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A CLASS OF MEASURES ON THE BOHR GROUP

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Let R_d denote the real line with the discrete topology. Let $B = \hat{R}_d$ be its dual. R, the real line is continuously isomorphic to a dense subgroup of B. Let μ be a finite positive measure defined for Borel subsets of B. Let χ_t denote the character on B corresponding to the real number t. We shall denote by H_S the subspace of $L_2(B, \mu)$ spanned by $\{\chi_t : t \leqq S\}$. Assume that $\bigcap_{-\infty<S<\infty} H_S = \{0\}$. In this case the subspaces H_S are strictly increasing in the sense that $H_S \subsetneqq H_{S'}$ whenever $S < S'$. The increasing subspaces generate a spectral measure E defined for intervals $a < x \leqq b$ by $E(a, b] =$ orthogonal projection on $H_b \ominus H_a$. We shall say that E has multiplicity 1 if there exists an element $w \in L_2(B, \mu)$ such that $\{E(\sigma)w : \sigma \in \mathscr{B}\}$ spans $L_2(B, \mu)$. Here $\mathscr{B}$ denotes the class of Borel subsets of R.

THEOREM 1. **Assume that**
(i) $\bigcap_S H_S = \{0\}$.
(ii) E has multiplicity 1.
Then μ sits on a coset of R in B.

The present work was suggested and is strongly influenced by papers of Helson and Lowdenslager [6, 7] and Helson [4, 5]. Other papers that were useful are also listed in the references.

2. In this section we shall state some general results about spectral measures pertinent to us and prove some results that shall be of use. Proof of Theorem 1 will be given in § 3.

Let H be a (complex) Hilbert space. Let E be a spectral measure defined on Borel subsets $\mathscr{B}$ of the real line. Values of E are orthogonal projections on subspaces of H. Suppose that E has multiplicity 1, i.e., there exists a vector $z \in H$ such that $\{E(\sigma)z : \sigma \in \mathscr{B}\}$ spans H. This implies that H is separable. Such a z is called cyclic vector for E. Write $m(\sigma) = \| E(\sigma) \|^2$, $\sigma \in \mathscr{B}$. Then m is a finite positive measure on $\mathscr{B}$. If we write $z(\sigma) = E(\sigma)z$, then $z(\sigma)$ is a vector valued measure taking values in H. It can be shown that under the assumption of multiplicity 1, every element $h \in H$ has an integral representation of the type $h = \int_{-\infty}^{\infty} \varphi_h(\lambda) dz(\lambda)$, with

$$\int_{-\infty}^{\infty} | \varphi_h(\lambda) |^2 dm(\lambda) < \infty$$

[§ 1, p. 264]. Further $E(\sigma)h = \int_{\sigma} \varphi_h(\lambda) dz(\lambda)$. Let $h_1, h_2, h_3, \cdots$ be a complete orthonormal set in H. Let measures m_i, $i = 1, 2, 3, \cdots$ be defined by $m_i(\sigma) = \|E(\sigma)h_i\|^2$. Let ν denote the measure $\nu(\sigma) = \sum_{i=1}^{\infty}(1/2^n)m_i(\sigma)$. The following lemma is known and easy to prove.

LEMMA 1. *If E has multiplicity* 1 *and z its cyclic vector, then measure m and ν are mutually absolutely continuous. Here m is the measure defined by $m(\sigma) = \|E(\sigma)z\|^2$.*

The next lemma is known and its proof can be found in ([9], p. 318). For any measure μ_0 we shall write μ_t for the measure $\mu_t(\sigma) = \mu(\sigma + t)$, $\sigma \in \mathscr{B}$, $t \in R$.

LEMMA 2. *μ_0 is a finite positive measure defined on Borel subsets of R. Assume that μ_0 and μ_t are mutually absolutely continuous. Then μ_0 and Lebesgue measure are mutually absolutely continuous.*

A function called cocycle presents itself in the proof of Lemma 5. It is necessary to show that, in our context, such function has a special form called coboundary.

DEFINITION 1. A function $A(t, \lambda)$ on $R \times R$ is called a cocycle if
(i) $|A(t, \lambda)| = 1$ for all t, λ.
(ii) $A(t, \lambda)$ is Borel measurable in λ for every fixed t.
(iii) $A(t + u, \lambda)A(t, \lambda)A(u, \lambda + t)$ for almost every $\lambda \in R$ (with respect to the Lebesgue measure).

The set of Lebesgue measure zero where (iii) does not hold may depend on (t, u).

DEFINITION 2. A cocycle $A(t, \lambda)$ is called a coboundary if it is of the type $B(\lambda + t)B^{-1}(\lambda)$ for some function B on R of absolute value 1.

We shall prove the following

LEMMA 3. *Every cocycle is a coboundary.*

Few remarks should be made before we prove this lemma. The proof of Lemma 3 is trivial if condition (iii) of cocycle held everywhere instead of almost everywhere. For in that case we need only put $\lambda = 0$ in (iii) and observe that

$$A(u, t) = A(t + u, 0)A^{-1}(t, 0) .$$

Cocycles and coboundaries occur very crucially in the works of Henry Helson and David Lowdensager although domain of definitions of these functions changes according to context in their work. In his book on invariant subspace [4], Helson proved Lemma 3 under an additional hypothesis which is equivalent to requiring that $A(t, \lambda)$ be jointly measurable in (t, λ). There are cases however where one has to deal with cocycles $A(t, \lambda)$ which are measurable in λ for every fixed t. The present paper is one such case.

Finally we remark that the idea of our proof is already contained in the papers of Mackey [9] and Helson [4, 5]. We state here, without proof, a theorem of Mackey [9, p. 317] which we shall need in the proof of Lemma 3.

THEOREM 2. *Let M_1 and M_2 be sigma finite measure spaces with measures μ_1 and μ_2. Suppose that there is a countably generated Borel field $\mathscr{C}$ of measurable subsets of M_2 such that every measurable subset of M_2 differs from some member of $\mathscr{C}$ by subset of a set of measure zero. Let f be a complex valued function on $M_1 \times M_2$ which is measurable and essentially bounded as a function on M_2 for each fixed point in M_1. Suppose that $\int_E f(x, y)d\mu_2(y)$ is measurable on M_1 for each fixed measurable subset E of M_2 of finite measure. Then there exists a function $f'(x, y)$ jointly measurable on $M_1 \times M_2$ such that for all $x \in M_1$*

$$f(x, y) = f'(x, y) \qquad \text{for almost all } y \in M_2 .$$

Proof of Lemma 3. Consider

$$F(u, t, \lambda) = A(t + u, \lambda)A^{-1}(t, \lambda)A^{-1}(u, \lambda) .$$

From the cocycle relations (iii) we see that for each fixed u, t

(1) $F(u, t, \lambda) = A(u, \lambda + t)A^{-1}(u, \lambda)$ a.e. λ .

(2) $F(u, t, \lambda) = A(t, \lambda + u)A^{-1}(t, \lambda)$ a.e. λ .

(1) and (2) show that $F(u, t, \lambda)$ is measurable in (t, λ) for each fixed u and measurable in (u, λ) for each fixed t. We show that F can be chosen to be measurable in all three variable (u, t, λ) and still satisfy (1) and (2). Let σ be a measurable set in (t, λ) of finite measure. Consider

$$\iint_\sigma F(u, t, \lambda)d\lambda dt .$$

We show that this integral moves continuously in u.

$$\begin{aligned} &\iint_\sigma | F(u, t, \lambda) - F(s, t, \lambda) | d\lambda dt \\ &\quad = \iint_\sigma | A(t + u, \lambda)A^{-1}(t, \lambda)A^{-1}(u, \lambda) \\ &\qquad - A(t + s, \lambda)A^{-1}(t, \lambda)A^{-1}(s, \lambda) | d\lambda dt \\ (*) &\quad = \iint_\sigma | A(t + u, \lambda)A^{-1}(u, \lambda) - A(t + s, \lambda)A^{-1}(s, \lambda) | d\lambda dt . \end{aligned}$$

This identity holds since $|A(t, \lambda)| = 1$. Again by cocycle identity (iii), (*) is equal to

$$\iint_\sigma |A(t, \lambda + u) - A(t, \lambda + s)| d\lambda dt \to 0 \quad \text{as } s \to u .$$

Thus for each (t, λ) measurable set σ of finite measure

$$\iint_\sigma F(u, t, \lambda) d\lambda dt$$

moves continuously in u. So by Theorem 2, we can replace $F(u, t, \lambda)$ by a measurable function in all three variables.

So we assume now that F is measurable in all three variables. Now

$$(^{**}) \quad A(t + u, \lambda)A^{-1}(t, \lambda)A^{-1}(u, \lambda) = A(u, \lambda + t)A^{-1}(u, \lambda)$$

for almost every λ for fixed (t, u). But the left hand side is measurable in (u, t, λ) and so by Fubini theorem there exists λ_0 such that (**) holds for almost every (u, t). So for almost every (u, t)

$$A(t + u, \lambda_0)A^{-1}(t, \lambda_0)A^{-1}(u, \lambda_0) = A(u, \lambda_0 + t)A^{-1}(u, \lambda_0)$$
$$A(t + u, \lambda_0)A^{-1}(t, \lambda_0) = A(u, \lambda_0 + t) .$$

Put $\lambda_0 + t = s$, then

$$A(s - \lambda_0 + u, \lambda_0)A^{-1}(s - \lambda_0, \lambda_0) = A(u, s) .$$

Write $B(x) = A(x - \lambda_0, \lambda_0)$. Obviously $A(u, s) = B(u + s)B^{-1}(s)$.

REMARK. We note that

$$B(u + s)B^{-1}(s)B^{-1}(u) = A(u + s - \lambda_0, \lambda_0)A^{-1}(s - \lambda_0, \lambda_0)A^{-1}(u - \lambda_0, \lambda_0)$$

is jointly measurable in (u, s).

DEFINITION. A spectral measure E on $\mathscr{B}$ is called stationary if there exists a commutative group T^t, $t \in R$, of unitary operators such that for every Borel set $\sigma \in \mathscr{B}$, $T^tE(\sigma)T^{-t} = E(\sigma + t)$.

LEMMA 4. *Let E be a stationary spectral of multiplicity* 1 *and let z be a cyclic vector for E. Then the measure m defined by $m(\sigma) = \|E(\sigma)z\|^2$ and Lebesgue measure are mutually absolutely continuous.*

Proof. Let $h_1, h_2, h_3, \cdots$ be a complete orthonormal set in H. Since T^t is unitary $T^th_1, T^th_2, \cdots$ is again a complete orthonormal set in H. By Lemma 1, m is equivalent to μ_t defined by $\mu_t(\sigma) = \sum_{n=1}^{\infty} (1/2^n)(E(\sigma)T^th_n, T^th_n)$ for every t. Now

$$\mu_t(\sigma) = \sum_{n=1}^{\infty} \frac{1}{2^n}(E(\sigma)T^t h_n, T^t h_n) = \sum_{n=1}^{\infty} (T^{-t}E(\sigma)T^t h_n, h_n)$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^n}(E(\sigma + t)h_n, h_n) = \mu_0(\sigma + t) .$$

Thus μ_0 and μ_t are mutually absolutely continuous for every t, and so by Lemma 2 μ_0 and Lebesgue measure are mutually absolutely continuous. So m and Lebesgue measure are mutually absolutely continuous.

LEMMA 5. *Let E be a stationary spectral measure of multiplicity 1. Then there exists a vector valued measure $z(\cdot)$ on $\mathscr{B}$ and a function A on R of absolute value 1 such that*

(i) *$z(A) \perp z(B)$ whenever $A \cap B =$ null set,*

(ii) *$\| z(A) \|^2 = L(A)$ where L stands for the Lebesgue measure,*

(iii) *every $h \in H$ has a representation of the type*

$$h = \int_{-\infty}^{\infty} \varphi_h(\lambda)dz(\lambda), \int_{-\infty}^{\infty} | \varphi_h(\lambda) |^2 d\lambda < \infty ,$$

(iv) $E(\sigma)h = \int_\sigma \varphi_h(\lambda)dz(\lambda)$,

(v) *$A^{-1}(\lambda)A(\lambda + t)$ is measurable in λ for every t,*

(vi) $T^t h = \int_{-\infty}^{\infty} \varphi_h(\lambda)A^{-1}(\lambda)A(\lambda + t)dz(\lambda + t)$.

Proof. Let w be a cyclic vector for E and m the measure defined by $m(\sigma) = \| E(\sigma)w \|^2$, $\sigma \in \mathscr{B}$. Let ρ denote the Radon-Nikodym derivative m with respect to the Lebesgue measure L. Since m and L are mutually absolutely continuous, $\rho > 0$ a.e. (L).

Define $z(\cdot)$ by $z(\sigma) = \iint_\sigma (1/\sqrt{\rho(\lambda)})dw(\lambda)$. Then properties (i) and (ii) for $z(\cdot)$ are easily verified. To prove (iii), let $h \in H$. Then h has a representation: $h = \int_{-\infty}^{\infty} \psi_h(\lambda)dw(\lambda)$ with $\int_{-\infty}^{\infty} | \psi_h(\lambda) |^2 dm(\lambda) < \infty$. $h = \int_{-\infty}^{\infty} \psi_h(\lambda)\sqrt{\rho(\lambda)}(1/\sqrt{\rho(\lambda)})dw(\lambda)$. Now $(1/\sqrt{\rho(\lambda)})dw(\lambda)$ is equal to $dz(\lambda)$. We write $\psi_h(\lambda)\sqrt{\rho(\lambda)} = \varphi_h(\lambda)$, and representation for h becomes $h = \int_{-\infty}^{\infty} \varphi_h(\lambda)dz(\lambda)$ with $\int_{-\infty}^{\infty} | \varphi_h(\lambda) |^2 d\lambda < \infty$. This proves (iii). Again $E(\sigma)h = \int_\sigma \psi_h(\lambda)dw(\lambda) = \int_\sigma \varphi_h(\lambda)dz(\lambda)$ can be seen to be true. This proves (iv). It remains to show (v) and (vi). Let us write $\tilde{z}_t(\sigma) = T^t z(\sigma - t)$. $\tilde{z}_t$ is again a countably additive measure on $\mathscr{B}$ with values in H. Since E is stationary, $\tilde{z}_t(\sigma) \in E(\sigma)H$. So $\tilde{z}_t(\sigma)$ has a representation of the type: $\tilde{z}_t(\sigma) = \int_\sigma A(t, \lambda)dz(\lambda)$ for some function $A(t, \lambda)$, measurable in λ for every t. Further

$$\|\tilde{z}_t(\sigma)\|^2 = \|T^t z(\sigma - t)\|^2 = \|z(\sigma - t)\|^2 = L(\sigma - t)$$
$$= L(\sigma) = \int_\sigma |A(t, \lambda)|^2 d\lambda .$$

Since this holds for every σ, $|A(t, \lambda)| = 1$ for almost every λ. We can write the relation between $\tilde{z}_t(\cdot)$ and $z(\cdot)$ formally as $d\tilde{z}_t(\lambda + t) = A(t, \lambda)dz(\lambda + t)$ which is the same as $T^t dz(\lambda) = A(t, \lambda)dz(\lambda + t)$. Now

$$\begin{aligned} T^{s+t}dz(\lambda) &= T^s T^t dz(\lambda) = T^s A(t, \lambda)dz(\lambda + t) \\ &= A(t, \lambda)A(s, \lambda + t)dz(\lambda + t + s) \\ &= A(s + t, \lambda)dz(\lambda + s + t) . \end{aligned}$$

From this equation we get $A(t, \lambda)A(s, \lambda + t) = A(s + t, \lambda)$. Thus $A(t, \lambda)$ is a cocycle. By Lemma 3 A is of the type $A(\lambda + t)/A(\lambda)$ for some functions A. We choose this A for A of Lemma 5. Obviously $|A(t)| = 1$ for every t. Further $A(t, \lambda) = A^{-1}(\lambda)A(\lambda + t)$ is measurable in λ for every t. This proves (v). Finally let $h = \int_{-\infty}^{\infty} \varphi_h(\lambda)dz(\lambda)$. Then

$$\begin{aligned} T^t h &= T^t \int_{-\infty}^{\infty} \varphi_h(\lambda)dz(\lambda) = \int_{-\infty}^{\infty} \varphi_h(\lambda)T^t dz(\lambda) = \int_{-\infty}^{\infty} \varphi_h(\lambda)A(t, \lambda)dz(\lambda + t) \\ &= \int_{-\infty}^{\infty} \varphi_h(\lambda)A^{-1}(\lambda)A(\lambda + t)dz(\lambda + t) . \end{aligned}$$

This proves (vi).

3. Let us return to the notation and terminology of Theorem 1. For $f \in L_2(B, \mu)$, write $T^t f = \chi_t f$, where χ_t is a character on B corresponding to the real number t. It is obvious that T^t is a commutative group of unitary operators on $L_2(B, \mu)$. Further following two identities can be easily verified:

(A) $T^t(H(b) \ominus H(a)) = H(b + t) \ominus H(a + t)$ where a, b $(a < b)$ are any two real numbers.

(B) For any $f \in L_2(B, \mu)$, $\|E(a, b]f - f\|^2 = \|T^t E(a, b]f - T^t f\|^2$.

(A) and (B) together imply that E is a stationary spectral measure, $T^t E(\sigma)T^{-t} = E(\sigma + t)$.

Proof of Theorem 1. The spectral measure E of Theorem 1 is stationary as shown in the above paragraph. By hypothesis E has multiplicity 1. By Lemma 5 there exists a vector valued measure $z(\cdot)$ with values in $L_2(B, \mu)$ and a function A on R of absolute value 1 satisfying (i)–(vi) of Lemma 5. $\chi_0 \in L_2(B, \mu)$ has a representation of the type $\chi_0 = \int_{-\infty}^{0} f(\lambda)dz(\lambda)$.

$$\chi_t = \chi_t\chi_0 = T^t\chi_0 = \int_{-\infty}^{0} f(\lambda)A^{-1}(\lambda)A(\lambda + t)dz(\lambda + t) .$$

$$(*) \qquad \chi_t A^{-1}(t) = \int_{-\infty}^{0} f(\lambda)A^{-1}(\lambda)A(\lambda + t)A^{-1}(t)dz(\lambda + t) .$$

We shall show that $\chi_t(b)A^{-1}(t)$ is measurable in t for almost every $b \in B$ (with respect to μ). First observe that, by remark following Lemma 3, $A^{-1}(\lambda)A(\lambda + t)A^{-1}(t)$ is measurable in (t, λ). Next the integral representation (*) for $\chi_t A^{-1}(t)$ exists in the sense that approximating sums of the type $\sum_i f(\lambda_i)A^{-1}(\lambda_i)A^{-1}(\lambda_i + t)A^{-1}(t)z(\sigma_i)$ converge to $\chi_t A^{-1}(t)$ in $L_2(B, \mu)$. Since μ is a finite measure the sequence of approximating sums converges to $\chi_t A^{-1}(t)$ almost everywhere on B with respect to μ. But each of the approximating sum is measurable in t for every $b \in B$. Hence for almost every $b \in B$ (with respect to μ) $\chi_t(b)A^{-1}(t)$ is measurable in t (see [2], p. 430). Now fix a $b' \in B$ such that $\chi_t(b')A^{-1}(t)$ is measurable in t. Consider $\chi_t(b' - b)$. $\chi_t(b' - b) = \chi_t(b')(\chi_t(b))^{-1} = \chi_t(b')A^{-1}(t)(\chi_t(b)A^{-1}(t))^{-1} =$ ratio of two measurable functions in t for almost every b. So $\chi_t(b - b')$ is measurable in t for almost every $b \in B$ (with respect to μ). But a measurable character on R is necessarily continuous. So $b' - b \in R$ for almost every $b \in B$ (with respect to μ). Hence $b \in b' + R$ for almost every $b \in B$ (with respect to μ). So μ sits on a coset of R in B.

Much more is true than simply the fact that μ sits on a coset of R in B. For example, μ restricted to the appropriate coset is absolutely continuous with respect to the Lebesgue measure on that coset and if f is its Radon-Nikodym derivative, then

$$\left|\int \frac{\log f(\lambda)}{1 + \lambda^2} d\lambda \right| < \infty \qquad \text{([2], p. 586)} .$$

A converse of Theorem 1 is true: If $\cap H_s = \{0\}$ and μ sits on a coset of R in B, then E has multiplicity 1. This is essentially a consequence of a result of O. Hanner [3] on representation of weakly stationary purely nondeterministic stationary stochastic process.

A finite regular measure ν on B is called analytic if $\int_B \chi_t(b)\nu(db) = 0$ for $t < 0$. Let μ denote the total variation measure of ν. It can be shown that the subspaces H_s in $L_2(B, \mu)$ have the property $\bigcap_{-\infty<s<\infty} H_s = \{0\}$. Let E be the spectral measure generated by $H_s - \infty < s < \infty$. If E has multiplicity 1 then by Theorem 1 μ sits on a coset of R in B and so ν sits on a coset of R in B.

Recently in collaboration with V. Mandrekar, we have studied finite regular measure μ on B for which $\bigcap_{-\infty<s<\infty} H_s = \{0\}$ without assuming that spectral measure E has multiplicity 1. These results will be published elsewhere.

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