ *such that*

(1) $Af = \lambda(f)$ *for all* $f \in X$ *and*

$$\|\lambda\| = \sup\{\|Af\|: f \in X, \|f\| \leqq 1\} < \infty .$$

(2) *The adjoint of* A, A^*, *takes* $M(T)$ *into* $M(S)$ *and is given by*

$$(A^*\mu)(E) = \int_T \lambda(x, E)\mu(dx) .$$

(3) *Under the natural imbeddings of* $B(S)$ *and* $B(T)$ *into* $M(S)^*$ *and* $M(T)^*$ *respectively we have for* $f \in B(S)$
$\lambda(f) = A^{**}f$ *where* A^{**} *is the adjoint of* A^* *restricted to* $M(T)$
Hence $A^{**}(B(S)) \subset B(T)$ *and* A^{**} *defines a continuous extension of* A *to* $B(S)$ *into* $B(T)$.

THEOREM 4. *Let* $\lambda: T \to M(S)$. *If* $\lambda(f) \in C(T)$ *for all* $f \in C_c(S)$ *and* λ *satisfies condition* E' *then* $\lambda(f)$ *is a continuous function on* T *for* $f \in C(S)$. *Conversely, if* S *is paracompact and* $\lambda(f)$ *is continuous for* $f \in C(S)$ *then* λ *satisfies condition* E.

THEOREM 5. *Let* $\lambda: T \to M(S)$ *and* A *the linear operator on* $C(S)$ *defined by* $Af = \lambda(f)$. *Then* A *is a continuous operator from* $C(S)_{\beta'}$ *into* $C(T)_{\beta'}$ *or* $C(T)_\beta$ *if and only if* $\|\lambda\| < \infty$, $\lambda(f) \in C(T)$ *for* $f \in C_c(S)$ *and* λ *satisfies condition* E'.

COROLLARY 1. *Let* $A: C_0(S) \to Y$ *where* Y *is as in Theorem* 3. *Then* A^{**} *is a continuous operator from* $C(S)_{\beta'}$ *into* $C(T)_{\beta'}$ *if and only if the kernel* λ *satisfies condition* E'. *Moreover* A^{**} *is the only extension of* A *to* $C(S)$ *given by a kernel and consequently is the only* β *or* β' *continuous extension of* A *to* $C(S)$.

Proof of Theorem 1. Let U be an open subset of S and let χ denote its characteristic function. Since $\lambda(x)$ is regular it follows that $\lambda(x, U) = \sup\{\lambda(f)(x): 0 \leqq f \leqq \chi, f \in C_c(S)^+\}$. Since $\lambda(f)$ is lower semi-continuous for each $f \in C_c(S)^+$, then $\lambda(\cdot, U)$ is lower semi-continuous and hence Borel-measurable. Let Σ denote the class of Borel sets E

for which $\lambda(\cdot, E)$ is measurable. Then Σ contains all open sets and is closed under countable unions of mutually disjoint sets $E \in \Sigma$ and, if $A, B \in \Sigma$ and $A \supset B$ then $A - B \in \Sigma$. It now follows from [6, p. 2] that $\Sigma = \sigma(\Delta)$ and (1) is proven.

We now prove (2). If U is an open set then as a consequence of the Riesz Representation Theorem we have

$$|\lambda|(x, U) = \sup\{|\lambda(f)(x)| : f \in C_c(S),\ ||f|| = 1 \text{ and support } (f) \subset U\} \quad \text{for each } x \in T .$$

This means that $|\lambda|(\cdot, U)$ is lower semi-continuous and as in the proof of (1) that $|\lambda|(\cdot, E)$ is measurable for each Borel set E.

We can suppose for the remainder of the proof that $\lambda(x)$ is a real signed measure for each $x \in T$ and we then have [5, p. 123] that $\lambda(x) = \lambda(x)^+ - \lambda(x)^-$ where $\lambda(x)^+, \lambda(x)^- \in M(S)^+$ and $|\lambda(x)| = \lambda(x)^+ + \lambda(x)^-$ for all $x \in T$. We show that λ^+, λ^- satisfy condition (1).

Let $f \in C_c(S)^+$ and set $\mu(x, E) = \int_E f(y)\lambda(x, dy)$. Then for each x, $\mu(x) \in M(S)$ and for

$$g \in C_c(S),\ \mu(g) = \int_S g(y)f(y)\lambda(x, dy) = \lambda(gf) .$$

Hence $\mu(g)$ is continuous for each $g \in C_c(S)$ and therefore from what we have just shown $|\mu|(\cdot, S)$ is lower-semicontinuous since S is open. But $|\mu|(x, S) = \int_S f(y)\,|\lambda|(x, dy)$ and therefore $|\lambda|(f)$ is lower semi-continuous for each $f \in C_c(S)^+$. Since $|\lambda|(x) = \lambda^+(x) + \lambda^-(x)$ and $\lambda(x) = \lambda^+(x) - \lambda^-(x)$ it now follows that for $f \in C_c(S)^+$, $\lambda^+(f)$ and $\lambda^-(f)$ are lower semi-continuous. But then it follows from (1) that $\lambda^+(\cdot, E)$, $\lambda^-(\cdot, E)$ and hence $\lambda(\cdot, E)$ are measurable for each Borel set E.

Condition (3) easily follows for we can approximate $\lambda(f)$ uniformly by means of measurable functions of the form $\sum_{i=1}^n a_i\lambda(\cdot, E_i)$.

REMARK 1. T need not be Hausdorff or locally compact in Theorem 1.

Proof of Theorem 2. It is well known that $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ defines a measure on S such that $\int_S f d\mu = \int_T \lambda(f) d\nu$ for $f \in B(S)$. Hence we will only show that μ is regular.

We can assume that ν is real and $||\nu|| = 1$. Further we can suppose that $\lambda(x) \in M(S)^+$ for each $x \in T$. For we can first assume that $\lambda(x)$ is a real signed measure, and writing $\lambda(x) = \lambda(x)^+ - \lambda(x)^-$, the proof of Theorem 1 shows that for $f \in C_c(S)^+, \lambda^+(f)$ and $\lambda^-(f)$ are lower semi-continuous. Hence we have the condition (1) of Theorem 1 and additionally, $||\lambda|| = \sup\{||\lambda(x)|| : x \in S\} < \infty$.

LEMMA 1. *Let U be an open set in S, χ its characteristic function. Let $X = \{f \in C_c(S): 0 \leqq f \leqq \chi\}$, $Y = \{g \in C_c(T): 0 \leqq g \leqq \lambda(\cdot, U)\}$. Then*

$$\sup\left\{\int_T g d\nu : g \in Y\right\} \leqq \sup\left\{\int_T \lambda(f) d\nu : f \in X\right\}.$$

Proof. Let $g \in Y$, $\varepsilon > 0$ and let g vanish outside the compact set K and fix $x \in K$.

Since $g \in Y$ then $g(x) - \varepsilon/2 < \lambda(x, U)$ and hence there is a function $f \in X$ such that $g(x) - \varepsilon/2 < \lambda(f)(x)$. Since $\lambda(f)$ is lower semicontinuous there is a neighborhood V of x such that for $t \in V$ one has $g(x) - \varepsilon/2 < \lambda(f)(t)$. But also there is a neighborhood V' of x such that if $t \in V'$ then $g(t) - \varepsilon < g(x) - \varepsilon/2$. Hence there is a neighborhood W of x such that for $t \in W$, $g(t) - \varepsilon < \lambda(f)(t)$. We extract a finite cover of sets W of K with associated functions $f \in X$. If we let h be the pointwise maximum of the corresponding functions f then $h \in X$ and for $t \in K$ we have

$$g(t) - \varepsilon < \lambda(h)(t).$$

Hence $\int_T g d\nu - \varepsilon < \int_T \lambda(h) d\nu$ and the proof is complete.

LEMMA 2. $\int_T \lambda(x, U)\nu(dx) \leqq \sup\left\{\int_T g d\nu : g \in Y\right\}$.

Proof. Let $\varepsilon > 0$ and n be an integer such that $n\varepsilon > \|\lambda\| \geqq (n-1)\varepsilon$. Then set

$$E_k = \{x \in T: k\varepsilon < \lambda(x, U) \leqq (k+1)\varepsilon\} \quad \text{for } k = 0, 1, \cdots, n-1.$$

Then $\{E_k\}$ is a partition of T by Borel sets and

$$(1) \qquad 0 \leqq \int_T \lambda(x, U)\nu(dx) - \sum_{k=0}^{n-1} k\varepsilon\nu(E_k) < \varepsilon.$$

Let

$$U_k = \{x: \lambda(x, U) > k\varepsilon\}.$$

Then U_k is an open set and $E_k = U_k - U_{k+1}$. Since ν is regular then for each k there is a compact set $K_k \subset E_k$ such that $\nu(E_k - K_k) < \varepsilon/n^2$. We can then find for each k an open set V_k with compact closure contained in U_k and containing K_k. Further there exist functions $f_k \in C_c(T)^+$ for $k = 0, \cdots, n-1$ such that $f_k(x) = k\varepsilon$ for $x \in K_k$, $f_k(x) = 0$ for $x \in T - V_k$ and $0 \leqq f_k(x) \leqq k\varepsilon$ for all $x \in T$. Therefore $f_k(x) \leqq k\varepsilon < \lambda(x, U)$ for $x \in U_k$ and hence $f_k \in Y$. We let

$$f(x) = \max\{f_k(x): 0 \leqq k \leqq n-1\}.$$

It follows that $f \in Y$ and

$$f(x) \leqq \sum_{k=0}^{n-1} k\varepsilon\chi_k(x),$$

where χ_k denotes the characteristic function of the set E_k. We then have

$$\begin{aligned} 0 &\leqq \int_T \sum_0^{n-1} k\varepsilon\chi_k d\nu - \int_T f d\nu \\ &\leqq \sum_0^{n-1} \int_{E_k} (k\varepsilon - f_k) d\nu \\ &= \sum_0^{n-1} \int_{E_k - K_k} (k\varepsilon - f_k) d\nu \\ &\leqq \sum_0^{n-1} \int_{E_k - K_k} k\varepsilon d\nu \\ &\leqq \sum_0^{n-1} k\varepsilon^2/n^2 \leqq \varepsilon^2 . \end{aligned}$$

But

$$\int_T \sum_0^{n-1} k\varepsilon\chi_k d\nu = \sum_0^{n-1} k\varepsilon\nu(E_k)$$

and applying (1) we have

$$0 \leqq \int_T \lambda(x, U)\nu(dx) - \int_T f d\nu \leqq \varepsilon^2 + \varepsilon$$

completing the proof.

LEMMA 3. *$\mu(U) = \sup\left\{\int_S f d\mu: f \in X\right\}$ and μ is regular.*

Proof. Combining Lemma 1 and Lemma 2 we have

$$\mu(U) \leqq \sup\left\{\int_T \lambda(f) d\nu: f \in X\right\}.$$

But $\int_S f d\mu = \int_T \lambda(f) d\nu$ and therefore

$$\mu(U) \leqq \sup\left\{\int_S f d\mu; f \in X\right\} \leqq \mu(U).$$

Now the mapping $f \to \int_S f d\mu$ defines a bounded linear form on the space $C_0(S)$ and hence there is a measure $\omega \in M(S)^+$ such that $\int_S f d\mu = \int_S f d\omega$ for all $f \in C_0(S)$ and since ω is regular

$$\omega(U) = \sup\left\{\int_S f d\omega : f \in X\right\} = \mu(U) .$$

This means the collection Σ of all Borel sets E for which $\omega(E) = \mu(E)$ contains all open sets and it follows from [6, p. 2] as in the proof of (1) Theorem 1 that Σ is the class of all Borel sets. Hence μ is the regular measure ω. It is easily seen that $|u|(E) \leqq \int_T |\lambda|(x, E)|\nu|(dx)$ and the proof is complete.

Proof of Theorem 3. From [1], [4] and the Riesz Representation Theorem, $X^* = M(S)$ and $Y^* \supset M(T)$. From [9, pp. 38-39]

$$A^*(M(T)) \supset M(S)$$

and the formula $\lambda(x) = A^*\mathring{x}$, where $\mathring{x}(E) = 1$ if $x \in E$, 0 if $x \notin E$, defines a map λ: $T \rightarrow M(S)$ satisfying (3) of Theorem 1 since $\|\lambda\| = \sup\{\|Af\|: \|f\| \leqq 1, f \in C_0(S)\} < \infty$ because the norm, β and β' bounded sets are the same (see [1] and [4]) and from [9, p. 45] A takes bounded sets into bounded sets. Furthermore $Af = \lambda(f)$ for $f \in X$ and if $\nu(E) = \int_T \lambda(x, E)\mu(dx)$ then

$$\int_S f d\nu = \int_T \lambda(f) d\mu = \int_T A f d\mu = \int_S f d(A^*\mu)$$

for all $f \in X$ and consequently $A^*\mu = \nu$ since ν is regular. Finally if A^{**} is the adjoint of A^* restricted to $M(T)$ then for $\mu \in M(T)$ and

$$f \in B(S)\ [A^{**}f](\mu) = f(A^*\mu) = \int_S f d(A^*\mu) = \int_T \lambda(f) d\mu = [\lambda(f)](\mu)$$

since $\lambda(f) \in B(T)$. This holds for all $u \in M(T)$ and consequently $A^{**}f = \lambda(f)$. Hence $A^{**}(B(S)) \subset B(T)$ and $\|A^{**}\| = \|\lambda\|$.

REMARK 2. If for each $t \in [0, \infty]$, $T(t)$ is a continuous operator from X to X and $T(t + u) = T(t)T(u)$ then $T(t + u)^{**} = T(t)^{**}T(u)^{**}$. If we then write $[T(t)f](x) = \int_S f(y)\lambda_t(x, dy)$, then by the above theorem $\lambda_t(f) = T(t)^{**}f$ for $f \in B(S)$. If χ is the characteristic function of the Borel set E we have

$$\lambda_{t+u}(\chi) = \lambda_t(\lambda_u(\chi))$$

or the Chapmann-Kolmogorov equation

$$\lambda_{t+u}(x, E) = \int_S \lambda_u(y, E)\lambda_t(x, dy) .$$

Consequently a transition function $\lambda_t(x, \cdot)$ can be obtained for a semigroup of β or β' continuous operators on the space $C(S)$ when S is locally compact.

REMARK 3. One can obtain a kernel λ satisfying (1) under the weaker condition that A have range $B(T)$ and domain $C_0(S)$. For the set of linear mappings $f \to \lambda(f)(x)$ for $x \in T$ is pointwise bounded and hence uniformly bounded since $C_0(S)$ is a Banach space.

Proof of Theorem 4. For each compact set $K \subset S$ there is a function $\varphi_K \in C_c(S)$ such that $\varphi_K \equiv 1$ on K. If $f \in C(S)$ then the net $\{\varphi_K f\} \subset C_c(S)$ converges β' to f since it is uniformly bounded and β convergent to f. Consequently $C_c(S)$ is β' dense in $C(S)$. If $x \in T$ and U is a neighborhood of x with compact closure then $\{\lambda(x_\alpha) : x_\alpha \in U\}$ is a β'-equicontinuous set of linear functionals on $C(S)$ for any net $\{x_\alpha\} \subset U$ converging to x. By hypothesis $\lambda(x_\alpha) \to \lambda(x)$ on $C_c(S)$. Since $C_c(S)$ is β' dense and $\{\lambda(x_\alpha)\}$ is β'-equicontinuous, $\lambda(x_\alpha) \to \lambda(x)$ on $C(S)$. Hence $\lambda(f)$ is continuous at x for all $f \in C(S)$.

Conversely if $\lambda(f) \in C(T)$ for $f \in C(S)$ then for any compact set $K \subset T$ $\{\lambda(x): x \in K\}$ is weak-* compact as as ubset of the dual of $C(S)_\beta$ and, as Conway [3] has shown, must be β-equicontinuous.

Proof of Theorem 5. Suppose that A is continuous from $C(S)_{\beta'}$ to $C(T)_{\beta'}$ or $C(T)_\beta$. Then $\|\lambda\| < \infty$ by Theorem 3 and if K is a compact set in T and V is the β neighborhood of 0 defined by some function $\varphi \in C_0(T)$ identically 1 on K there is a β' neighborhood of 0, U, such that $A(U) \subset V$. That is, $|\lambda(f)(x)| \leqq 1$ for all $f \in U$ and $x \in K$. Consequently λ satisfies condition E'.

Conversely, let us show A is continuous from $C(S)_{\beta'}$ into $C(T)_{\beta'}$ Let V be a β' neighborhood of 0 in $C(T)$ and $r > 0$. We show there is a β neighborhood U of 0 in $C(S)$ such that $A^{-1}(V) \supset B_r \cap U$ thus showing that $A^{-1}(V)$ is a β' neighborhood.

Let $p = r\|\lambda\|$. There is a $\phi \in C_0(T)$ such that

$$V \supset B_p \cap \{g: P_\phi(g) \leqq 1\} \quad \text{and} \quad \phi \geqq 0 .$$

Let $K = \{t: |\phi(t)| \geqq 1/(p+1)\}$. Since λ satisfies condition E' there is a β' neighborhood U_0 in $C(S)$ such that $|\lambda(f)(x)| \leqq 1$ for all $f \in U_0$ and $x \in K$. Let $W = \{f \in C(S): \|\phi\| f \in U\}$. Then $A^{-1}(V) \supset B_r \cap W$ for if $f \in B_r \cap W$ then $Af \in B_p$ and $|\phi(x)[Af](x)| < p/(p+1)$ for $x \notin K$ while for $x \in K$, $|\phi(x)[Af](x)| \leqq \|\phi\| |[Af](x)| \leqq 1$ since $\|\phi\| f \in U_0$. Hence

$$A^{-1}(V) \supset A^{-1}(B_p) \cap A^{-1}\{g: P_\phi(g) \leqq 1\} \supset B_r \cap (B_r \cap W) = B_r \cap W .$$

We then choose a β neighborhood U such that $W \supset B_r \cap U$ completing the proof.

REMARK 4. If A is continuous from $C(S)_\beta$ into $C(T)_{\beta'}$ it follows that λ satisfies E.

The proof of Corollary 1 is almost immediate. As a consequence of Theorem 3 and Theorem 5 continuity from $C(S)_{\beta'}$ to $C(T)_{\beta'}$ is equivalent to condition E'. If A' is an extension of A to $C(S)$ into $C(T)$ given by a kernel μ then $\mu = \lambda$ on $C_0(S)$ and consequently $\mu = \lambda$ on $C(S)$ and $A = A'$. Since by Theorem 3 any β or β' continuous extension is given by a kernel this shows that A^{**} is unique.

It should be noted that if S is paracompact and A is any operator on $C(S)$ into $C(T)$ given by a bounded kernel λ then by Theorems 4 and 5, A is continuous from $C(S)_\beta$ to $C(T)_{\beta'}$.

We conclude with a brief remark on operators from $M(T)$ into $M(S)$. Suppose B is such a linear operator and B^* its adjoint on $B(S)$. Define $\lambda: T \to M(S)$ by $\lambda(x) = B\mathring{x}$ where $\mathring{x}$ is the measure defined in the proof of Theorem 3. If B is bounded and $B^*(C_c(S)) \subset C(T)$ then $B^*(B(S)) \subset B(T)$ by Theorem 1. By Theorem 2,

$$(B\mu)(E) = \int_T (B\mathring{x})(E)\mu(dx) .$$

If λ satisfies condition E' then by Theorem 5 B is the adjoint of the continuous operator B^* from $C(S)_{\beta'}$ to $C(T)_{\beta'}$. Thus B is completely determined by its action on the point measures $\{\mathring{x}: x \in T\}$.

REMARK 5 (added January 13, 1967). One can amplify Remark 4 by observing that if, moreover, λ satisfies E then Theorem 5 remains true with β' replaced by β. For then A is continuous from $C(S)_{\beta'}$ to $C(T)_\beta$ and using condition E, [3], part (2) of Theorem 3 and [9, p. 39] it follows that A^* takes β-equicontinuous sets of $M(T)$ into β-equicontinuous sets of $M(S)$ making A continuous on $C(S)_\beta$ into $C(T)_\beta$.

REMARK 6. It has recently come to the author's attention that a version of Theorem 2 can be found on page 176 of the recent book by P. A. Meyer, *Probability and Potentials*, Blaisdell, Waltham, Massachusetts, 1966, under the conditions that S be σ-compact, $\lambda: S \to M(S)^+$, $\lambda(f)$ be continuous for all $f \in C_c(S)^+$ and that ν have compact support.

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