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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS

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In earlier papers of this author and D. S. Passman, some properties of finite groups with r. b. n **were discussed, where we say that a group** G **has** r.b. n (representation bound n) **if all the absolutely irreducible characters of** G **have degree** $\leqq n$. **In the present paper, the situation where** $p \mid |G|$ **is a prime which in some sense is large when compared with** n **is explored. An earlier result of this nature to which we refer states that if** G **has** r.b. $(p-1)$ **then an** S_p **subgroup of** G **is normal and abelian. Here we get a weak result of this general type for groups with** r.b. (p^2-p-1). **For smaller representation bounds, more information can be obtained. Our main result is:**

THEOREM. **Let** G **have** r.b. $(2p-3)$ **for a prime** p. **Then either**

(i) **An** S_p **of** G **is normal and abelian,**

(ii) G **is solvable and has** p**-length** 1

or

(iii) $G = P \times H$ **where** P **is an abelian** p**-group and** $p^2 \nmid |H|$.

We begin in this section with some preliminary observations and results. Let G be a group of order u and let ε be a primitive uth root of unity. Put $F = Q[\varepsilon]$ where Q is the field of rational numbers. If σ is an automorphism of F over Q, then since the values of any character χ are in F, we may define $\chi^\sigma(g) = [\chi(g)]^\sigma$. By a theorem of Brauer (see [2], Th. 41.1) F is a splitting field for G and χ is the character of some representation X over F. Therefore, χ^σ is the character of X^σ and σ permutes the characters of G, taking irreducible characters into irreducible characters. Furthermore, if $\chi = \chi^\sigma$ then X is similar to X^σ and hence σ permutes the eigenvalues of $X(g)$ for each $g \in G$.

Now fix a prime $p \neq 2$ dividing u so that $u = p^a m$ and $p \nmid m$. Then if ε_0 is a primitive mth root of unity, the Galois group $\mathfrak{G}$ of F over $Q[\varepsilon_0] = F_0$ is isomorphic to that of $Q[\delta]$ over Q where δ is a primitive p^ath root of unity. Since $p \neq 2$ this group is cyclic and we let σ be a generator of $\mathfrak{G}$. Now σ fixes every root of unity in F which has index prime to p and any two roots which have the same power of p as index are conjugate under $\mathfrak{G} = \langle\sigma\rangle$. We have $\sigma(\varepsilon) = \varepsilon^r$ where r is an integer with $r^{|\mathfrak{G}|} \equiv 1 \bmod u$ and since $\varepsilon_0 = \sigma(\varepsilon_0) = \varepsilon_0^r$, $r \equiv 1 \bmod m$.

The group $\mathfrak{G}$ may also be allowed to act on the elements of G by $\sigma \cdot x = x^r$ for $x \in G$. Since $r^{|\mathfrak{G}|} \equiv 1 \bmod |G|$, this is really a group action. Since the action preserves the conjugacy classes in G, we may regard $\mathfrak{G}$ as acting on the classes. If χ is any irreducible character of G then $\chi^{\sigma}(x) = \chi(x^{r}) = \chi(\sigma \cdot x)$ and thus the hypothesis of the following lemma of Brauer (see [1]) hold.

(1.1) LEMMA (*Brauer*). *Let $\mathfrak{G}$ be a group which acts to permute both the classes and the irreducible characters of a group G in such a manner that $\chi^{\sigma}(x) = \chi(\sigma \cdot x)$ where $\sigma \in \mathfrak{G}$, $x \in G$ and $\sigma \cdot x$ is any element of $\sigma \cdot K$, K being the class containing x. Then the numbers of orbits in the two actions of $\mathfrak{G}$ are equal and if $\mathfrak{G}$ is cyclic, the numbers of fixed points are also equal.*

In the present situation, if x is a p-regular element of G then $\sigma \cdot x = x^{r} = x$ since $x^{m} = 1$ and $r \equiv 1 \bmod m$. Therefore, if G is not a p-group, there are at least two classes fixed by the action of G. Thus in this case there is some nonprincipal irreducible character χ with $\chi = \chi^{\sigma}$. This observation is especially useful in connection with the following.

(1.2) LEMMA. *Let G be a group with a faithful irreducible character $\chi = \chi^{\sigma}$. If $\chi(1) < (k+1)(p-1)$ for some k, $0 \leqq k \leqq p-1$, then G has an elementary abelian S_p subgroup of order $\leqq p^{k}$.*

Proof. Let P be an S_p of G. If P is not abelian, let $1 \neq z \in P' \cap \mathfrak{Z}(P)$. Since z is in the kernel of every linear constituent of the restriction $\chi \mid P$, the restriction must have some irreducible constituent ψ of degree $\geqq p$ such that $\psi(z) = \psi(1)\delta$ where δ is a root of unity of index a power of p. There are no less than $p-1$ roots of unity of index equal to that of δ and they are all conjugate under the action of $\langle \sigma \rangle$. Since σ permutes the eigenvalues associated with $\chi(z)$ we have $\chi(1) \geqq \psi(1)(p-1) \geqq p(p-1)$. This contradiction shows that P is abelian.

If λ is a linear constituent of $\chi \mid P$ and $x \in P^{\#}$, suppose that $\lambda(x) = \delta$, a root of unity of index $p^{v} > 1$. Then δ has $p^{v-1}(p-1)$ distinct algebraic conjugates under the action of $\langle \sigma \rangle$ and all of these are eigenvalues associated with $\chi(x)$. Thus $p^{v-1}(p-1) \leqq \chi(1) < p(p-1)$ and hence $v = 1$. Since this is true for all the linear constituents of $\chi \mid P$, x has order p and P is elementary abelian.

Now λ has exactly $p-1$ conjugates under the action of $\langle \sigma \rangle$ and each has the same kernel of index p in P. The number of distinct kernels of constituents of $\chi \mid P$ is therefore $\leqq \chi(1)/(p-1) \leqq k$. Since the intersection of the kernels is the identity, we have $|P| \leqq p^{k}$ and the result follows.

As has been observed by W. Feit, what is essentially the case

$k = 1$ of (1.2) may be used in place of his own very much deeper theorem in the proof of Theorem E of [4].

We next apply these results to groups with r.b. $[(k+1)(p-1)-1]$ where $0 \leqq k \leqq p-1$. We remark first, however, that if G has r.b. n then so does every homomorphic image, and by a simple application of Frobenius Reciprocity, ([2], Th. 38.8) so does every subgroup.

(1.3) PROPOSITION. Let G have r.b. $(k+1)(p-1)-1$ where $0 \leqq k \leqq p-1$. If P is an S_p of G and $\mathfrak{O}_p(G)$ is the maximum normal p-subgroup of G, then $P/\mathfrak{O}_p(G)$ is elementary abelian of order $\leqq p^k$.

Proof. We suppose the statement is false and let G be a counterexample of minimal order. Certainly $p \neq 2$ and G is not a p-group and thus some nonprincipal irreducible character χ of G is fixed by σ. Let $N = \operatorname{Ker} \chi$.

If $\mathfrak{O}_p(G) > 1$, then by minimality the proposition holds for $G/\mathfrak{O}_p(G)$ and we get a contradiction. If N is a p'-group then $P \cong S_p(G/N)$ which is elementary of order $\leqq p^k$ by Lemma 1.2 and again there is a contradiction and thus $p \mid |N|$. If $\chi(1) = 1$ then by Lemma 1.2 $p \nmid [G:N]$ and $P \subseteqq N < G$. By minimality, $P/\mathfrak{O}_p(N)$ is elementary of the appropriate order and since $\mathfrak{O}_p(N) \subseteqq \mathfrak{O}_p(G) = 1$, we have a contradiction. Therefore $\chi(1) > 1$ and G/N is nonabelian. Since the S_p PN/N of G/N is abelian, $U = PN < G$. By minimality we can conclude that $\mathfrak{O}_p(U) \neq 1$. Since $\mathfrak{O}_p(U) \cap N \subseteqq \mathfrak{O}_p(N) \subseteqq \mathfrak{O}_p(G) = 1$, $\mathfrak{O}_p(U) \cap N = 1$ and since both are normal in U, $\mathfrak{O}_p(U)$ centralizes N.

Put $M = \mathfrak{C}_G(N)$ and $Z = M \cap N$. Since Z is central in both M and N, if either M/Z or N/Z has a normal S_p then so does M or N respectively. Since p divides both $|M|$ and $|N|$ and G has no normal p-subgroup this cannot happen. Therefore both M/Z and N/Z fail to have normal S_p subgroups and by Theorem E of [4], each of M/Z and N/Z have some irreducible character of degree $\geqq p$ and thus $MN/Z \cong (M/Z) \times (N/Z)$ has an irreducible character of degree $\geqq p^2$ which contradicts MN having r.b. $p(p-1)$. This proves the result.

2. In this section we present some results which will be used to prove our theorem. We begin with some character theoretic observations.

If $H \triangle G$ and χ is an irreducible character of G then $\chi \mid H = a \sum_{i=1}^{t} \theta_i$ where the θ_i are distinct and form a complete orbit of irreducible characters of H under the action of G. This action is defined by $\theta^g(h) = \theta(ghg^{-1})$ for $g \in G$ and $h \in H$. Clearly, all $\theta_i(1)$ are equal and thus each divides $\chi(1)$. If T_i is the subgroup of G fixing θ_i we call it the *inertia group* of θ_i in G and $[G:T_i] = t$. If $\chi \mid H$ is irreducible and β is any irreducible character of G/H, viewed as a character of G then by Proposition 1.1 of [3], $\beta\chi$ is an irreducible character of G.

(2.1) LEMMA. *Let G have a normal p-complement H and suppose that for every irreducible character of H, the inertia group in G is all of G. Then G has a normal S_p.*

Proof. G/H is a group which acts both on the irreducible characters and on the classes of H and the hypotheses of Brauer's Lemma (1.1) are satisfied because of the definitions of the actions. In the present situation, all of the characters are fixed and thus the same is true of all the classes.

Suppose that $P_1, P_2, \cdots, P_r$ are the S_p's of G and that $h \in H$. In the action of P_1 on H, the class containing h is fixed and since its cardinality is prime to p, some element in the class is centralized by P_1 and hence h centralizes some P_i and $H = \bigcup_{i=1}^{q} \mathfrak{C}_H(P_i)$. Since all of the P_i are conjugate by elements of H, the subgroups $\mathfrak{C}_H(P_i)$ are all conjugate in H. Since no group is a union of conjugate proper subgroups, $H = \mathfrak{C}_H(P_1)$ and thus $P_1 \triangle G$.

(2.2) LEMMA. *Let G have* r.b. $(2k-1)$ *and let $H \triangle G$ have an irreducible character θ of degree k. Then G/H is abelian.*

Proof. Let χ be an irreducible constituent of the induced character θ^G. By Frobenius Reciprocity, θ is a constituent of $\chi \mid H$ and thus $\theta(1) \mid \chi(1)$. Since $\chi(1) < 2k$ and is divisible by k, we have $\chi(1) = k$ and $\chi \mid H$ is irreducible. Let β be any irreducible character of G/H viewed in G. Then $\beta\chi$ is irreducible and $2k > (\beta\chi)(1) = \beta(1)k$ and hence $\beta(1) = 1$ and all irreducible characters of G/H are linear. The result follows.

(2.3) PROPOSITION. Let G have r.b. $(2p-1)$ and suppose that G does not have a normal S_p. If G is p-solvable, then it is solvable.

Proof. If the statement is false, let G be a minimal counterexample. If G has any normal p-subgroup U then G/U fails to have a normal S_p and being p-solvable it is solvable by the minimality of $|G|$. Since U is solvable, this is a contradiction.

Suppose that $K \triangle G$ has a normal p-complement H and that $p \mid |K|$. Now K fails to have a normal S_p since it would be normal in G and thus by Lemma 2.1 we may conclude that K does not fix some irreducible character θ of H. Let ψ be an irreducible constituent of θ^K. Then $\psi \mid H = a \sum_{i=1}^{t} \theta_i$ and $t > 1$. Thus $p \mid t$ because K/H is a p-group and we have $2p > \psi(1) = at\,\theta(1) = amp\,\theta(1)$. Hence $\theta(1) = 1$ and

$\psi(1) = p$. By Lemma 2.2 applied to K, G/K is abelian and if $K < G$ then by minimality K is solvable and thus so is G. Therefore G contains no proper normal subgroup which has a normal p-complement and order divisible by p.

Now we let $H = \mathfrak{O}_{p'}(G)$. Since G is p-solvable, $\mathfrak{O}_p(G/H)$ is not trivial and thus its inverse image in G has order divisible by p. Since it has a normal p-complement H, it is all of G and G and all its subgroups have normal p-complements. Since H has the nonprincipal linear character θ, $H' < H$. Let P be an S_p of G. Then $PH' < G$ and if H' is not solvable then $P \triangle PH'$ and P centralizes H'. Therefore $\mathfrak{C}_G(H') \triangle G$ and has order divisible by p. Since it has a normal p-complement $\mathfrak{C}_G(H') = G$ by the preceding observations and thus H' is abelian. Hence in any case H' is solvable and therefore so is G and the result is proved.

3. In this section we prove the theorem. We begin with a proposition which when combined with Proposition 2.3 will yield the result.

(3.1) PROPOSITION. Let G have r.b. $(2p - 3)$ and let $U = \mathfrak{O}_p(G)$. If U is not an abelian direct factor of G then G is p-solvable and has p-length 1.

Proof. If the statement is false let G be a counterexample of minimum order. Certainly $p \neq 2$. If U is not abelian then it has an irreducible character of degree p and by Lemma 2.2, G/U is abelian and we have a contradiction. Thus U is abelian and therefore is not a direct factor of G. Note that by Proposition 1.3, $p^2 \nmid [G : U]$.

Now suppose that $H \triangle G$ where $H > 1$ is a p'-group. Then G/H is not p-solvable with p-length 1 and thus by the minimality of $|G|$, $\mathfrak{O}_p(G/H)$ is a direct factor of G/H. Clearly $UH/H \subseteq \mathfrak{O}_p(G/H)$ and we must have equality since any larger normal p-subgroup than UH/H would be a normal S_p of G/H which does not exist. Thus there exists $K \triangle G, K \supseteq H$ with $KU = G$ and $K \cap UH = H$. Hence $K \cap U = K \cap UH \cap U = H \cap U = 1$ and U is a direct factor of G. This contradiction shows that H cannot exist.

Let $C = \mathfrak{C}_G(U) \supseteq U$. If $p \nmid [C : U]$ then U is a central S_p of C which therefore has a normal p-complement which is a p' group normal in G and hence is trivial. In this case $C = U$. We show that this is the only possibility, for if $p \mid [C : U]$ then C contains P, an abelian S_p of G. Now $P \cap \mathfrak{Z}(\mathfrak{N}_G(P)) \supseteq U$ and we must have equality or else $P \subseteq \mathfrak{Z}(\mathfrak{N}_G(P))$ and C has a normal p-complement which cannot occur. Therefore, by a corollary of Grun's theorem (13.5.5 of [5]), C has a characteristic subgroup D with $C/D \cong U$. If $D \cap U \neq 1$ then $D \cap U$ is a central S_p of D since $p^2 \nmid |D|$ and thus D has a normal p-complement, a contradiction. Hence $D \cap U = 1$ and therefore $\mathfrak{O}_p(D) = 1$. Since $p \mid |D|$, we may conclude from Proposition 1.3 that D has an irreducible character of degree $\geqq p-1$. By Lemma 2.2 with $k = p-1$, G/D is abelian and the commutator $[G, U] \subseteqq D$. Since $[G, U] \subseteqq U$ and $U \cap D = 1$, $U \subseteqq \mathfrak{Z}(G)$ and $C = G$. Since $UD = C = G$, U is a direct factor of G and we have a contradiction. Thus indeed $U = C$.

Suppose that $K < G$ is any subgroup of index prime to p. Then K contains a full S_p of G and $U \subseteqq K$. If K contains a normal p' subgroup then it would centralize U which is not the case and thus if K is p-solvable of p-length 1 it has a normal S_p. By the minimality of G the only other possibility is that U is a direct factor of K which again contradicts $U = \mathfrak{C}_G(U)$. This shows that the normalizers of the S_p's of G are the only maximal subgroups with index prime to p.

Now let $V \subseteqq U$ be a minimal normal subgroup of G. Let $\lambda \neq 1$ be a linear character of V and let T be the inertia group of λ in G. If χ is an irreducible constituent of λ^G then $\chi \mid V = a\sum_{i=1}^{t}\lambda_i$ and $t \mid \chi(1)$, $\chi(1) < 2(p-1)$. If $p \mid [G:T]$ then $t = [G:T] = p$. If $p \nmid [G:T]$ and $T < G$ then $T \subseteqq \mathfrak{N}_G(P)$ for some S_p, P of G. In this case

$$2(p-1) > t = [G:T] = [G:\mathfrak{N}(P)][\mathfrak{N}(P):T] = (kp+1)[\mathfrak{N}(P):T]$$

by Sylow's theorem. Thus $[\mathfrak{N}(P):T] = 1 = k$ and $t = p+1 = \chi(1)$. The remaining possibility is $T = G$. In that case, if $M = \operatorname{Ker}\chi$, V is central mod M and $[G, V] \subseteqq M \cap V$. Since $V \not\subseteqq M$, we conclude from the minimality of V that $M \cap V = 1$ and thus $V \subseteqq \mathfrak{Z}(G)$. We show that this cannot occur.

Suppose then that $V \subseteqq \mathfrak{Z}(G)$. If U/V is a direct factor of G/V then it is central and $[G, U] \subseteqq V$. If $x \in G$ has order q, prime to p we have for $u \in U$, $u^x = vu$ for some $v \in V$. Thus $u^{x^i} = v^i u$ and $u = u^{x^q} = v^q u$ and $v^q = 1$. Since the order of v is a power of p, $v = 1$ and $u^x = u$. Since $u \in U$ was arbitrary, $x \in \mathfrak{C}(U) = U$. This contradiction shows that G/V is p-solvable of p-length 1. If L is the inverse image in G of $\mathfrak{O}_{p'}(G/V)$ then L has the central S_p V and therefore has a normal p-complement which must be trivial. Thus $\mathfrak{O}_{p'}(G/V) = 1$ and G/V has a normal S_p. Since V is a p-group, this is a contradiction and thus indeed $V \not\subseteqq \mathfrak{Z}(G)$.

We have now shown that all nonprincipal linear charaters of V are permuted into orbits of size p or $p+1$ by the action of G. Therefore $|V| = 1 + Ap + B(p+1)$ where A and B are nonnegative integers. Since $p \mid |V|$, $B \neq 0$ and hence some orbit of size $p+1$ does exist and G has an irreducible character χ with $\chi(1) = p+1$ and $V \not\subseteqq \operatorname{Ker}\chi$.

By the minimality of V it is elementary abelian and every nonidentity element has order p. If some $v \in V$ is in a conjugacy class of G containing exactly p elements, then since $\mathfrak{C}_G(v) = \mathfrak{C}_G(v^i)$ for all

i, $1 \leqq i \leqq p-1$, every nonidentity element in the group $\langle v \rangle$ has exactly p conjugates in G. By a theorem of Burnside (see 12.3.1 of [5]), either v is central mod Ker χ or else $\chi(v) = 0$. The first possibility cannot occur for otherwise $[v, G] \subseteqq V \cap \operatorname{Ker} \chi = 1$ and $v \in \mathfrak{Z}(G)$ which contradicts v having p conjugates. Thus χ vanishes on $\langle v \rangle^{\#}$ and the character inner product $[\chi \mid \langle v \rangle, 1_{\langle v \rangle}] = (1/p)\chi(1)$ which is not an integer. This contradiction shows that no $v \in V$ has exactly p conjugates in G.

No $v \in V$ is central in G since otherwise, by the minimality of V, $\langle v \rangle = V$ is central. Thus if $v \in V$ has exactly $s < p$ conjugates in G, $\mathfrak{C}_G(v)$ is a proper subgroup of G of p' index and hence in contained in the normalizer of some S_p of G. Since the normalizer has index $p+1$ this contradicts $[G : \mathfrak{C}(v)] = s$ and hence all $v \in V$ are permuted into orbits of size $\geqq p + 1$ by the action of G. There are therefore $\leqq 1 + (Ap + B(p+1))/(p+1)$ orbits and by Brauer's Lemma (1.1) this is equal to $1 + A + B$, the number of orbits of characters. We conclude that each orbit has exactly $p+1$ elements in both cases and if $1 \neq v \in V$ then $[G : \mathfrak{C}(v)] = p + 1$. Therefore, $\mathfrak{C}_G(v) = \mathfrak{N}_G(P_i)$ for some S_p P_i of G. Thus $V^{\#} = \bigcup_{i=1}^{p+1} \mathfrak{C}_V(\mathfrak{N}_G(P_i))^{\#}$ and this union is disjoint. If $|V| = p^a$ and $|\mathfrak{C}_V(\mathfrak{N}_G(P_i))| = p^b$ then $p^a - 1 = (p+1)(p^b - 1)$ and this implies that $a = 2$ and $|V| = p^2$.

Put $W = \mathfrak{C}_G(V) = \bigcap \mathfrak{N}_G(P_i)$. Then $U \subseteqq W < G$, $W \triangle G$ and G/W may be regarded as a subgroup of $GL(2, p)$ in its action on V. The homomorphism δ taking each element of $GL(2, p)$ to its determinant in $GF(p) - 1$ may be applied to G/W. If its kernel is proper it is a subgroup of index dividing $p - 1$ and yet would be contained in $\mathfrak{N}_G(P_i)$ which has index $p + 1$. This shows that δ is trivial on G/W.

Since $(p+1) \mid [G : W]$, $2 \mid [G : W]$ and G/W has an element τ of order 2. Since $\delta(\tau) = 1$, both eigenvalues of τ as viewed in $GL(2, p)$ are -1 and τ is central in G/W. Thus a preimage x of τ in G normalizes each $\mathfrak{N}(P_i)$ and thus is in W. Since $\tau \neq 1$ this is a contradiction and the result is proved.

We are now ready to give the

Proof of the theorem. If an S_p of G is normal and abelian we have (i). If it is normal and nonabelian then it has an irreducible character of degree p and by Lemma 2.2 its quotient group is abelian and thus G is solvable and we have case (ii). If G does not have a normal S_p let $P = \mathfrak{O}_p(G)$. If P is not an abelian direct factor of G then by Proposition 3.1, G is p-solvable and has p-length 1. By Proposition 2.3 then, G is solvable and again we have case (ii) of the theorem. The remaining possibility is that P is an abelian direct factor of G and since $p^2 \nmid [G : P]$ by Proposition 1.3 we have case (iii) and the proof is complete.

Since (ii) is the only case of the theorem in which an S_p of G is not abelian we get the following:

(3.2) COROLLARY. *If G has* r.b. $(2p-3)$ *and has a nonabelian S_p subgroup then G is solvable.*

We also state another interesting consequence of the theorem.

(3.3) COROLLARY. *If a p-solvable group has* r.b. $(2p-3)$ *then it has p-length* 1.

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