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SOME RESULTS ON AMPLENESS AND DIVISORIAL SCHEMES

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The purpose of this note is twofold. Part I consists of an example of an algebraic scheme which is the union of two closed, quasi-projective subscheme, but which is not itself quasi-projective. The main result of Part II is a structure theorem for coherent sheaves over divisorial schemes and, as an application, the proof that Theorem 2 of Borel-Serre's paper "Le Théorèm de Riemann-Roch", which is stated only for quasi-projective, nonsingular schemes, can be extended to arbitrary nonsingular schemes. (See the Remark on page 108 of the mentioned paper.)

The example given in Part I shows furthermore that, if $\mathscr{L}$ is an invertible sheaf over a noncomplete scheme X, which induces ample sheaves over the irreducible components of X, $\mathscr{L}$ need not be ample. That $\mathscr{L}$ is ample if X is complete is shown by Grothendieck in Theorem 2.6.2., Chapter III of "Elements de Geometrie Algebrique". The example we give consists of the union of two quasi-affine closed subschemes (whence their respective sheaves of local rings are ample). Since the union itself is not quasi-projective, its sheaf of local rings is not ample.

The result obtained in Part II is but a first step towards Riemann-Roch-type theorems for arbitrary nonsingular schemes. To the author's knowledge, no suitable definition of a ring structure for equivalence classes of sheaves (i.e. a satisfactory intersection theory for equivalence classes of cycles) has been found as yet over an arbitrary nonsingular scheme. (See [3] and the remark on page 143 of [4] "On ne peut pas . . .".)

The essential part of the proof of Theorem 3.3. in Part II was communicated to the author by Steven Kleiman, to whom the author is indebted for this and other conversations.

The notation and terminology we use are, unless otherwise specifically stated, those of [7] and [5]. We consider only algebraic schemes, with an arbitrary, algebraically closed ground field. For the sake of convenience we drop the adjective "algebraic" and speak simply of schemes. Also, all rings we consider are understood to be commutative and with unity, and all ring homomorphisms to be such that $1 \rightarrow 1$.

When we refer to, say, Lemma 2.3 without further identification we mean Lemma 2.3 of the present work, to be found as the third Lemma of § 2.

PART I.

1. We begin with some simple preliminary results which are included here for completeness sake.

DEFINITION 1.1. Let $\mathscr{C}$ be an arbitrary category, A, B, C three objects of $\mathscr{C}$, $u \in \mathrm{Hom}(A, C)$, $v \in \mathrm{Hom}(B, C)$ two fixed morphisms. The triplet (D, φ, ψ), where $D \in \mathscr{C}$, $\varphi \in \mathrm{Hom}(D, A)$, $\psi \in \mathrm{Hom}(D, B)$ is called a "fibre product" of A and B with respect to u and v if the following conditions are satisfied:

(1) $u \circ \varphi = v \circ \psi$

(2) For every $D' \in \mathscr{C}$ the function

$$\mathrm{Hom}(D', D) \longrightarrow \mathrm{Hom}(D', A) X_P \mathrm{Hom}(D', B)$$

given by $\alpha \rightarrow (\varphi \circ \alpha, \psi \circ \alpha)$ is bijective, where $\mathrm{Hom}(D', A) X_P \mathrm{Hom}(D', B)$ denotes the subset of $\mathrm{Hom}(D', A) X \mathrm{Hom}(D', B)$ consisting of those pairs of morphisms (u', v') such that $u \circ u' = v \circ v'$.

It is easy to see that the category of rings and ring homomorphisms has fibre products, and that fibre products are unique up to isomorphisms.

If $(X, 0_X)$ is a reducible scheme, it is always possible to find two distinct sheaves of ideals of 0_X, say $\mathscr{I}_1$ and $\mathscr{I}_2$, such that, if $(X_1, 0_{X_1})$ and $(X_2, 0_{X_2})$ denote the closed subschemes defined by $\mathscr{I}_1$ and $\mathscr{I}_2$ respectively, we have

(i) $X_i \neq X, X = X_1 \cup X_2$

(ii) $\mathscr{I}_1 \cap \mathscr{I}_2 = 0$.

This is obvious when $(X, 0_X)$ is reduced. Otherwise we use the fact that, in a noetherian ring A, (0) is the intersection of sufficient powers of the minimal prime ideals of A. Since conditions (i) and (ii) are in fact local, and X is compact, we are done.

LEMMA 1.2. *Let A be a ring, I_1, I_2 two ideals of A such that $I_1 \cap I_2 = 0$. Let*

$$u: A/I_1 \longrightarrow A/I_1 + I_2 \qquad \varphi_1: A \longrightarrow A/I_2$$
$$v: A/I_2 \longrightarrow A/I_1 + I_2 \qquad \varphi_2: A \longrightarrow A/I_2$$

be the canonical morphisms. Then $(A, \varphi_1, \varphi_2)$ is the fibre product of A/I_1 and A/I_2 with respect to u and v.

Proof. We have the commutative diagram

$$\begin{array}{ccc} A/I_1 & \xrightarrow{u} & A/I_1 + I_2 \\ {\scriptstyle\varphi_1}\uparrow & & \uparrow{\scriptstyle v} \\ A & \xrightarrow[\varphi_2]{} & A/I_2 \end{array}$$

which shows that the first property of a fibre product is satisfied by $(A, \varphi_1, \varphi_2)$. As to the second, let $\bar{a}_i \in A/I_i$ be such that $u(\bar{a}_1) = v(\bar{a}_2)$ and let $a_i \in A$ such that $\varphi_i(a_i) = \bar{a}_i$. Clearly

$$\begin{aligned}(u \circ \varphi_1)(a_1 - a_2) &= (v \circ \varphi_2)(a_1 - a_2) \\ &= (u \circ \varphi_1)(a_1) - (v \circ \varphi_2)(a_2) = 0 .\end{aligned}$$

Therefore $a_1 - a_2 \in I_1 + I_2$, whence there exist elements b_1, b_2 in I_1, I_2 respectively, such that $a = a_1 + b_1 = a_2 + b_2$. Clearly $\varphi_i(a) = \bar{a}_i$. If $b \in A$ is such that $\varphi_i(b) = \bar{a}_i$, then $a - b \in I_1 \cap I_2 = 0$. The lemma is proved.

Let now $(X, 0_X)$ be an arbitrary scheme, and let $\mathscr{L}$ be an invertible sheaf over X, i.e. a locally free sheaf of rank 1.

This means that we have a finite, open cover of X, say $\mathscr{U} = (U_j)_{j \in J}$, and isomorphisms

$$u_j \colon \mathscr{L} \mid U_j \longrightarrow 0_X \mid U_j .$$

We may also assume that the U_j's are affine. If $(Y, 0_Y)$ is a closed subscheme of X defined by a sheaf of ideals $\mathscr{I}$ of 0_X, the exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow 0_X \xrightarrow{\varphi} 0_Y \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow \mathscr{L} \otimes \mathscr{I} \longrightarrow \mathscr{L} \xrightarrow{\tilde{\varphi}} \mathscr{L} \otimes 0_Y \longrightarrow 0$$

whence a homomorphism

$$H^0(X, \mathscr{L}) \longrightarrow H^0(Y, \mathscr{L} \otimes 0_Y) .$$

The homomorphisms

$$u'_j \colon \mathscr{L} \otimes 0_Y \mid U_j \cap Y \longrightarrow 0_Y \mid U_j \cap Y$$

defined by

$$u'_j(g \otimes a) = (\varphi \circ u_j)(g) \cdot a \qquad g \in \mathscr{L},\ a \in 0_Y$$

are easily seen to be 0_Y-isomorphisms, whence $\mathscr{L} \otimes 0_Y$ is an invertible sheaf over Y. One can also immediately check that the diagram

$$(1)\qquad \begin{array}{ccc} H^0(U_j, \mathscr{L}) & \longrightarrow & H^0(U_j, 0_X) \\ \downarrow & & \downarrow \\ H^0(U_j \cap Y, \mathscr{L} \otimes 0_Y) & \longrightarrow & H^0(U_j \cap Y, 0_Y) \end{array}$$

is commutative, where the vertical arrows are induced by φ and $\tilde{\varphi}$ respectively, and the horizontal ones by u_j and u'_j.

We now extend Lemma 1.2, which is of a local nature, to a global situation.

THEOREM 1.3. *Let $(X, 0_X)$ be a scheme, $\mathscr{I}_1, \mathscr{I}_2$ two sheaves of ideals of 0_X such that $\mathscr{I}_1 \cap \mathscr{I}_2 = 0$. Let $(X_i, 0_{X_i})$ be the closed subscheme of X defined by $\mathscr{I}_i$, $i = 1, 2$, $(Y, 0_Y)$ the closed subscheme defined by $\mathscr{I}_1 + \mathscr{I}_2$. Let $\mathscr{L}$ be an invertible sheaf over X. The diagram*

$$(2)\qquad \begin{array}{ccc} H^0(X, \mathscr{L}) & \longrightarrow & H^0(X, \mathscr{L} \otimes 0_{X_1}) \\ \downarrow & & \downarrow \\ H^0(X, \mathscr{L} \otimes 0_{X_2}) & \longrightarrow & H^0(X, \mathscr{L} \otimes 0_Y) \end{array}$$

identifies $H^0(X, \mathscr{L})$ as the fibre product of the $H^0(X, \mathscr{L} \otimes 0_{X_i})$'s over $H^0(X, \mathscr{L} \otimes 0_Y)$.

Proof. Let $\mathscr{U} = (U_j)$ be an open affine cover of X such that there exist isomorphisms

$$u_j\colon \mathscr{L} \mid U_j \longrightarrow 0_X \mid U_j \;.$$

We let

$$u_j^{(i)}\colon \mathscr{L} \otimes 0_{X_i} \mid U_j \cap X_i \longrightarrow 0_{X_i} \mid U_j \cap X_i$$
$$\tilde{u}_j\colon \mathscr{L} \otimes 0_Y \mid U_j \cap Y \longrightarrow 0_Y \mid U_j \cap Y$$

be the induced isomorphisms, as was explained above.

We need to show that, given sections $s_i \in H^0(X, \mathscr{L} \otimes 0_{X_i})$, $i = 1, 2$, which induce the same section $t \in H^0(X, \mathscr{L} \otimes 0_Y)$, there exists a unique section $s \in H^0(X, \mathscr{L})$ which induces s_1 and s_2 respectively.

Let $f_{ij} = u_j^{(i)}(s_i) \in H^0(U_j \cap X_i, 0_{X_i})$. From (1) we see that f_{1j} and f_{2j} induce the some element of $H^0(U_j \cap Y, 0_Y)$. Hence, by Lemma 1.2 (since U_j is affine and $U_j = (U_j \cap X_1) \cup (U_j \cap X_2)$) there exists a unique section $f_j \in H^0(U_j, 0_X)$ which induces f_{1j} and f_{2j} respectively. We shall show that the family $\{u_j^{-1}(f_j)\}_{j \in J}$ defines an element s of $H^0(X, \mathscr{L})$. Let

$$b_{jk} = (u_j \circ u_k^{-1})(1) \in H^0(U_j \cap U_k, 0_X) \;.$$

Then $f_j - b_{jk}f_k$ is an element of $H^0(U_j \cap U_k, 0_X)$ and it induces the zero element of $H^0(U_j \cap U_k \cap X_i, 0_{X_i})$, $i = 1, 2$. This is seen by observing that the family $\{f_j\}_{j \in J}$ induces the family $\{f_{ij}\}_{j \in J}$ over X_i, $i = 1, 2$. By Lemma 1.2 we have $f_j - b_{jk}f_k = 0$, which shows our contention. It is now obvious that s induces s_1 and s_2, and, by using Lemma 1.2, that only one such s exists. The theorem is proved.

It is well known that, if the X_i's are affine, so is X. This is in fact a consequence of a theorem of Chevalley's (See [5], Ch. II, Th. 6.7.1 and corollaries). Combining Theorem 1.3, Lemma 1.2 and the uniqueness of fibre products, we obtain the same result, without using the heavy machinery involved in the proof of Chevally's theorem, that is, Serre's characterization of affine schemes. ([5], Ch. II, Th. 5.2.1), ([8], Th. 1).

The above points out that, to obtain an example of a nonquasi-projective scheme with quasi-projective irreducible components, one must consider components which are neither projective nor affine. We exhibit such example in the next section.

2. We recall some of the properties of one of Nagata's examples of a nonquasi-projective surface, [6].

Let k' be the rational field, a and b two trascendentals over k' such that $a^3 + b^3 = 1$, k the algebraic closure of $k'(a, b)$. We let A_3 denote the affine, three-dimensional space over k, and let V be the affine cone defined by

$$X^3 + Y^3 + 3(aX^2 + bY^2)Z + 3(a^2X + b^2Y)Z^2 = 0 . \tag{3}$$

We let $\mathfrak{B} = k[x, y, z]$ be the coordinate ring of V, $K = k(x, y, z)$ its function field. The function $\sigma: K \to K$ given by

$$\begin{aligned} t &= \sigma(x) = (a^2x + b^2y)/b^2x^2 \\ u &= \sigma(y) = [tx - (a/b)^2]t \\ v &= \sigma(z) = \{[tx - (a/b)^2]^3 + 1\}/3b^2z \end{aligned}$$

defines an involution of K, and the point (t, u, v) is the generic point of a cone V^σ whose equation is again (3). For every closed subset H of V we denote by H^σ the closed subset of V^σ whose defining ideal in $\sigma(\mathfrak{B})$ is $\sigma(\mathfrak{A})$, $\mathfrak{A} \subset \mathfrak{B}$ being the ideal defining H.

We denote by F the divisor of V associated to the ideal

$$\mathfrak{G} = x \cdot \mathfrak{B} + (y^2 + 3byz + 3b^2z^2) \cdot \mathfrak{B} ,$$

D the divisor of V associated to the ideal $\mathfrak{P} = x \cdot \mathfrak{B} + y \cdot \mathfrak{B}$. In [6] Nagata proves the following three statements:

(4) (i) σ defines a birational, biregular isomorphism of the two open subsets $V - F$, $V^\sigma - F^\sigma$ of V, V^σ respectively.
(ii) $k[x, y, z, t, u, v]$ is the coordinate ring of the affine variety $V - F = V^\sigma - F^\sigma$.
(iii) $\mathfrak{P} \cap \sigma(\mathfrak{P}) = k$.

For every element $f \in K$ and every open subset U of V (U^σ of V^σ) we denote by $(f)_U$, $((f)_{U^\sigma})$, the divisor of f on U (on U^σ). For example, we have $(x)_V = F + D$, $(x)_{V-F} = D$. Also, when there is no danger of confusion, we shall use the same symbol for an effective divisor and its support. If $f \in K$ we have

$$((f)_V)^\sigma = (\sigma(f))_{V^\sigma} .$$

Having introduced all the necessary notations we prove

THEOREM 2.1. *There exist two elements f_1, f_2 of $\mathfrak{S}$ such that*
(i) *f_1, f_2 generate the unit ideal in $k[x, y, z, t, u, v]$.*
(ii) *If H is any prime divisor of $V - F$ the following holds $\operatorname{ord}_H(f_i) > 0$ if, and only if $\operatorname{ord}_{H^\sigma}(f_i) > 0$, $i = 1, 2$.*

Proof. Take $f_1 = x(a^2x + b^2y)$. In fact $a^2x + b^2y = b^2x^2t$ whence

$$((a^2x + b^2y))_{V-F} = 2D + D^\sigma$$

and

$$(a^2x + b^2y)_V = 2D + D^\sigma .$$

Therefore

$$(x(a^2x + b^2y))_V = 3D + D^\sigma + F .$$

To construct f_2 we start with any element $g \in \mathfrak{S}$ such that

$$(5)\qquad \begin{array}{ll} \text{(a)}\ (g)_V = nF + mD' & n > 0, m > 0 \\ \text{(b)}\ D' \cap D = D' \cap D^\sigma = \text{the vertex of } V . & \end{array}$$

For example, a simple computation shows that $y^2 + 3byz + 3b^2z^2$ satisfies (a) and (b) above. Now we have

$$\left(\frac{x^n}{g}\right)_V = nD - mD'$$

$$\left(\frac{t^n}{\sigma(g)}\right)_{V^\sigma} = nD^\sigma - mD'^\sigma$$

$$\left(\frac{t^n}{\sigma(g)}\right)_V = nD^\sigma - mD'^\sigma + qF \qquad q \in \boldsymbol{Z} .$$

Let

$$\alpha = (a^2x + b^2y)^n(g/x^n)^2(\sigma(g)/t^n)\ .$$

Then

$$(\alpha)_V = 2mD' + mD'^{\sigma} - qF\ .$$

Let r be a positive integer such that $rn - q > 0$. Then

$$(\alpha g^r)_V = m(r + 2)D' + mD'^{\sigma} + (rn - q)F\ .$$

Since V is normal $f_2 = \alpha g^r \in \mathfrak{G}$, and part (b) of (5) shows that f_1, f_2 generate the unit ideal in $k[x, y, z, t, u, v]$. The theorem is proved.

COROLLARY 2.2. *The elements $\sigma(f_i)$, $i = 1, 2$, of $\sigma(\mathfrak{G})$ also satisfy the requirements of Theorem* 2.1.

Proof. It suffices to observe that by (ii) of the theorem, the varieties of zeros of f_i and $\sigma(f_i)$ over $V - F$ are the same, $i = 1, 2$.

We now proceed to construct a nonquasi-projective scheme which has quasi-projective components.

We let A_3, A_3^{σ} denote two copies of three-dimensional affine space over k. We let

$$X_1' = A_3 - F$$
$$X_2' = A_3^{\sigma} - F^{\sigma}$$

and identify, using σ, the two closed subsets $V - F$, $V^{\sigma} - F^{\sigma}$ of X_1', X_2' respectively.

We obtain in this way a continuous mapping τ of the topological space $X_1' \amalg X_2'$ onto a topological space X which, in the usual quotient topology, consists of two irreducible components, X_1, X_2, homeomorphic to X_1', X_2' respectively, i.e.

$$X_1' \amalg X_2' \xrightarrow{\tau} X_1 \cup X_2 = X\ .$$

THEOREM 2.3. *X can be given the structure of a scheme in such a way that X_1, X_2 have induced structures isomorphic to X_1', X_2' respectively, and $X_1 \cap X_2 \approx V - F \approx V^{\sigma} - F^{\sigma}$.*

Proof. Let g_1, g_2 be elements of $k[X, Y, Z]$ which induce on V the functions f_1, f_2 respectively. Similarly, let h_1, h_2 be elements of $k[T, U, V]$ which induce $\sigma(f_1)$, $\sigma(f_2)$ on V^{σ}. Finally, let

$$g_3 = X^3 + Y^3 + 3(aX^2 + bY^2)Z + 3(a^2X + b^2Y)Z^2$$
$$h_3 = T^3 + U^3 + 3(aT^2 + bU^2)V + 3(a^2T + b^2U)V^2\ .$$

It follows easily from Theorem 2.1 that the following are open subsets of X, which form a cover of X

$$\begin{aligned} U_1 &= \tau[(X_1')_{g_3}] \\ U_2 &= \tau[(X_2')_{h_3}] \\ U_3 &= \tau[(X_1')_{g_1} \cup (X_2')_{h_1}] \\ U_4 &= \tau[(X_1')_{g_2} \cup X_2')_{h_2}] \,. \end{aligned}$$

Furthermore, $\tau \mid \tau^{-1}(U_i)$ is a homeomorphism for $i = 1, 2$ whence $U_1(U_2)$ can be given the structure of the affine scheme $(X_1')_{g_3}, ((X_1')_{h_3})$.

We now show that U_3 and U_4 can be given affine structures in such a way that X becomes a scheme. We do so for U_3, the procedure for U_4 being exactly the same. By Theorem 2.1 and Corollary 2.2 we have that $(V - F)_{f_1} = (V - F)_{\sigma(f_1)}$ and therefore the two rings

$$k[x, y, z, t, u, v]_{f_1} \qquad \text{and} \qquad k[x, y, z, t, u, v]_{\sigma(f_1)}$$

are identical. Furthermore $(V - F)_{f_1}$ is a closed subset of $(X_1')_{g_1}$, whence we have an epimorphism

$$k[X, Y, Z]_{g_1} \longrightarrow k[x, y, z, t, u, v]_{f_1}$$

and similarly an epimorphism

$$k[T, U, V]_{h_1} \longrightarrow k[x, y, z, t, u, v]_{\sigma(f_1)} \,.$$

The affine ring of U_3 is then the fibre product of $k[X, Y, Z]_{g_1}$ and $k[T, U, V]_{h_1}$ over $k[x, y, z, t, u, v]_{f_1}$.

One now easily sees that $U_3 \cap U_4$ has as affine ring the fibre product of $k[X, Y, Z]_{g_1 \cdot g_2}$ and $k(T, U, V)_{h_1 \cdot h_2}$ over $k[x, y, z, t, u, v]_{f_1 \cdot f_2}$ $(=k[x, y, z, t, u, v]_{\sigma(f_1)\sigma(f_2)}$. Therefore, X is a scheme and the theorem is proved.

THEOREM 2.4. *X is not quasi-projective.*

Proof. Let $\varphi\colon X_1 \amalg X_2 \to X$ be the canonical surjection, $j_i\colon X_i \to X_1 \amalg X_2$, $i = 1, 2$, $j\colon X_1 \cap X_2 \to X$ the canonical injections.

Assume that there exists on X an invertible ample sheaf $\mathscr{L}$. Let $\mathscr{L}_i = (\varphi \circ j_i)^*(\mathscr{L})$, $i = 1, 2$, $\tilde{\mathscr{L}} = j^*(\mathscr{L})$. Then, as in Theorem 1.3, we have the commutative diagram

$$(6) \qquad \begin{array}{ccc} H^0(X, \mathscr{L}^{\otimes n}) & \longrightarrow & H^0(X_1, \mathscr{L}_1^{\otimes n}) \\ \downarrow & & \downarrow \\ H^0(X_2, \mathscr{L}_2^{\otimes n}) & \longrightarrow & H^0(X_1 \cap X_2, \tilde{\mathscr{L}}^{\otimes n}) \end{array} .$$

Since every divisor of $A_3 - F, A_3^\sigma - F^\sigma$ is principal, we have

$\mathscr{L}_i^{\otimes n} \approx 0_{X_i}$, whence (6) becomes

$$\begin{array}{ccc} H^0(X, \mathscr{L}^{\otimes n}) & \longrightarrow & H^0(X_1, 0_{X_1}) \\ \downarrow & & \downarrow \alpha_1 \\ H^0(X_2, 0_{X_2}) & \xrightarrow[\alpha_2]{} & H^0(X_1 \cap X_2, 0_{X_1 \cap X_2}) \end{array}$$

and, by Theorem 1.3, $H^0(X, \mathscr{L}^{\otimes n})$ is the fibre product of $H^0(X_i, 0_{X_i})$ over $H^0(X_1 \cap X_2, 0_{X_1} \cap X_2)$. Now we have, with the notations of Theorem 1.3,

$$H^0(X_1, 0_{X_1}) = k[X, Y, Z]$$
$$H^0(X_2, 0_{X_2}) = k[T, U, V]$$
$$H^0(X_1 \cap X_2, 0_{X_1 \cap X_2}) = k[x, y, z, t, u, v]$$

and, by (4, iii), $\alpha_1(H^0(X_1, 0_{X_1})) \cap \alpha_2(H^0(X_2, 0_{X_2})) = k$, which clearly contradicts the ampleness of $\mathscr{L}$. Hence X is not quasi-projective.

REMARK. As was mentioned in the introduction, the above example shows that an invertible sheaf may induce ample sheaves over every irreducible component of a noncomplete scheme X, and yet fail to be ample. (See [4], Ch. III, Th. 2.6.2.)

PART II.

Let X be a scheme. As in [1], we denote by $F(X)$ the free abelian group generated over by the coherent sheaves over X. In $F(X)$ we consider the subgroup H generated by all elements of the form $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, where $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ are sheaves over X such that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$. We denote by $K(X)$ the group $F(X)/H$. We then repeat the above construction considering only *locally free* sheaves, and we obtain another group, which we denote by $K_1(X)$, and a natural homomorphism $\varepsilon: K_1(X) \to K(X)$. ε is induced by the injection $\mathfrak{r}: F_1(X) \to F(X)$, where $F_1(X)$ is the free abelian group generated over $\boldsymbol{Z}$ by the locally free sheaves over X. In [1] the following theorem is proved ([1], Th. 2).

THEOREM 3.1. *If X is a nonsingular, irreducible, quasi-projective scheme, then the homomorphism ε is an isomorphism.*

REMARK. Throughout the proof, as is pointed out in the remark on page 108 of [1], the hypothesis of the quasi-projectivity of X is used only in the proof of the following statement: If X is quasi-projective, every coherent is a quotient of a locally free sheaf. To extend Theorem 3.1 to arbitrary nonsingular schemes it is then sufficient to prove that, if X is a nonsingular scheme, every coherent sheaf over X is a quotient of a locally free sheaf. More generally, we shall prove that the above statement holds when X is a divisorial scheme.

The essential steps of the proof of Theorem 3.3 are due to S. Kleiman.

We review first the notion of a divisorial scheme. Let X be a schme, $\mathscr{L}$ an invertible sheaf over X. For any element s of $H^0(X, \mathscr{L})$ we define

$$X_S = \{x \in X \mid s(X) \notin n_x\}$$

where n_x denotes the unique maximal submodule of $\mathscr{L}_x$. In [2] the author proved that X_S is an open subset of X. ([2], Prop. 2.1)

DEFINITION 3.1. The scheme X is called divisorial if, for every $x \in X$ there exists an invertible sheaf $\mathscr{L}$, an element $s \in H^0(X, \mathscr{L})$ such that X_S is affine and $x \in X_S$. Let

$$X_{\mathscr{L}} = \{x \in X \mid x \in X_S,\ s \in H^0(X, \mathscr{L}^{\otimes n}),\ \text{affine},\ n = 1, 2, \cdots\}$$

Then X is divisorial if, and only if, there exists a finite number $\mathscr{L}_1, \cdots, \mathscr{L}_t$ of invertible sheaves over X such that

$$X = \bigcup_{i=1}^{t} X_{\mathscr{L}_i}\,. \qquad \text{(See [2], Corollary 3.1).}$$

We now prove

LEMMA 3.2. *Let X be a scheme, $\mathscr{F}$ a coherent sheaf over X, $\mathscr{L}$ an invertible sheaf voer X. For a sufficiently high integer n, the sheaf $\mathscr{F} \otimes \mathscr{L}^{\otimes n} \mid X_{\mathscr{L}}$ is generated by a finite number of elements of $H^0(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*

Proof. Since X is compact, there exists an integer d and sections $s_1, \cdots, s_t \in H^0(X, \mathscr{L}^{\otimes d})$ such that

(i) X_{S_i} is affine

(ii) $X_{\mathscr{L}} = \bigcup_{i=1}^{t} X_{S_i}$

(iii) $\mathscr{L} \mid X_{S_i} \cong 0_X \mid X_{S_i}$.

Let $M_i = H^0(X_{S_i}, \mathscr{F})$, $A_i = H^0(X_{S_i}, 0_X)$. The A_i-module M_i is finitely generated, and we let s_{ij}, $j = 1, \cdots, t_i$ be a set of generators. By (9.3.1) of Chapter I of [5], there exists a sufficiently high integer n such that the sections $s_{ij} \otimes s_i^{\otimes n}$ extend to sections s_{ij}^* of $H^0(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$. Since $\mathscr{L} \mid X_{S_i} \cong 0_X \mid X_{S_i}$, the A_i-modules M_i and $H^0(X_{S_i}, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ are canonically isomorphic. Hence the sections s_{ij}^*, $j = 1, \cdots, t_i$, generate $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over X_{S_i}. The theorem is proved.

We are now in the position of proving our main result.

THEOREM 3.3. *Let X be a divisorial scheme, $\mathscr{F}$ a coherent sheaf over X. Then $\mathscr{F}$ is isomorphic to the quotient of a locally free sheaf over X.*

Proof. Since X is divisorial, there exist a finite number of invertible sheaves $\mathscr{L}_1, \cdots, \mathscr{L}_p$ such that $X = \bigcup_{i=1}^p X_{\mathscr{L}_i}$. By Theorem 3.2. there exist integers $n, m_i, i = 1, \cdots, p$, and exact sequences

$$0_X^{m_i} \longrightarrow \mathscr{F} \otimes \mathscr{L}_i^{\otimes n} \longrightarrow \mathscr{K}_i \longrightarrow 0$$

with Supp $\mathscr{K}_i \subset X - X_{\mathscr{L}_i}$. Hence we have exact sequences

$$\mathscr{L}_i^{\otimes -n} \otimes 0_X^{m_i} \longrightarrow \mathscr{F} \longrightarrow \mathscr{L}_i^{\otimes -n} \otimes \mathscr{K}_i \longrightarrow 0 .$$

Since $\bigcup_{i=1}^p (X - X_{\mathscr{L}_i}) = \varnothing$ we obtain an exact sequence

$$\sum_i \mathscr{L}_i^{\otimes -n} \otimes 0_X^{m_i} \longrightarrow \mathscr{F} \longrightarrow 0$$

which proves the theorem, since $\mathscr{L}_i^{\otimes -n} \otimes 0_X^{m_i}$ are obviously locally free sheaves.

For completeness' sake we state as a theorem the now immediate generalization of Theorem 3.1.

THEOREM 3.4. *Let X be a nonsingular scheme. Then the homomorphism $\varepsilon: K_1(X) \rightarrow K(X)$ is an isomorphism.*

Proof. Apply Theorem 4.2 of [2].

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