PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 2, 1967

THE CHARACTERISTIC FUNCTION OF A HARMONIC FUNCTION IN A LOCALLY EUCLIDEAN SPACE

WILLIAM LEE JOHNSON

We define the characteristic function for each harmonic function having prescribed singularities in a locally Euclidean space and the class of harmonic functions with bounded characteristic. The main result is that any harmonic function of bounded characteristic can be represented as the difference of two positive harmonic functions with prescribed singularities. Thus the well-known theory of the characteristic functions associated with meromorphic functions has an analogue for harmonic functions in locally Euclidean spaces.

1. Preliminaries.

2. Let V be a locally Euclidean n-space $(n > 2)$. By definition V is an n-manifold for which the defining homeomorphisms η of open sets O with n-balls of R^n are isometries. We shall use the same symbol z for the point of V and for its parametric image. Properties of a function $u(z)$ are always to be understood in terms of the parameter. Expressions such as "an n-ball centered at a" and "$|z - a|$, the distance between z and a" refer to the parametric representation.

3. Let C denote the unit ball in R^n and P a coordinate hyperplane. A region $G \subset V$ is a *bordered region* if

(B–1) $B = \partial G$ is compact,

(B–2) for any $z \in B$ there is a neighborhood $N(z)$ and a diffeomorphism ϕ of $N(z)$ with C such that $\phi(N \cap B) = C \cap P$ and $\phi(N \cap G)$ is one of the two half-balls of $C - P$.

A bordered region $G \subset V$ is *regular* if

(R–1) $\bar{G}$ is compact,

(R–2) $B = \partial(V - \bar{G})$,

(R–3) all components of $V - G$ are noncompact.

The *flux* of a function $u \in C^1$ on the border B of a bordered region G is

$$\int_B (\partial u/\partial n) dS ,$$

where dS is an area element on B and $\partial/\partial n$ is the exterior normal derivative.

4. The standard properties of harmonic functions which are true for locally Euclidean spaces are used without qualification. In particular we have Green's formulae, the mean value properties, the Poisson formula, and Harnack's inequality.

The characteristic singularity ([2], p. 241) for a function $u \in H$ in $G - a$ is

$$s(z) = |z - a|^{2-n}/\omega_n(n-2) , \tag{1}$$

where ω_n is the area of the unit sphere in R^n. The flux of $s(z)$ across a sphere centered at a is -1.

The capacity function p_G of $\bar{G}$ with singularity (1) at a has a constant value on ∂G such that the regular part of p_G tends to zero at a. For $u \in H$ in G, we have

$$u(a) = \int_{\partial G} u(z)(\partial p(z, a)/\partial n)dS . \tag{2}$$

2. Harmonic functions of bounded characteristic.

5. Let L denote the class of harmonic functions regular in V except for singularities of the type

$$\lambda_j |z - z_j|^{2-n}/\omega_n(n-2) , \qquad j = 1, \cdots, m , \tag{3}$$

where the λ_j are real numbers, and the z_j are arbitrary points of V. The subclass of positive functions in L is denoted by LP.

Given a function $h \in L$ in V which is finite at $a \in U$, choose a regular region $\Omega \subset V$ containing a. Let $x_\Omega{}^+(z)$ be the solution of the Dirichlet problem in Ω with boundary values $h^+ = \max(h, 0)$ on $\partial\Omega$. By (2)

$$x_\Omega^+(z) = \int_{\partial\Omega} h^+(t)(\partial p_\Omega(t, z)/\partial n)dS , \tag{4}$$

where the capacity function p_Ω has its singularity at z. Similarly $x_\Omega^-(z) \in H$ in Ω with boundary values $h^- = \max(-h, 0)$ on $\partial\Omega$, and

$$x_\Omega^-(z) = \int_{\partial\Omega} h^-(t)(\partial p_\Omega(t, z)/\partial n)dS . \tag{5}$$

Let the positive and negative singularities of h be a_i and b_j respectively, and define

$$\begin{aligned} y_\Omega^+(z) &= \textstyle\sum_{a_i \in \Omega} \lambda_i g_\Omega(z, a_i) , & \lambda_i > 0 , \\ y_\Omega^-(z) &= \textstyle\sum_{b_j \in \Omega} \lambda_j g_\Omega(z, b_j) , & \lambda_j < 0 ; \end{aligned} \tag{6}$$

$$u_\Omega^+(z) = x_\Omega^+(z) + y_\Omega^+(z) \,, \tag{7}$$
$$u_\Omega^-(z) = x_\Omega^-(z) + y_\Omega^-(z) \,.$$

Here $g_\Omega(z, t)$ is the Green's function of Ω with singularity at t.

We call $C(\Omega, h) = u_\Omega^+(a)$ the *characteristic function* of h with respect to a and Ω. The class LC of functions of *bounded characteristic* consists of $h \in L$ with

$$C(\Omega, h) \leqq M$$

for some $M < \infty$ and all $\Omega \subset V$. Note that $C(\Omega, -h) = u_\Omega^-(a)$. It is a consequence of Theorem 1 below that the class of LC is independent of the point a chosen.

3. The decomposition theorem.

6. THEOREM 1. *A necessary and sufficient condition for $h \in LC$ in V is that*

$$h = u - v \,,$$

where $u, v \in LP$ in V.

We first prove that $C(\Omega, h)$ is an increasing function of Ω.

LEMMA 1. *Let $h \in L$ in V and $\Omega \subset \Omega'$ regular subregions of V with $a \in \Omega$. Then*

$$u_\Omega{}^+(z) \leqq u_{\Omega'}{}^+(z) \,.$$

Proof. The function $h - y_\Omega^+ + y_\Omega^- \in C^0$ in $\bar{\Omega}$, $\in H$ in Ω, and has boundary values h on $\partial\Omega$. From (2) we have

$$h(z) - y_\Omega^+(z) + y_\Omega^-(z) = \int_{\partial\Omega} h(t)(\partial p_\Omega(t, z)/\partial n)dS \,. \tag{8}$$

Here we assume that no singularity of h lies on $\partial\Omega$. An appeal to a continuity argument will give the same result if this is not the case.

On separating h into h^+ and h^-, the right side of (8) is $x_\Omega^+(z) - x_\Omega^-(z)$. Hence

$$h(z) = u_\Omega^+(z) - u_\Omega^-(z) \,. \tag{9}$$

Since $u_\Omega^-(z) \geqq 0$, (9) gives for all $\Omega \subset V$

$$h^+(z) \leqq u_\Omega^+(z) \,. \tag{10}$$

By virtue of (4), (10), and (2), we have

$$\begin{aligned} x_\Omega^+(z) &= \int_{\partial\Omega} h^+(t)(\partial p_\Omega(t, z)/\partial n)dS \\ &\leqq \int_{\partial\Omega} u_{\Omega'}^+(t)(\partial p_\Omega(t, z)/\partial n)dS \\ &= \int_{\partial\Omega} [u_{\Omega'}^+(t) - y_\Omega^+(t)](\partial p_\Omega(t, z)/\partial n)dS \\ &= u_{\Omega'}^+(z) - y_\Omega^+(z) \,. \end{aligned}$$

We conclude that $u_\Omega^+(z) \leqq u_{\Omega'}^+(z)$.

7. LEMMA 2. *If $h \in LC$ in V, then $-h \in LC$.*

Proof. Choose $a \in V$ with $h(a) \neq \infty$. The characteristic function of $-h$ is $u_\Omega^-(a)$. Using (9) with argument a we see that the boundedness of $u_\Omega^+(a)$ guarantees the same for $u_\Omega^-(a)$, and $-h \in LC$.

8. *Proof of Theorem* 1. Let $h \in LC$ in V. The function $u_\Omega^+(z)$, which increases with Ω by Lemma 1, is harmonic on $\Omega - \{a_i\}$. The limit function $u(z)$ is either harmonic or $+\infty$ in $V - \{a_i\}$. The former must hold since $u_\Omega^+(a)$ is bounded for all Ω by assumption. Analogously, $u_\Omega^-(z)$ tends to $v(z) \in H$ in $V - \{b_j\}$. Formula (9) implies in the limit that

$$h(z) = u(z) - v(z)$$

in V, and $u, v \in LP$.

To establish the converse, suppose that $h(z) = u_1(z) - v_1(z)$ with $u_1, v_1 \in LP$. Since $u_1(z) \geqq 0$, (9) implies that all positive singularities of u_Ω^+ are among those of u_1. Thus $u_1 - u_\Omega^+$ is superharmonic in Ω and takes its minimum on $\partial\Omega$. This minimum is nonnegative, and, consequently, $u_\Omega^+(z) \leqq u_1(z)$ in Ω. If $u_1(a)$ is finite, then $h \in LC$.

If $u_1(a) = \infty$, let $u_2(z) = u_1(z) - \lambda g_\Omega(z, a)$, where λ is the order of singularity of u_1 at a. Similarly let $v_2(z) = v_1(z) - \mu g_\Omega(z, a)$, where μ is the order of the singularity of $v_2(z)$ at a. Then

$$h(z) = u_2(z) - v_2(z) + (\lambda - \mu)g_\Omega(z, a) \,.$$

If $\lambda \leqq \mu$, then $u_\Omega^+(a) \leqq u_2(a) < \infty$, and $h \in LC$. If $\lambda > \mu$, then $-h \in LC$, and $h \in LC$ by Lemma 2.

4. Extremal decomposition.

9. THEOREM 2. *Let $h \in LC$ in V and let u, v be the functions constructed in the proof of Theorem 1. For any decomposition $h = u_1 - v_1$, with $u_1, v_1 \in LP$, we have $u \leqq u_1$, $v \leqq v_1$.*

Proof. For $\Omega \subset V$ we have

$$h(z) = u_1(z) - v_1(z) = u_\Omega^+(z) - u_\Omega^-(z) .$$

By the reasoning of 8,

$$u_\Omega^+(z) \leqq u_1(z) ,$$
$$u_\Omega^-(z) \leqq v_1(z) .$$

Since the inequalities hold for all $\Omega \subset V$, the limit functions u and v are dominated by u_1 and v_1 respectively.

10. Suppose there is an $h \in LP$ in V. For any $a \in \Omega \subset V$ the Green's function with singularity at a exists. Since g_Ω vanishes on $\partial\Omega$, $h - g_\Omega$ is superharmonic. The g_Ω increase with Ω, and we conclude that the Green's function g_V of V exists.

11. The extremal functions u and v of Theorem 2 have a further decomposition.

THEOREM 3. *$h \in LC$ in V if and only if*

$$h = (x^+ + y^+) - (x^- + y^-), \tag{11}$$

where the x-functions are regular harmonic in V and

$$y^+ = \textstyle\sum_{a_i \in V} \lambda_i g_V(z, a_i),\ y^- = \sum_{b_j \in V} \lambda_j g_V(z, b_j) . \tag{12}$$

REMARK. The Green's function for V exists by 10 and will be constructed in the course of the proof.

Proof. If h has the asserted decomposition, then $h \in LC$ by Theorem 1.

Conversely, let $h \in LC$ and choose regular regions $\Omega_0 \subset \Omega \subset V$. The function $y_\Omega^+ - y_{\Omega_0}^+$ is harmonic in Ω_0 and nonnegative on $\partial\Omega_0$. By the minimum principle $y_{\Omega_0}^+(z) \leqq y_\Omega^+(z)$ in Ω_0. Similarly $y_{\Omega_0}^-(z) \leqq y_\Omega^-(z)$ in Ω_0. Let y^+ and y^- be the respective limit functions as $\Omega \to V$. The singularities of y_Ω^+ are among those of u, and $y_\Omega^+(z) \leqq u(z)$ in Ω. The limit function y^+ is therefore harmonic in $V - \{a_i\}$. Also $y^- \in H$ in $V - \{b_j\}$.

We show next that $\lim_{\Omega \to V} y_\Omega^+(z) = \sum_{a_i \in V} \lambda_i g_V(z, a_i)$. The proof of the corresponding result for y_Ω^- is similar and will be omitted. We have

$$\begin{aligned} y_\Omega^+(z) &= \textstyle\sum_{a_i \in \Omega} \lambda_i g_\Omega(z, a_i) \leqq \sum_{a_i \in \Omega} \lambda_i g_V(z, a_i) \\ &\leqq \textstyle\sum_{a_i \in V} \lambda_i g_V(z, a_i) , \end{aligned}$$

and consequently

$$\limsup y_\Omega^+(z) \leqq \sum_{a_i \in V} \lambda_i g_V(z, a_i) .$$

On the other hand,

$$\begin{aligned}\sum_{a_i \in \Omega_0} \lambda_i g_V(z, a_i) &= \lim_{\Omega \to V} \sum_{a_i \in \Omega_0} \lambda_i g_\Omega(z, a_i) \\ &\leqq \liminf \sum_{a_i \in \Omega} \lambda_i g_\Omega(z, a_i) = \liminf y_\Omega^+(z).\end{aligned}$$

On passing to the limit $\Omega_0 \to V$ we obtain

$$\sum_{a_i \in V} \lambda_i g_V(z, a_i) \leqq \liminf y_\Omega^+(z) ,$$

and y^+ has the asserted decomposition.

We recall that $x_\Omega^+ = u_\Omega^+ - y_\Omega^+ \to u - y^+$. Since the x_Ω^+ are regular harmonic, it follows that the same is true of the limit function x^+ in V. Similarly $x^- \in H$ in V.

5. Classification theory.

12. Given a locally Euclidean n-space V and a class of functions T defined in V, we say that $V \in O_T$ if the only functions of class T in V are constants. This definition has been used in the classification of Riemann surfaces ([1], [3], [4]) and of locally Euclidean spaces [6]. We know [6] that the O-classes for HB, bounded harmonic functions, and for HD, harmonic functions with finite Dirichlet integral, are related by

$$O_{HB} \subset O_{HD}$$

Since any bounded function becomes positive by addition of a suitable constant, we also have

$$O_{HP} \subset O_{HB}$$

where HP is the class of positive harmonic functions. By definition $V \in O_G$ if V has no Green's function.

By considering the classes LC and LP defined in 5 we can incorporate the corresponding O-classes into the inclusion chain. We find beginning in 13 that

$$O_G \subset O_{LC} \subset O_{LP} \subset O_{HP} \subset O_{HB} \subset O_{HD} . \tag{13}$$

The first three classes are in fact equal, and the inclusions

$$O_G \subset O_{HP} \text{ and } O_{HP} \subset O_{HB}$$

are strict. The strictness of the last inclusion in (13) is an open question.

13. Let $h \in LC$ in V. To show that $O_G \subset O_{LC}$ we may assume that h has at least one positive singularity at $a \in V$. For any regular $\Omega \subset V$ with $a \in \Omega$, let $g_\Omega(z, a)$ be the Green's function of Ω with singularity at a. There exists a decomposition $h = u - v$, where $u, v \in LP$ in V. The function $u - g_\Omega$ is superharmonic in Ω and nonnegative on $\partial\Omega$. Hence $g_\Omega \leqq u$ throughout Ω. Since g_Ω increases with Ω, we conclude that g_V exists and that $O_G \subset O_{LC}$.

Since $LP \subset LC$, we have $O_{LC} \subset O_{LP}$. It is clear that $HP \subset LP$ and hence $O_{LP} \subset O_{HP}$.

14. Since L is a class of functions which have singularities of the type $|z - a|^{2-n}$, a space V has L functions if and only if it has a Green's function. The inclusion $O_G \subset O_{LP}$ is thus an equality, and the first three 0-classes in (13) are equal.

15. The inclusion $O_G \subset O_{HP}$ is obviously strict. Also O_{HP} is strictly contained in O_{HB}, as is easily seen by considering $V = R^n - \{0\}$. A single point is a removable singularity for HB functions, an easy consequence of Harnack's inequality. Hence an HB function in V has a harmonic extension to all of R^n and must therefore be constant. On the other hand, the Green's function for R^n with singularity at 0 is a nonconstant HP function in V. Thus $V \in O_{HB}$ and the inclusion $O_{HP} \subset O_{HB}$ is strict.

References

1. L. Ahlfors and L. Sario, *Riemann Surfaces,* Princeton University Press, 1960.
2. R. Courant and D. Hilbert, *Methods of Mathematical Physics,* Vol. II, Interscience, New York, 1962.
3. L. Sario, *Über Riemannsche Flächer mit hebbarem Rand,* Ann. Acad. Sci. Fenn., Ser. A. I. **50** (1948).
4. ———, *Sur la classificatication des surfaces de Riemann,* C. R. Unzideme Congr. Math. Scand., Trondheim (1949), 229-238.
5. ———, *On locally meromorphic functions with singlevalued moduli,* Pacific J. Math, **13** (1963), 709-724.
6. ———, *Classification of locally Euclidean spaces,* Nagoya Math. J. **25** (1965), 87-111.

Received December 1, 1965, and in revised Form October, 12, 1966. This material is included in the author's doctoral dissertation written at the University of California, Los Angeles. The author is indebted to the Army Research Office (Durham) for support under Contract DA-ARO(D)-31-124-G40 and to Prof. Leo Sario for his guidance and inspiration.

U. S. Naval Ordnance Test Station
China Lake, California