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DISJOINT BASIC SUBGROUPS

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This paper arose from consideration of the following questions. First, what characterizes those infinite Abelian reduced p-groups which possess disjoint basic subgroups? Second, are there properties that a basic subgroup must possess to insure the existence of a basic subgroup disjoint from it?

We show that a necessary and sufficient condition for an infinite Abelian reduced p-group G to contain disjoint basic subgroups is that $|G| =$ final rank G. Futhermore, in such a group a necessary and sufficient condition for a basic subgroup B to have a basic subgroup disjoint from it is that B is a lower basic subgroup of G.

Throughout this paper the word "group" will mean "Abelian group" and the notation used will be that of L. Fuchs in [1] with the exception that $A \oplus B$ will denote the direct sum of the groups A and B, and $A + B$ will be the, not necessarily direct, sum.

We will use the following theorem:

THEOREM A. (*Mitchell and Mitchell in* [4]) *Let G be an infinite reduced Abelian p-group and B a basic subgroup of G such that $G/B = \sum_{\alpha \in I} (G_\alpha/B)$ where $G_\alpha/B \cong Z(p^\infty)$ for all $\alpha \in I$. Then $G = H \oplus K$ and $B = H \oplus L$ where L is a basic subgroup of K such that $r(K/L) = r(G/B) = |I|$ and $|K| = maximum \{\aleph_0, |I|\}$.*

We first prove the following lemmas:

LEMMA 1. *Let G be a p-group without elements of infinite height, and such that final rank $(G) = |G|$. Let B be a lower basic subgroup of G. Then there exists a basic subgroup, B', of G which is disjoint from B.*

Proof. Let $B = \sum_{\alpha \in I} \langle y_\alpha \rangle$, and let $G/B = \sum_{\beta \in J} C_\beta$ where each $C_\beta \cong Z(p^\infty)$. Let $\{\{y_\alpha \mid \alpha \in I\}, \{c_{\beta,n} \mid \beta \in J, n = 1, 2, \cdots\}\}$ be a quasibasis for G. Since B is a lower basic subgroup of G and final rank $(G) = |G|$, we have $|J| = r(G/B) = |G| \geqq |B| \geqq |I|$. If indeed we have $|J| > |I|$ a pure subgroup H of G can be chosen such that $H \supset B$, $H^1 = 0$, and final rank $(H) = |H| = |I|$. We can then prove that there is a basic subgroup B' of H which is disjoint from B and H being pure in G will insure B' is a basic subgroup of H. Thus it suffices to complete the proof when $|J| = |I|$ and we will assume moreover $I = J$.

Now for each $\alpha \in I$ choose from $\{c_{\alpha,n}\}_{n=1}^{\infty}$ the element $c_{\alpha,2E(y_\alpha)}$. Define $B' = \langle\{y_\alpha - p^{E(y_\alpha)}c_{\alpha,2E(y_\alpha)}\}_{\alpha\in I}\rangle$. We now claim that B' is the desired basic subgroup of G which is disjoint from B. To see this we prove the following:

(i) First claim that

$$B' = \sum_{\alpha\in I} \langle y_\alpha - p^{E(y_\alpha)} c_{\alpha,2E(y_\alpha)}\rangle .$$

Suppose that

$$0 = \sum_{i=1}^{n} a_i(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})}) ,$$

then

$$\sum_{i=1}^{n} a_i y_{\alpha_i} = \sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})} .$$

Since the

$$E(y_\alpha - p^{E(y_\alpha)} c_{\alpha,2E(y_\alpha)}) = E(y_\alpha) ,$$

we would be finished if $\sum_{i=1}^{n} a_i y_{\alpha_i} = 0$, so we can assume that $\sum_{i=1}^{n} a_i y_{\alpha_i} \neq 0$, and without loss of generality a_i is not equal to 0 mod $o(y_{\alpha_i})$. Now the height $h_G(\sum_{i=1}^{n} a_i y_{\alpha_i}) = r$, where r is the largest positive integer such that p^r divides each a_i, since $\sum_{i=1}^{n} a_i y_{\alpha_i}$ is an element of $B = \sum_{\alpha\in I}\langle y_\alpha\rangle$. But,

$$h_G\Big(\sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})}\Big) \geqq \underset{i=1,2,\cdots,n}{\text{minimum}} \{r + E(y_{\alpha_i})\} > r ,$$

which contradicts the equality

$$\sum_{i=1}^{n} a_i y_{\alpha_i} = \sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})} .$$

Therefore, we must have that

$$B' = \sum_{\alpha\in I} \langle y_\alpha - p^{E(y_\alpha)} c_{\alpha,2E(y_\alpha)}\rangle .$$

(ii) Next we will show that B' is a pure subgroup of G. Let $z \in B'[p]$, and write

$$z = \sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})-1}(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})}) ,$$

where each a_i is relatively prime to p. Now we have that

$$\begin{aligned} h_{B'}(z) &= \underset{i=1,\cdots,n}{\text{minimum}} \{h_{B'}[a_i p^{E(y_{\alpha_i})-1}(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i,2E(y_{\alpha_i})})]\} \\ &= \underset{i=1,\cdots,n}{\text{minimum}} \{E(y_{\alpha_i}) - 1\} . \end{aligned}$$

But

$$h_G(z) = h_G\left[\left(\sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})-1} y_{\alpha_i}\right) - \left(\sum_{i=1}^{n} a_i p^{2E(y_{\alpha_i})-1} c_{\alpha_i, 2E(y_{\alpha_i})}\right)\right],$$

and

$$h_G\left(\sum_{i=1}^{n} a_i p^{2E(y_{\alpha_i})-1} c_{\alpha_i, 2E(y_{\alpha_i})}\right) \geqq \text{minimum}\,\{2E(y_{\alpha_i}) - 1\}$$
$$> \underset{i=1,\cdots,n}{\text{minimum}}\,\{E(y_{\alpha_i}) - 1\},$$

and

$$h_G\left(\sum_{i=1}^{n} a_i p^{E(y_{\alpha_i})-1} y_{\alpha_i}\right) = \underset{i=1,\cdots,n}{\text{minimum}}\,\{E(y_{\alpha_i}) - 1\}.$$

Since the height of the sum of two elements with different heights is just the height of the smaller, we have that

$$h_G(z) = \underset{i=1,\cdots,n}{\text{minimum}}\,\{E(y_{\alpha_i}) - 1\}.$$

Therefore $h_G(z) = h_{B'}(z)$ for each element $z \in B'[p]$, and hence by Lemma 7, page 20, in [3], we have that B' is pure.

(iii) We will now complete the proof that B' is a basic subgroup of G by showing that B' cannot be extended to a larger pure direct sum of cyclic groups. Suppose that $B' \oplus \langle z \rangle$ is a pure direct sum of cyclic groups. Since $\{\{y_\alpha \mid \alpha \in I\}, \{c_{\alpha,n} \mid \alpha \in I, n = 1, 2, \cdots\}\}$ is a quasi-basis for G, we can write

$$z = \sum_{i=1}^{n} a_i y_{\alpha_i} + \sum_{i=1}^{k} s_j c_{\alpha_j, r_j}.$$

Now we also know that $c_{\alpha_j, r_j} = pc_{\alpha_j, r_j+1} + b_j$ where $b_j \in B = \sum_{\alpha \in I} \langle y_\alpha \rangle$, hence we can write

$$z = \sum_{j=1}^{k} s_j pc_{\alpha_j, r_j+1} + \sum_{i=1}^{m} t_i y_{\alpha_i}.$$

Now write

$$z = \sum_{j=1}^{k} s_j pc_{\alpha_j, r_j+1} + \sum_{i=1}^{m_1} t_i' y_{\alpha_i} + \sum_{i=1}^{m_2} t_i'' y_{\alpha_i}$$

where each t_i' is divisible by p, and each t_i'', is relatively prime to p. We are assuming that $H = B' \oplus \langle z \rangle$ is pure in G, and hence, $h_G(z) = h_H(z) = 0$, and since H is a direct sum of cyclic groups we must also have that $h_H(b' + z) = 0$, for any $b' \in B$. Consider the following element of B',

$$\sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}).$$

Now we have

$$z - \sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})$$
$$= \sum_{j=1}^{k} s_j p c_{\alpha_j, r_j+1} + \sum_{i=1}^{m_1} t_i' y_{\alpha_i} + \sum_{i=1}^{m_2} t_i'' p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})} ,$$

thus

$$h_G\Big(z - \sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})\Big) \geqq 1 ,$$

but this contradicts the assumption that H is a pure subgroup of G. Thus B' is a basic subgroup of G.

To complete the proof of the theorem, we need only show that $B \cap B' = 0$. To see this suppose that

$$\sum_{j=1}^{k} s_j(y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) = \sum_{i=1}^{n} a_i y_{\alpha_i} .$$

Consider

$$\sum_{i=1}^{n} a_i y_{\alpha_i} + B = 0 + B = \sum_{j=1}^{k} s_j(y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) + B$$
$$= \sum_{j=1}^{k} s_j p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})} + B$$
$$= \sum_{j=1}^{k} s_j c_{\alpha_j, E(y_{\alpha_j})} + B ,$$

so that $s_j c_{\alpha_j, E(y_{\alpha_j})} + B = 0 + B$ for $j = 1, \cdots, k$ since each is from a different summand of G/B. But this means that s_j is divisible by $p^{E(y_{\alpha_j})}$ for $j = 1, \cdots, k$. Thus

$$\sum_{j=1}^{k} s_j(y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) = 0 ,$$

and so $B \cap B' = 0$.

LEMMA 2. *Let G be a reduced p-group such that final rank $(G) = |G|$. Let B be a lower basic subgroup of G. Then there exists a basic subgroup B' of G which is disjoint from B.*

Proof. Let H be a high subgroup of G which contains B. By Theorem 5 in [2] H is pure and a basic subgroup of H is a basic of G. Thus rank (G/B) = rank (H/B) + rank (G/H), and we consider the following cases:

Case (i). Suppose that rank (H/B) = rank (G/B), then we know that final rank $(H) \geqq$ rank (H/B) = rank (G/B) = final rank $(G) = |G| \geqq |H|$. Thus final rank $(H) = |H|$, and Lemma 1 completes the proof.

Case (ii). Suppose that rank $(G/B) >$ rank (H/B). Since $|G| =$ final rank (G), and final rank $(G) =$ rank (G/B), we know that rank (G/B) is infinite. But if rank $(G/B) >$ rank (H/B), and is infinite, then the $|G^1[p]|$ is infinite, and hence we have $|G^1[p]| =$ rank $(G/H) >$ rank (H/B). Now rank $(G/B) =$ rank $(H/B) +$ rank $(G/H) =$ rank $(H/B) + |G^1[p]|$, and thus $|G^1[p]| =$ rank $(G/B) = |G|$. So that $|G^1[p]| \geqq |B|$, and for the purposes of this proof we can assume that $|G^1[p]| = |B|$. Let $G^1[p] = \sum_{\alpha\in I}\langle y_\alpha\rangle$, and let $B = \sum_{\alpha\in I}\langle x_\alpha\rangle$. For each $\alpha \in I$ choose z_α such that $y_\alpha = p^{E(x_\alpha)-1}z_\alpha$, which can be done since each y_α has infinite height. Now consider the subgroup $B' = \langle\{x_\alpha + z_\alpha\}_{\alpha\in I}\rangle$. We claim that B' is a basic subgroup of G which is disjoint from B. To see this we prove:

(i) First we must show that $B' = \sum_{\alpha\in I}\langle x_\alpha + z_\alpha\rangle$. Suppose

$$\sum_{i=1}^{n} a_i(x_{\alpha_i} + z_{\alpha_i}) = 0 ,$$

where $a_i \not\equiv 0 \bmod (o(x_{\alpha_i}))$, and $a_i < o(x_{\alpha_i})$. Notice that $o(x_\alpha) = o(x_\alpha + z_\alpha)$ since $p(y_\alpha) = 0 = p^{E(x_\alpha)}z_\alpha$. Let k_i be the largest positive integer such that p^{k_i} divides a_i. Let $r = \text{maximum}_{i=1,\dots,n}\{E(x_{\alpha_i}) - k_i\}$, and consider

$$\begin{aligned} 0 = p^{r-1}\Big(\sum_{i=1}^{n} a_i(x_{\alpha_i} + z_{\alpha_i})\Big) &= \sum_{i=1}^{n} a_i p^{r-1}x_{\alpha_i} + \sum_{i=1}^{n} a_i p^{r-1}z_{\alpha_i} \\ &= \sum_{i=1}^{n} a_i p^{r-1}x_{\alpha_i} + \sum_{i=1}^{n} a_i' y_{\alpha_i} . \end{aligned}$$

Hence we have

$$\sum_{i=1}^{n} a_i' y_{\alpha_i} = -\sum_{i=1}^{n} a_i p^{r-1}x_{\alpha_i} ,$$

but this means that an element of infinite height is equal to an element of finite height which is contradiction. Thus $B' = \sum_{\alpha\in I}\langle x_\alpha + z_\alpha\rangle$.

(ii) We must show that B' is pure. Let $s \in B'[p]$, and write

$$s = \sum_{i=1}^{n} a_i p^{E(x_{\alpha_i})-1}(x_{\alpha_i} + z_{\alpha_i})$$

where a_i is relatively prime to p for each i. Since $B' = \sum_{\alpha\in I}\langle x_\alpha + z_\alpha\rangle$, we know that $h_{B'}(s) = \text{minimum}_{i=1,\dots,n}\{E(x_{\alpha_i}) - 1\}$. Now consider

$$\begin{aligned} h_G(s) &= h_G\Big(\sum_{i=1}^{n} a_i p^{E(x_{\alpha_i})-1}x_{\alpha_i} + \sum_{i=1}^{n} a_i p^{E(x_{\alpha_i})-1}z_{\alpha_i}\Big) \\ &= h_G\Big(\sum_{i=1}^{n} a_i p^{E(x_{\alpha_i})-1}x_{\alpha_i} + \sum_{i=1}^{n} a_i y_{\alpha_i}\Big) \\ &= h_G\Big(\sum_{i=1}^{n} a_i p^{E(x_{\alpha_i})-1}x_{\alpha_i}\Big) \\ &= \underset{i=1,\cdots,n}{\text{minimum}}\{E(x_{\alpha_i}) - 1\} . \end{aligned}$$

Thus B' is a pure subgroup of G.

(iii) To complete the proof that B' is a basic subgroup of G, we need only prove that the quotient G/B' is divisible. If every element $s + B' \in (G/B')[p]$ has infinite height then G/B' is divisible. Thus we can assume $h_{G/B'}(s + B') = n$, a finite integer, and we can assume that $o(s) = o(s + B')$. Now since G/B is divisible we know that $s + B$ has infinite height in G/B. Consider the following cases:

Case (a). Suppose that $s \in B$, then

$$s = \sum_{i=1}^{m} a_i p^{E(x_{\alpha_i})-1} x_{\alpha_i}$$

where a_i is relatively prime to p for each i. Now define the following element of B', let

$$b' = \sum_{i=1}^{m} a_i p^{E(x_{\alpha_i})-1}(x_{\alpha_i} + z_{\alpha_i}) .$$

But

$$s - b' = \sum_{i=1}^{m} a_i y_{\alpha_i} , \qquad \text{and} \qquad \sum_{i=1}^{m} a_i y_{\alpha_i}$$

has infinite height in G so that $h_{G/B'}(s + B')$ is infinite. Therefore G/B' must be divisible.

Case (b). Suppose that $s \notin B$, then there exists an element $\sum_{i=1}^{m} a_i x_{\alpha_i} \in B$, such that

$$s + \sum_{i=1}^{m} a_i x_{\alpha_i} = p^{n+1} g$$

since $h_{G/B}(s + B)$ is infinite. Now write

$$\sum_{i=1}^{m} a_i x_{\alpha_i} = \sum_{j=1}^{r} c_j x_{\alpha_j} + \sum_{k=1}^{t} d_k x_{\alpha_k} ,$$

where c_j is divisible by $p^{E(x_{\alpha_j})-1}$, and d_k is not divisible by $p^{E(x_{\alpha_k})-1}$. Thus

$$s + \sum_{j=1}^{r} c_j x_{\alpha_j} + \sum_{k=1}^{t} d_k x_{\alpha_k} = p^{n+1} g$$

and so multiplication by p yields

$$\sum_{k=1}^{t} p d_k x_{\alpha_k} = p^{n+2} g .$$

By choice of the x_{α_k}'s we know $p d_k x_{\alpha_k} \neq 0$. Thus we must have

$$h_G\left(\sum_{k=1}^{t} d_k x_{\alpha_k}\right) \geqq n + 1 .$$

Therefore by letting $c'_j = c_j/p^{E(x_{\alpha_j})-1}$ we have that

$$s + \sum_{j=1}^{r} p^{E(x_{\alpha_j})-1} c'_j x_{\alpha_j} = p^{n+1} g' .$$

Consider the element $b' \in B'$ such that

$$b' = \sum_{j=1}^{r} c'_j p^{E(x_{\alpha_j})-1}(x_{\alpha_j} + z_{\alpha_j}) = \sum_{j=1}^{r} c'_j p^{E(x_{\alpha_j})-1} x_{\alpha_j} + \sum_{j=1}^{r} c'_j y_{\alpha_j} .$$

Then

$$s - b' = p^{n+1} g' - \sum_{j=1}^{r} c'_j y_{\alpha_j} = p^{n+1} g''$$

since $\sum_{j=1}^{r} c'_j y_{\alpha_j}$ has infinite height in G. But this implies that

$$h_{G/B'}(s + B') \geqq n + 1$$

which contradicts the assumption that $h_{G/B'}(s + B') = n$. Thus G/B' must be divisible.

(iv) To complete the proof of the theorem, we need only show that $B \cap B' = 0$. To see this, suppose that

$$\sum_{i=1}^{n} a_i x_{\alpha_i} = \sum_{j=1}^{k} s_j(x_{\alpha_j} + z_{\alpha_j}) \neq 0$$

so

$$\sum_{i=1}^{n} a_i x_{\alpha_i} - \sum_{j=1}^{k} s_j x_{\alpha_j} = \sum_{j=1}^{k} s_j z_{\alpha_j} \neq 0 ,$$

and by multiplying both sides of this equation by an appropriate power of p we get an element of infinite height on one side and an element of finite height on the other side, which is a contradiction. Thus $B \cap B' = 0$, and the proof is finished.

The following theorem gives a sufficient condition for a group G to possess disjoint basic subgroups.

THEOREM 3. *Let G be a reduced Abelian p-group. If final rank $(G) = |G|$, then G contains two disjoint basic subgroups.*

Proof. Since every p-group has a lower basic subgroup, then Lemma 2 will complete the proof.

The next corollary shows that the restriction final rank $(G) = |G|$, can be removed if instead of disjoint basic subgroups, one is seeking two basic subgroups whose intersection is bounded.

COROLLARY 4. *Let G be a reduced Abelian p-group. Then there exists two basic subgroups of G whose intersection is bounded.*

Proof. By Theorem 31.5, page 106, in [1], we can write $G = H \oplus K$, where K is bounded direct sum of cyclic groups, and final rank $(H) = |H|$. Now by Theorem 3 there exists A and B which are disjoint basic subgroups of H. Now $A \oplus K$ and $B \oplus K$ are basic subgroups of G whose intersection is bounded.

THEOREM 5. *Let G be a reduced Abelian p-group, and suppose that A and B are two disjoint basic subgroups of G. Then rank $(G/A) =$ rank $(G/B) = |G|$.*

Proof. Suppose that rank $(G/A) < |G|$, then by Lagrange's Theorem and since basic subgroups are isomorphic we know that $|G| = |B| = |A|$. By Theorem A we have $G = L \oplus F$ and $A = A' \oplus F$, where $|L| = \text{maximum}\,\{\aleph_0, \text{rank}\,(G/A)\}$. Since A and B are disjoint basic subgroup of G we know G cannot be bounded. Now $(G/A)[p] \supset [(A \oplus B)/A][p]$ and $|[(A \oplus B)/A][p]| = |B|$ which must be at least $\aleph_0$. Thus rank $(G/A) \geqq \aleph_0$, and therefore

$$|L| = \text{rank}\,(G/A) < |G| .$$

We can write each $x \in B$ as $x = y_x + f_x$, where $y_x \in L$ and $f_x \in F$. Since $|B| = |A| = |G| > |L|$ and B is a subgroup, there must exist some $y \in B$ such that $y \in F$, but $F \subset A$ which contradicts $A \cap B = 0$. Thus rank $(G/A) = |G|$, and similarity rank $(G/B) = |G|$.

We are now in a position to state the results of the original questions in Theorem 6 and Theorem 7.

THEOREM 6. *A necessary and sufficient condition for a reduced Abelian p-group to possess disjoint basic subgroups is that final rank $(G) = |G|$.*

Proof. If final rank $(G) = |G|$ then Theorem 3 completes the proof. If A and B are disjoint basic subgroups of G then by Theorem 5 we have $r(G/A) = r(G/B) = |G|$. But final rank $(G) \geqq$ rank (G/M) for any basic subgroup M of G. Thus final rank $(G) \geqq$ rank $(G/A) = |G|$, and since $|G| \geqq$ final rank (G) we have final rank $(G) = |G|$.

THEOREM 7. *If G is a reduced Abelian p-group such that final rank $(G) = |G|$, and A is a basic subgroup of G, then there is a basic subgroup of G which is disjoint from A if and only if A is a lower basic subgroup of G.*

Proof. If A is a lower basic subgroup then Lemma 2 assures the existence of a disjoint basic subgroup. If G possesses a basic subgroup B disjoint from A then by Theorem 5 we have rank $(G/A) = |G|$ and by hypothesis final rank $(G) = |G|$ thus rank $(G/A) =$ final rank (G) and A is a lower basic subgroup.

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