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DECOMPOSITION SPECTRA OF RINGS OF CONTINUOUS FUNCTIONS

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Let S be a subset of a completely regular Hausdorff space X. Sufficient conditions on S and X are obtained for the ring of continuous real-valued functions on S to be isomorphic to an inverse limit of quotient rings of the ring of continuous functions on X, or, alternatively, of the ring of bounded continuous functions on X. An application to the theory of rings of quotients of rings of continuous functions is given.

A decomposition spectrum of a set with some kind of structure is an inverse system of quotient structures of the same type. Decomposition spectra have been discussed recently by various authors: For topological spaces by Flachsmeyer [2], Pasynkov [4], and Vegrin [6]; and for ordered sets by Rinow [5]. Vegrin also considers briefly decomposition spectra of rings of continuous functions; however, the question he investigates is different from those considered here.

DEFINITION. A *decomposition spectrum* of a ring A is an inverse system of quotient rings of A.

The ring of all continuous real-valued functions on a completely regular Hausdorff space X will be denoted by $C(X)$, and the subring of bounded functions by $C^*(X)$.

The inverse limit of a decomposition spectrum of a ring is, of course, a ring. In the papers on decomposition spectra mentioned above, it often turns out that the inverse limit is an *extension* of the original structure (topological space or ordered set). Now if S is a subset of X, $C(S)$ is often an extension of $C(X)$; and $C(X)$ is always an extension of $C^*(X)$. This suggests the following questions: (1) For a subset S of X, when is $C(S)$ isomorphic to the inverse limit of some decomposition spectrum of $C(X)$? (2) When is $C(X)$ isomorphic to the inverse limit of some decomposition spectrum of $C^*(X)$?

The first question has the trivial answer: When S is C-embedded in X, that is when every function in $C(S)$ can be extended to a function in $C(X)$; for then, in fact, $C(S)$ is isomorphic to a quotient ring of $C(X)$, since the restriction mapping is a homomorphism onto $C(S)$. (This observation also leads naturally to the first question.) So some particular answers are: When S is compact, or when S is closed and X is normal, or when S is open-and-closed. The second question has the trivial answer: When X is compact; for in that case,

$C(X) = C^*(X)$. We show below that there are some nontrivial answers to both questions.

For any $f \in C(X)$ and any subset B of X, the restriction of f to B will be designated by $f \mid B$, and the image of B under f will be written $f[B]$. The constant function in $C(X)$ whose value is r will be denoted by $\mathbf{r}$, and the greatest lower bound of f and g in the lattice $C(X)$ will be symbolized by $f \wedge g$. When we say that a collection $\{B_\gamma\}$ of subsets of a space Y containing a point p determines the topology of Y at p, we mean that if f is a real-valued function on Y and $f \mid B_\gamma$ is continuous for all B_γ, then f is continuous at p.

The following lemma is used in obtaining all of our results on decomposition spectra. A parallel statement to the one given explicitly is indicated by the symbols in square brackets.

LEMMA. *Let S be a subset of a completely regular Hausdorff space X. Suppose there exists a collection $\{T_\gamma\}$ of subsets of S with the following properties*:

(1) *$\{T_\gamma\}$ is closed under finite unions*;

(2) *For each $p \in S$, the collection of all sets in $\{T_\gamma\}$ containing p determines the topology of S at p*;

(3) *For each $f \in C(S)$ and each T_γ, the function $f \mid T_\gamma$ can be extended to a function in $C(X)$ $[C^*(X)]$.*

Then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$ $[C^*(X)]$.

Proof. One obtains the proof of the parallel statement by replacing "$C(X)$" with "$C^*(X)$" throughout the following proof.

From (1), $\{T_\gamma\}$ is directed by the relation $\supset$. For each γ, let I_γ be the ideal $\{h \in C(X): h[T_\gamma] = \{0\}\}$. Now each $C(X)/I_\gamma$ is isomorphic to $\{g \mid T_\gamma: g \in C(X)\}$, so we shall view each element of $C(X)/I_\gamma$ as an element of $\{g \mid T_\gamma: g \in C(X)\}$. Thus, if $T_\gamma \supset T_\delta$, then the natural homomorphism defined by $g \mid T_\gamma \rightarrow g \mid T_\delta$ for $g \in C(X)$ may be considered a homomorphism of $C(X)/I_\gamma$ onto $C(X)/I_\delta$. Also, the transitivity property is clearly satisfied by restriction mappings. Hence $\{C(X)/I_\gamma\}$ and the natural homomorphisms comprise a decomposition spectrum of $C(X)$.

Now let $f \in C(S)$ be given. We define an element $(f_\gamma) \in \varprojlim (C(X)/I_\gamma)$ as follows: For each γ, f_γ is the image in $C(X)/I_\gamma$ of a function in $C(X)$ whose restriction to T_γ coincides with $f \mid T_\gamma$; the existence of such a function is ensured by (3). Then $(f_\gamma) \in \varprojlim (C(X)/I_\gamma)$, because $T_\gamma \supset T_\delta$ implies $f_\gamma \mid T_\delta = (f \mid T_\gamma) \mid T_\delta = f \mid T_\delta = f_\delta$. The mapping $\sigma: f \rightarrow (f_\gamma)$ embeds $C(S)$ in $\varprojlim (C(X)/I_\gamma)$, since $f \neq g$ implies $f(p) \neq g(p)$ for some $p \in S$, whence $f_\gamma \neq g_\gamma$ for any γ such that $p \in T_\gamma$. Furthermore, σ is an isomorphism, because $(f + g)_\gamma = f_\gamma + g_\gamma$ and $(fg)_\gamma = f_\gamma g_\gamma$ for each γ.

To prove that σ is surjective, let $(b_\gamma) \in \varprojlim (C(X)/I_\gamma)$ be given. Then b_γ is the restriction of a function in $C(X)$ to T_γ, and, since $\gamma < \delta$ implies that b_γ maps to b_δ under the natural homomorphism, b_γ is an extension of b_δ. By (2), $\{T_\gamma\}$ covers S, so (b_γ) may be associated with a function b on S. Since b is continuous on each T_γ, (2) implies that $b \in C(S)$; and $\sigma(b) = (b_\gamma)$.

THEOREM 1. *If X is a first countable space and S is any subset of X, then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$ $[C^*(X)]$.*

Proof. Let $\{T_\gamma\}$ be the collection of all subsets of S consisting of a finite number of points of S together with sequences converging to those points. It is clear that (1) holds; (2) holds because X, and hence S, is first countable; and (3) holds because each T_γ is compact. Hence the Lemma is applicable.

COROLLARY 1. *If X is a first countable space, then $C(X)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

THEOREM 2. *If X is a locally compact space, and S is any open subset of X, then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

Proof. Let $\{T_\gamma\}$ be the collection of finite unions of some family of compact neighborhoods of the points of S. It is evident that (1) and (2) hold; and (3) holds because each T_γ is compact. Hence the Lemma is applicable.

COROLLARY 2. *If X is a locally compact space, then $C(X)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

THEOREM 3. *If X is any completely regular Hausdorff space, and S is any open subset of X, then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$.*

Proof. Let $p \in S$. By complete regularity, there exists an $h_p \in C(X)$ such that $h_p[X - S] = \{0\}$, $h_p(p) = 2$, and $\mathbf{0} \leqq h_p \leqq \mathbf{2}$. Hence the nonnegative function $g_p = h_p \wedge \mathbf{1}$ has the properties $g_p[X - S] = \{0\}$ and $g_p[U_p] = \{1\}$, where U_p is a neighborhood of p. Choose one such neighborhood for each point of S, and let $\{T_\gamma\}$ be the collection of finite unions of these neighborhoods. Again, (1) and (2) are clear. To see that (3) holds, consider a particular $T_\gamma = U_{p_1} \cup \cdots \cup U_{p_n}$. The nonnegative function $g_{p_1} + \cdots + g_{p_n}$ is zero on $X - S$ and at least one on T_γ, whence $g_\gamma = (g_{p_1} + \cdots + g_{p_n}) \wedge \mathbf{1}$ has the properties $g_\gamma[X - S] = \{0\}, g_\gamma[T_\gamma] = \{1\}$, and $\mathbf{0} \leqq g_\gamma \leqq \mathbf{1}$. If $f \in C(S)$, we define $f_\gamma \in C(X)$ by $f_\gamma[X - S] = \{0\}$ and $f_\gamma(x) = \tan((g_\gamma(x))(\arctan(f(x))))$ for $x \in S$. Then $f_\gamma \mid T_\gamma = (\tan \circ \arctan \circ f) \mid T_\gamma = f \mid T_\gamma$, as required. Thus (3) holds, and again the Lemma is applicable.

REMARK. If S is a cozero-set in X, say $S = \{x \in X: h(x) \neq 0\}$, where $h \in C(X)$, then the decomposition spectrum can be formed from an ω^*-sequence of quotient rings of $C(X)$. For each positive integer n, we set $T_n = \{x \in X: |h(x)| \geqq 1/n\}$. There exists a function $g_n \in C(X)$ such that $g_n[X - S] = \{0\}, g_n[T_n] = \{1\}$, and $\mathbf{0} \leqq g_n \leqq \mathbf{1}$, since T_n is completely separated from $X - S$ [3; 1.15]. The proof that the collection $\{T_n\}$ satisfies (3) then concludes as in the proof of Theorem 3. Now (1) is evident, and (2) holds because each $p \in S$ is in the interior of some T_n; so the Lemma is applicable to this situation too.

We now give an application of Theorem 3. First recall that the maximal ring of quotients of a commutative ring A with identity may be obtained as the direct limit of the A-homomorphisms of dense ideals of A into A [1; 1.9], and that the classical ring of quotients may be obtained similarly from the A-homomorphisms of dense principal ideals [1; 1.10]. Fine, Gillman, and Lambek have shown that (1) the maximal ring of quotents of $C(X)$ has a representation as the direct limit of rings $C(U)$, U ranging over the dense open sets in X, and that (2) the classical ring of quotients of $C(X)$ has a representation as the direct limit of rings $C(U)$, U ranging over the dense cozero-sets in X [1; 2.6]. Combining Theorem 3 with these facts yields the following result.

COROLLARY 3. *If X is any completely regular Hausdorff space, theu both the maximal and classical rings of quotients of $C(X)$ have representations as a direct limit of inverse limits of quotient rings of $C(X)$.*

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