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RELATIVE FUNCTOR REPRESENTABILITY

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This paper deals with the general problem of determining conditions under which the representability of a given functor $G: \mathrm{A} \to \mathrm{E}ns$ implies the representability of a subfunctor $F: \mathrm{B} \to \mathrm{E}ns$ of the restriction of G to a subcategory B of A. With suitable conditions on A and B a set of necessary and sufficient conditions for the representability of such a functor F can be obtained. A few examples are given which indicate the connection between this case of relative or induced representability and universal algebra.

If $\boldsymbol{A}$ is a suitably restricted category, then a theorem giving a set of necessary and sufficient conditions for a functor $G: \boldsymbol{A} \to \boldsymbol{Ens}$ to be representable can be proved starting from the same concepts developed for relative representability in the first section. This result on absolute representability is similar to one of Benabou and has as corollary the theorem of Freyd giving a set of conditions for the existence of adjoint functors. We use the convention throughout that the functor $T: \boldsymbol{A} \to \boldsymbol{B}$ has an adjoint $S: \boldsymbol{B} \to \boldsymbol{A}$ or that T is a coadjoint of S if the Hom functors $\boldsymbol{A}(S-, -)$ and $\boldsymbol{B}(-, T-): \boldsymbol{B}^{op} \times \boldsymbol{A} \to \boldsymbol{Ens}$ are naturally isomorphic.

1. Minimal factorizations. Suppose that $G: \boldsymbol{A} \to \boldsymbol{Ens}$ is a functor and that $\boldsymbol{Ens}$ is the category of sets. Let $\boldsymbol{A}_G$ be the category whose objects are those pairs (A, x) with $A \in \boldsymbol{A}$ and $x \in GA$ and whose morphisms $\alpha: (A, x) \to (B, y)$ are those morphisms $\alpha: A \to B$ of $\boldsymbol{A}$ such that $(G\alpha)x = y$.

$\boldsymbol{A}$ has *minimal G factorizations* if for each (A, x) in $\boldsymbol{A}_G$ there is a subobject $\kappa: K \to A$ of A in $\boldsymbol{A}$ minimal with respect to the property that $\kappa: (K, k) \to (A, x)$ for some $k \in GK$. In addition it is required that if α and β are morphisms $(K, k) \to (B, y)$ then $\alpha = \beta$. We call $x = G(\kappa)k$ a minimal G factorization of x.

$\boldsymbol{A}$ is *well powered* if the class of subobjects of each object is a set. The term *co-well powered* is defined dually. $\boldsymbol{A}$ is *complete* if each set of objects of $\boldsymbol{A}$ has a product and any pair of morphisms $\alpha, \beta: A \to B$ has an equalizer.

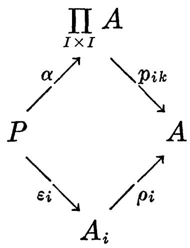
$G: \boldsymbol{A} \to \boldsymbol{Ens}$ is *continuous* if G preserves products and equalizers.

LEMMA. *If $\boldsymbol{A}$ is a well powered complete category and if $G: \boldsymbol{A} \to \boldsymbol{Ens}$ is continuous, then $\boldsymbol{A}$ has minimal G factorizations.*

Proof. Let $\lambda: E \to A$ be the intersection of the set $S =$

$\{\rho_i: A_i \dashrightarrow A\}_{i \in I}$ of all subobjects of A such that $\rho_i: (A_i, x_i) \dashrightarrow (A, x)$ for some $x_i \in GA_i$.

Let $P = \prod_{i \in I}(A_i)$ with projections ε_i. For the product $\prod_{I \times I}(A)$ with projections p_{ik} there exists a unique pair of morphisms $\alpha, \beta: P \dashrightarrow \prod_{I \times I}(A)$ such that



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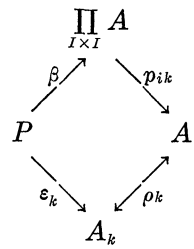
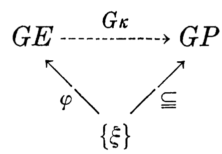


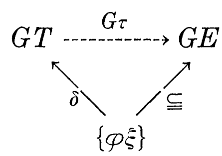
commute for each $i, k \in I$. There exists an equalizer κ of α, β with $\lambda = \rho_k \varepsilon_k \kappa: E \dashrightarrow A$.

For each $\rho_i \in S$ let $x_i \in GA_i$ be an element such that $\rho_i: (A_i, x_i) \dashrightarrow (A, x)$. Since G preserves products there is a unique $\xi \in \prod_I GA_i = G(\prod A_i)$ such that $x_i = (G\varepsilon_i)\xi$ for each $i \in I$. Thus $\varepsilon_i: (P, \xi) \dashrightarrow (A_i, x_i)$. From the diagram it follows that $p_{ik}\alpha, p_{ik}\beta: (P, \xi) \dashrightarrow (A, x)$ for each $(i, k) \in I \times I$. Thus $(G\alpha)\xi = (G\beta)\xi$ and since G preserves equalizers there is a unique φ such that



commutes. Thus $\lambda = \rho_i \varepsilon_i \kappa: (E, \varphi\xi) \dashrightarrow (A, x)$ as required.

If $\mu, \omega: (E, \varphi\xi) \dashrightarrow (B, y)$, then let $\tau: T \dashrightarrow E$ be the equalizer in A of μ, ω. Since G preserves equalizers there is a unique δ such that



commutes. Thus $\tau: (T, \delta\varphi\xi) \dashrightarrow (E, \varphi\xi)$ must be an equivalence since otherwise we would have a contradiction to the minimality of $\lambda: E \dashrightarrow A$.

A morphism $x: R \dashrightarrow A$ *admits an image* if there exists a smallest subobject $\kappa: K \dashrightarrow A$ such that x has a factorization $R \dashrightarrow K \dashrightarrow A$ with $R \dashrightarrow K$ epic.

COROLLARY. *In a well powered, complete category every morphism $x: R \dashrightarrow A$ admits an image.*

Proof. The minimal $G = \boldsymbol{A}(R, -)$ factorization of x is the one required.

COROLLARY. (*Freyd*) *If* $\boldsymbol{A}$ *is well powered and complete and* $J: \boldsymbol{A} \dashrightarrow \boldsymbol{B}$ *is continuous, then for every morphism* $y: B \dashrightarrow JA$ *there is a minimal subobject of* A *which allows* y.

Proof. Let $G = \boldsymbol{B}(B, J-)$.

2. Absolute representability. The functor $G: \boldsymbol{A} \dashrightarrow \boldsymbol{Ens}$ is *representable* if there exists an object U in $\boldsymbol{A}$ such that the Hom functor $\boldsymbol{A}(U, -)$ is naturally isomorphic to G. Equivalently, $G: \boldsymbol{A} \dashrightarrow \boldsymbol{Ens}$ is representable if $\boldsymbol{A}_G$ has an initial point (cf. Mac Lane [7]).

THEOREM. *If* $\boldsymbol{A}$ *is a well powered and complete category, then* $G: \boldsymbol{A} \dashrightarrow \boldsymbol{Ens}$ *is representable if and only if*

(i) G *is continuous.*

(ii) *There exists a set of objects* (A_i, a_i) *in* $\boldsymbol{A}_G$, *indexed by* I, *such that for each* (A, x) *in* $\boldsymbol{A}_G$ *there exists* $(A_i, a_i) \dashrightarrow (A, x)$ *for some* i *in* I.

Proof. If $\phi: \boldsymbol{A}(R, -) \dashrightarrow G$ is a natural isomorphism for some $R \in \boldsymbol{A}$, then $(R, \phi_R(1_R))$ is initial in $\boldsymbol{A}_G$ and the continuity of G follows from that of $\boldsymbol{A}(R, -)$.

For the converse we will find an initial point for $\boldsymbol{A}_G$. Let $B = \prod_{j \in I} (A_j)$ with projections ε_j. There exists a unique $b \in GB$ with j-th component a_j since G is continuous. Thus we have $\varepsilon_j: (B, b) \dashrightarrow (A_j, a_j)$ in $\boldsymbol{A}_G$. By the Lemma there exists $\kappa: (B', b') \dashrightarrow (B, b)$ giving a minimal G factorization of b. If $(A, x) \in \boldsymbol{A}_G$, then by hypothesis there exists $\varphi: (A_i, a_i) \dashrightarrow (A, x)$ for some $i \in I$. Thus $\varphi\varepsilon_i\kappa: (B', b') \dashrightarrow (A, x)$. If $\beta: (B', b') \dashrightarrow (A, x)$, then $\beta = \varphi\varepsilon_i\kappa$ by the definition of minimal G factorizations. Hence (B', b') is the required initial point.

This theorem has the following result of Freyd [4] as a corollary.

COROLLARY. *Let* $\boldsymbol{A}$ *be a well powered and complete category and let* $J: \boldsymbol{A} \dashrightarrow \boldsymbol{B}$ *be a functor. Then* J *has an adjoint if and only if*

(i) J *is continuous*

(ii) *For* B *in* $\boldsymbol{B}$ *there is a set of objects* $S_B \subseteqq \boldsymbol{A}$ *such that for* $B \longrightarrow JA$ *with* $A \in \boldsymbol{A}$ *there is* $a_i: B \dashrightarrow JA_i$ *with* $A_i \in S_B$ *and* $\alpha: A_i \dashrightarrow A$ *such that*

$$\begin{array}{ccc} B & \overset{a_i}{\dashrightarrow} & JA_i \\ & \searrow & \downarrow J\alpha \\ & & JA \end{array}$$

commutes in $\boldsymbol{B}$*.*

3. **Relative representability.** Let

$$\begin{array}{l} G\colon \boldsymbol{A} \dashrightarrow \boldsymbol{Ens} \\ \quad \uparrow \subseteq \\ F\colon \boldsymbol{B} \dashrightarrow \boldsymbol{Ens} \end{array}$$

be a diagram of categories and functors such that $FB \subseteq GB$ for all B in $\boldsymbol{B}$ and $F\beta$ is the restriction of $G\beta$ to FB for $\beta\colon B \dashrightarrow B'$ in $\boldsymbol{B}$. Such a functor F will be called a *subfunctor* of the restriction of G to $\boldsymbol{B}$. Then $\rho \in GB$ is *F distinguished* if $B \in \boldsymbol{B}$ and $\rho \in FB$.

Now we come to a useful set of conditions *sufficient* to ensure the representability of a subfunctor $F\colon \boldsymbol{B} \dashrightarrow \boldsymbol{Ens}$ of the restriction of a representable functor $G\colon \boldsymbol{A} \dashrightarrow \boldsymbol{Ens}$ to $\boldsymbol{B} \subseteq \boldsymbol{A}$.

THEOREM. *Let* $\boldsymbol{A}$ *be well and co-well powered and complete, and let* $\boldsymbol{B}$ *be a full subcategory of* $\boldsymbol{A}$ *containing a copy of* $\prod B_i$ *for each set* B_i *of its objects. Suppose that* F *is a product preserving subfunctor of the restriction of* $\boldsymbol{A}(R, -)$ *to* $\boldsymbol{B}$*, with the property that if* ρ *is* F *distinguished, than* ρ *has an image*

$$\begin{array}{ccc} R & \overset{\rho}{\dashrightarrow} & B \\ & {\scriptstyle\rho'}\searrow \quad \nearrow & \\ & B' & \end{array}$$

where ρ' *is* F *distinguished. Then* F *is representable by a natural equivalence* $\psi\colon F \dashrightarrow \boldsymbol{B}(R', -)$ *such that the diagram*

$$\begin{array}{ccc} F & \overset{\psi}{\dashrightarrow} & \boldsymbol{B}(R', -) = \boldsymbol{A}(R', -) \mid \boldsymbol{B} \\ & {\scriptstyle\subseteq}\searrow \quad \swarrow{\scriptstyle\sigma^*} & \\ & \boldsymbol{A}(R, -) \mid \boldsymbol{B} & \end{array}$$

commutes for an epic $\sigma\colon R \dashrightarrow R'$ *in* $\boldsymbol{A}$*.*

Proof. If $\{B_k\}_{k \in K}$ is a set of objects in $\boldsymbol{B}$, then there exists $\prod_K (B_k)$ in $\boldsymbol{B}$ with projections $\varepsilon_k\colon \prod (B_k) \dashrightarrow B_k$. Then for each $\rho\colon R \dashrightarrow \prod (B_k)$ for which $\varepsilon_k\rho\colon R \dashrightarrow B_k$ is F distinguished for all $k \in K$, it follows that ρ is F distinguished since F is product preserving.

Let $S = \{X_j: R \dashrightarrow B_j\}_{j \in J}$ be the set of all F distinguished quotient objects of R. There exists a unique $\mu: R \dashrightarrow \prod_{j \in J} B_j$ such that

$$\begin{array}{ccc} R & \overset{\mu}{\dashrightarrow} & \prod_{j \in J} B_j \\ & \searrow^{X_j} & \downarrow{\scriptstyle \varepsilon_j} \\ & & B_j \end{array}$$

commutes in $\boldsymbol{A}$ for each $j \in J$ where ε_j is the projection. But $X_j = \varepsilon_j \mu$ is F distinguished. Hence μ is F distinguished. Let

$$\begin{array}{ccc} R & \overset{\mu}{\dashrightarrow} & \prod B_j \\ & {}_{\mu'}\searrow \quad \nearrow_{i_\mu} & \\ & R' & \end{array}$$

be an image of μ. Then μ' is F distinguished and epic in $\boldsymbol{A}$. Thus $(R', \mu') \in \boldsymbol{B}_F$.

If (C, α) is an object of $\boldsymbol{B}_F$, then let

$$\begin{array}{ccc} R & \overset{\alpha}{\dashrightarrow} & C \\ & {}_{\alpha'}\searrow \quad \nearrow_{i_\alpha} & \\ & C' & \end{array}$$

be an image of α. α' is epic and is F distinguished since α is F distinguished. Hence α' represents a member of S. Let $\varepsilon_{\alpha'}: \prod B_j \dashrightarrow C'$ be the corresponding projection. Thus we obtain $i_\alpha \varepsilon_{\alpha'} i_\mu: (R', \mu') \dashrightarrow (C, \alpha)$ in $\boldsymbol{B}_F$ from the preceding three diagrams noting that

$$R \overset{\mu}{\dashrightarrow} \prod B_j \overset{\varepsilon_{\alpha'}}{\dashrightarrow} C' = R \overset{\alpha'}{\dashrightarrow} C' .$$

But diagram II gives a minimal $\boldsymbol{A}(R, -)$ factorization of μ. Hence if $i_\alpha \varepsilon_{\alpha'} i_\mu$ and τ are morphisms $(R', \mu') \dashrightarrow (C, \alpha)$ in $\boldsymbol{A}_F$ then $i_\alpha \varepsilon_{\alpha'} i_\mu = \tau$. Thus (R', μ') is initial in $\boldsymbol{B}_F$. If $\psi: F \dashrightarrow \boldsymbol{B}(R', -)$ is the corresponding equivalence, then for $\sigma = \mu'$ it is clear that the required diagram commutes.

The category $\boldsymbol{B} \subseteqq \boldsymbol{A}$ is *closed under subobjects* means that if $B \in \boldsymbol{B}$ and $A \dashrightarrow B$ is a monomorphism in $\boldsymbol{A}$, then $A \in \boldsymbol{B}$.

Finally we obtain a set of *necessary and sufficient* conditions for relative representability.

THEOREM. *Let $\boldsymbol{A}$ be well and co-well powered and complete and let $\boldsymbol{B}$ be a full subcategory of $\boldsymbol{A}$ closed under subobjects and containing a copy of $\prod B_i$ for each family B_i of its objects. Assume that any* *morphism which is epic and monic is invertible.*

If F is a subfunctor of the restriction of $\boldsymbol{A}(R, -)$ to $\boldsymbol{B}$, then F is representable by a natural equivalence $\psi\colon F \dashrightarrow \boldsymbol{B}(R', -)$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\psi} & \boldsymbol{B}(R', -) = \boldsymbol{A}(R', -) \mid \boldsymbol{B} \\ & {\scriptstyle\subseteq}\searrow \quad \swarrow{\scriptstyle\sigma^*} & \\ & \boldsymbol{A}(R, -) \mid \boldsymbol{B} & \end{array}$$

commutes for an epic $\sigma\colon R \dashrightarrow R'$ in $\boldsymbol{A}$, if and only if

(i) *For each ρ which is F distinguished with image*

$$\begin{array}{ccc} R & \xrightarrow{\rho} & B \\ {\scriptstyle\rho'}\searrow & & \nearrow \\ & B' & \end{array}$$

it follows that ρ' is F distinguished.

(ii) *F is product preserving.*

Proof. Let ρ be F distinguished with image

$$\begin{array}{ccc} R & \xrightarrow{\rho} & B \\ {\scriptstyle\rho'}\searrow & & \nearrow{\scriptstyle i_\rho} \\ & B' & \end{array}$$

Now $\rho\, F$ distinguished is equivalent to the existence of a factorization

$$\begin{array}{ccc} R & \xrightarrow{\rho} & B \\ {\scriptstyle\sigma}\searrow & & \nearrow{\scriptstyle\rho_0} \\ & R' & \end{array}$$

The morphism ρ_0 has an image

$$\begin{array}{ccc} R' & \xrightarrow{\rho_0} & B \\ {\scriptstyle\rho_0'}\searrow & & \nearrow{\scriptstyle i_\rho'} \\ & B_\rho' & \end{array}$$

with ρ_0' epic in $\boldsymbol{A}$. From the minimality of i_ρ it follows that there exists φ such that

$$\begin{array}{ccccc} R & \xrightarrow{\sigma} & R' & \xrightarrow{\rho_0} & B \\ {\scriptstyle \rho'}\downarrow & & {\scriptstyle \rho_0'}\downarrow & \nearrow{\scriptstyle i_\rho'} & \\ B' & \xrightarrow{\varphi} & B_\rho' & & \end{array}$$

commutes. The morphisms σ and ρ_0' are epic. Thus φ is monic and epic and hence an equivalence. The object $B' \in \boldsymbol{B}$ since $\boldsymbol{B}$ is closed under $\boldsymbol{A}$ subobjects and ρ' factors through σ. Hence ρ' is F distinguished. The converse follows from the preceding theorem.

4. Applications. Let $\Omega = \cup \Omega(n)$ be a disjoint union of sets indexed by the nonnegative integers. Then Ω is called an *operator set*. A is an Ω algebra if A is a set with functions $\omega: A^n \dashrightarrow A$ defined for each $\bar{\omega} \in \Omega(n)$. $\alpha: A \dashrightarrow B$ is a morphism of Ω algebras if α is a set mapping such that

$$\begin{array}{ccc} A^n & \xrightarrow{\omega} & A \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B^n & \xrightarrow{\omega} & B \end{array}$$

commutes for each $\omega \in \Omega(n)$ and each integer n. For fixed Ω let (Ω) be the category of all Ω algebras and their homomorphisms. For further details see Cohn [2].

LEMMA. (a) *There is only one θ algebra structure on the cartesian product of θ algebras so that each of the projections becomes a morphism of θ algebras.*

(b) *If $f: C_1 \dashrightarrow C_2$ is any function, and $g: C_2 \dashrightarrow C_3$ is a θ algebra monomorphism such that gf is a θ algebra morphism, then f is a θ algebra morphism.*

THEOREM. *Let*

$$\begin{array}{ccc} (\Omega) & \xrightarrow{S} & \boldsymbol{Ens} \\ {\scriptstyle \subseteq}\uparrow & & \uparrow{\scriptstyle T} \\ \boldsymbol{B} & \xrightarrow{J} & \boldsymbol{C} \end{array}$$

be a commutative diagram of categories where $\boldsymbol{C}$ is a full subcategory of θ algebras for some operator set θ, $\boldsymbol{B}$ is a variety of Ω algebras, and S, T are forgetful functors. Then J has an adjoint.

Proof. The forgetful functor $S: (\Omega) \dashrightarrow \boldsymbol{Ens}$ has an adjoint W by a result of Cohn [2]. Thus there is a natural equivalence

$$\varphi\colon \boldsymbol{Ens}(TC, S-) \dashrightarrow (\Omega)(WTC, -)\ .$$

It is sufficient to show that $\boldsymbol{C}(C, J-)$ is representable for each $C \in \boldsymbol{C}$. A natural equivalence between $\boldsymbol{C}(C, J-)$ and $F =$ a subfunctor of the restriction of $(\Omega)(WTC, -)$ to $\boldsymbol{B}$ is determined by $\alpha \dashrightarrow \varphi(T\alpha)$ for $\alpha \in \boldsymbol{C}(C, JB)$.

Let $\varphi(T\alpha)$ be an F distinguished morphism with image

$$\begin{array}{ccc} WTC & \xrightarrow{\ \varphi(T\alpha)\ } & B\ . \\ & {}_{\xi}\searrow \quad \nearrow_{i} & \\ & B' & \end{array}$$

$\boldsymbol{B}$ is closed under subobjects. Thus $B' \overset{i}{\dashrightarrow} B \in \boldsymbol{B}$. Under φ^{-1} the diagram becomes

$$\begin{array}{ccc} TC & \xrightarrow{\ T\alpha\ } & SB\ . \\ & {}_{\varphi^{-1}\xi}\searrow \quad \nearrow_{Si} & \\ & SB' & \end{array}$$

$Si = TJi$. Hence $T\alpha$ and the monomorphism TJi are θ algebra homomorphisms and thus so is $\varphi^{-1}\xi$ by part (b) of the lemma. Thus $\varphi^{-1}\xi = T\alpha'$ for some $\alpha' \in \boldsymbol{C}$ and $\xi = \varphi(T\alpha')$ is F distinguished.

Let $\prod B_j \dashrightarrow B_j$ be a product in $\boldsymbol{B}$. $J(\prod B_j)$ is the set theoretic cartesian product of the θ algebras JB_j since $S = TJ$ is forgetful on $\boldsymbol{B}$. Thus $J(\prod B_j) = \prod JB_j$ by part (a) of the lemma. Hence J and thus F preserve products.

It should be noted that the same result holds by the same type of argument if there are elements of Ω corresponding to infinitary as well as finitary operations.

The preceding theorem has the following result of Lawvere [6] as a corollary.

COROLLARY. *Every algebraic functor has an adjoint.*

Proof. An algebraic functor $\delta^{(f)}\colon \delta^{(N)} \dashrightarrow \delta^{(M)}$ is determined by a morphism $f\colon \boldsymbol{M} \dashrightarrow \boldsymbol{N}$ of algebraic theories. $U_N = U_M\delta^{(f)}$ for $U_N\colon \delta^{(N)} \dashrightarrow \boldsymbol{Ens}$ the underlying set functor. A commutative diagram

$$\begin{array}{ccc} (\Omega) & \xrightarrow{\ S\ } & \boldsymbol{Ens} \\ \subseteq\uparrow & {}^{U_N}\nearrow & \uparrow U_M \\ \delta^{(N)} & \xrightarrow[\ \delta^{(f)}\ ]{} & \delta^{(M)} \end{array}$$

is thus obtained. This completes the proof since $\delta^{(N)}$ is a variety of Ω algebras for some operator set Ω.

Let $\boldsymbol{B}$ be the category of associative R algebras and let $\boldsymbol{C}$ be the category of Jordan algebras over R. We suppose that (Γ) is the category of all sets having the same number of n-ary operations defined as $\boldsymbol{B}$ for each $n \geqq 0$. If $M(C, B)$ is the set of Jordan representations $C \dashrightarrow B$, then the representability of $M(C, -)\colon \boldsymbol{B} \dashrightarrow \boldsymbol{Ens}$ follows from that of $\boldsymbol{Ens}(TC, S-)\colon (\Gamma) \dashrightarrow \boldsymbol{Ens}$ by the relative representability theorem for $S\colon (\Gamma) \dashrightarrow \boldsymbol{Ens}$ and $T\colon \boldsymbol{C} \dashrightarrow \boldsymbol{Ens}$ the forgetful functors. In terms of universal algebra the representability of $M(C, -)$ is equivalent to the usual result that there exists a Jordan representation $C \dashrightarrow UC$ which is universal for any Jordan representation $C \dashrightarrow B$.

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