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# INTEGRAL KERNEL FOR ONE-PART FUNCTION SPACES

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**Let $X$ be a separable compact Hausdorff space, and let $B$ be a linear space of continuous real functions on $X$, where $1 \in B$ and $B$ separates the points of $X$. Let $\Gamma$ denote the Silov boundary of $B$ in $X$, and assume that $\varDelta = X \sim \Gamma \neq 0$. Further assumptions on $B$ are made which are in the nature of axioms for an abstract potential theory. These assumptions are more global than is usual, and in particular a sheaf axiom is not assumed, nor is the existence of a base of regular neighborhoods. Instead the assumptions are concerned with equicontinuity properties of $B$ on $\varDelta$, and the consequences of $\varDelta$ being a single Gleason part of $X$. With suitable hypotheses on $B$ and $\varDelta$ there is an integral kernel representation of the following sort: $u(x) = \int_\Gamma u(\theta)Q(x, \theta)d\mu(\theta)$, where $Q$ is a jointly measurable function on $\varDelta \times \Gamma$ which is "in $B$" (i.e., abstractly harmonic) as a function of $x$ for each fixed $\theta \in \Gamma$.**

2. Topologies on $\varDelta$. Let $\mathfrak{J}$ denote the given compact topology of $X$, usually considered as relativized to $\varDelta$. Since $X$ is compact, $\mathfrak{J}$ is the weak topology induced by $B$. Let $\|x\|_*$ be the norm of $B^*$ transferred to points of $\varDelta$ by considering them as evaluation functionals. Let $\mathfrak{J}_*$ be the metric topology on $\varDelta$ obtained from the norm $\| \quad \|_*$ of $B^*$. Clearly $\mathfrak{J}_* \supset \mathfrak{J}$. We will later introduce other topologies on $\varDelta$ which are germane in the presence of additional assumptions on $B$.

Let ball $B = \{u \in B\colon \|u\| \leqq 1\}$, and let

$$B^+(z) = \{u \mid \varDelta\colon u \in B, u > 0, u(z) = 1\}$$

be the section of $B^+$ normalized at some $z \in \varDelta$. We will be concerned with conditions implying the equicontinuity of ball $B$ and $B^+(z)$. We remark that Loeb and Walsh [7] have recently shown that equicontinuity of $B^+(z)$ can be taken as the convergence axiom of Brelot's axiomatic potential theory.

THEOREM 1. *If $B^+(z)$ is equicontinuous on $\varDelta$, then ball $B$ is equicontinuous on $\varDelta$. Ball $B$ is equicontinuous on $\varDelta$ if and only if $\mathfrak{J} = \mathfrak{J}_*$ on $\varDelta$.*

*Proof.* Suppose that $B^+(z)$ is equicontinuous on $\varDelta$ (with respect to $\mathfrak{J}$) and that $\|u\| \leqq 1$. Then $v = (u + 2)/(u(z) + 2) \in B^+(z)$. Given $\varepsilon > 0$ and $x \in \varDelta$ there is a neighborhood $U$ of $x$ such that

$|w(y) - w(x)| < \varepsilon$ for all $w \in B^+(z)$ and all $y \in U$. In particular

$$\left|\frac{u(y) + 2}{u(z) + 2} - \frac{u(x) + 2}{u(z) + 2}\right| < \varepsilon$$

for all $y \in U$, and consequently $|u(y) - u(x)| < 3\varepsilon$ if $y \in U$ and $\|u\| \leqq 1$. Hence ball $B$ is equicontinuous.

We have already observed that $\mathfrak{F} \subset \mathfrak{F}_*$. If $x_n \to x$ implies that $u(x_n) \to u(x)$ uniformly for $\|u\| \leqq 1$ (equicontinuity of ball $B$), then certainly $\|x_n - x\|_* \to 0$. That is, equicontinuity of ball $B$ implies $\mathfrak{F} = \mathfrak{F}_*$. The converse is clear.

We recall (see [1]) that $X$ is decomposed into parts by the equivalence relation $x \sim y$ if and only if $1/a \leqq u(x)/u(y) \leqq a$ for some $a \geqq 1$ and all positive $u \in B$. If $x \sim y$, let $R(x, y)$ be the infimum of the numbers $a$ which satisfy the inequality. Then $\log R(x, y) = d(x, y)$ is a metric on each part. We call $d$ the "part metric", and let $\mathfrak{F}_d$ be the part metric topology. It will simplify the exposition without any real loss of generality to assume that $\Delta$ is a single part. Otherwise the statements below would hold for individual parts within $\Delta$.

THEOREM 2. *If $\Delta$ is a part, then $\mathfrak{F}_d \supset \mathfrak{F}_* \supset \mathfrak{F}$, and $B^+(z)$ is equicontinuous if and only if $\mathfrak{F} = \mathfrak{F}_d$.*

*Proof.* Suppose $x_n, x \in \Delta$ and $d(x_n, x) \to 0$; i.e., $R(x_n, x) \to 1$. Given $\varepsilon > 0$ there is $N$ such that

$$\left|\frac{u(x_n)}{u(x)} - 1\right| = \frac{|u(x_n) - u(x)|}{u(x)} < \varepsilon$$

for all $u > 0$, if $u \geqq N$. If $\|v\| \leqq 1$, and $u = v + 2$, then $1 \leqq u \leqq 3$, $u(x_n) - u(x) = v(x_n) - v(x)$, and

$$\left|\frac{v(x_n) - v(x)}{v(x) + 2}\right| < \varepsilon$$

if $n \geqq N$. Therefore $|v(x_n) - v(x)| < 3\varepsilon$ if $n \geqq N$ and $\|v\| \leqq 1$, and $\|x_n - x\|_* \to 0$ if $d(x_n, x) \to 0$.

It is shown in [2, Th. 1] that $d(x_n, x) \to 0$ if and only if $u(x_n) \to u(x)$ uniformly for all $u \in B^+(z)$. If $B^+(z)$ is equicontinuous on $\Delta$, then by definition we have such convergence uniformly over $B^+(z)$ whenever $x_n \to x$ (in $\mathfrak{F}$). Hence $\mathfrak{F} \supset \mathfrak{F}_d$ if $B^+(z)$ is equicontinuous.

We will say that $B$ is a $(U)$-space if for each $x \in \Delta$ the evaluation functional $e_x \in B^*$ has a unique maximal (in the sense of [9, §§ 4, 6]) representing probability measure $\mu_x$ on $\Gamma$; recall that this measure is in an appropriate sense supported by the Choquet boundary $bX$ of $X$ with respect to $B$. Clearly $B$ is a $(U)$-space whenever the base

$\{F: F \in B^*, \|F\| = 1 = F(1)\}$ of the positive cone in $B^*$ is a simplex [9, § 9], since that means that every positive linear functional on $B$ has a unique maximal representing measure. It is known (see [6, p. 63, (14b)]) that this occurs if $B$ has the Riesz decomposition property and if and only if its uniform closure does, so $B$ is a $(U)$-space whenever it is a Dirichlet space [2, p. 294]. If $B$ is a $(U)$-space and $\Delta$ is a part, then the maximal representing measures for the point of $\Delta$ are all mutually absolutely continuous with bounded derivatives both ways; for in the argument in [4] in which representing measures are constructed, there would be no loss in generality in taking the measures $\alpha$ and $\beta$ to be maximal, whence (since the maximal measures form a cone [9, p. 65]) $\mu_x$ and $\mu_y$ as constructed there would also be maximal—but uniqueness guarantees that those are our $\mu_x$ and $\mu_y$. Let $\mu = \mu_z$ represent the point $z \in \Delta$, and write $d\mu_x = g_x d\mu$ for $x \in \Delta$. We then have $\Delta$ identified with a subset $\{g_x : x \in \Delta\}$ of $L_\infty(\mu)$ so that $u(x) = \int_\Gamma u g_x d\mu$ for all $u \in B$ and all $x \in \Delta$. Let $\| \quad \|_\infty$ be the $L_\infty(\mu)$ norm, and write $\|x - y\|_\infty = \|g_x - g_y\|_\infty$ to transfer this norm-metric to $\Delta$. Let $\mathfrak{J}_\infty$ be the resulting topology on $\Delta$.

THEOREM 3. *If $\Delta$ is a part and $B$ is a $(U)$-space, then $\mathfrak{J}_\infty = \mathfrak{J}_d \supset \mathfrak{J}_* \supset \mathfrak{J}$. If in addition $B^+(z)$ is equicontinuous on $\Delta$, then $\mathfrak{J} = \mathfrak{J}_d = \mathfrak{J}_d = \mathfrak{J}_\infty = \mathfrak{J}_*$.*

*Proof.* If $u \in B^+(z)$, then

$$\begin{aligned} |u(x_n) - u(x)| &= \left|\int u(x_n - x) d\mu\right| \\ &\leqq \|x_n - x\|_\infty \int u d\mu \\ &= \|x_n - x\|_\infty u(z) . \end{aligned}$$

Since $u(z) = 1$ for $u \in B^+(z)$, $u(x_n) \to u(x)$ uniformly for $u \in B^+(z)$ if $\|x_n - x\|_\infty \to 0$. Hence $d(x_n, x) \to 0$ if $\|x_n - x\|_\infty \to 0$ by Theorem 1 of [2].

Now we show that $d$-convergence implies $L_\infty$ convergence. Since $B$ is a $(U)$-space and $R(x, y)$ is the infimum of the constants $c$ usable in the proof of [4, Th. 1], any two Radon-Nikodým derivatives $g_x, g_y$ must satisfy

$$\frac{1}{R(x, y)} \leqq \frac{g_x}{g_y} \leqq R(x, y)$$

almost everywhere with respect to $\mu$. We also have, comparing $g_x$ and $g_z \equiv 1$ in the inequality above, that $0 \leqq g_x \leqq R(x, z) = \exp d(x, z)$ holds a.e. $\mu$. For $x, y \in \Delta$, we have

$$| g_x - g_y | \leqq g_y[R(x, y) - 1] \leqq R(y, z)[R(x, y) - 1]$$

holding a.e. $\mu$. Since $d(x, y) \rightarrow 0$ is equivalent to $R(x, y) \rightarrow 1$, and $R(y, z)$ is fixed, we have that $\| g_x - g_y \|_\infty \rightarrow 0$ if $x \rightarrow y$ in $\mathfrak{J}_d$. Hence $\mathfrak{J}_d = \mathfrak{J}_\infty$ on $\Delta$.

The final statement of the theorem follows from the equivalence of $\mathfrak{J} = \mathfrak{J}_d$ with equicontinuity of $B^+(z)$.

**3. An integral kernel for $B$.** The second half of the proof of Theorem 3 is a modification of that used by Nakai [8] in the case that $B$ consists of all harmonic functions on a Riemann surface with an ideal boundary which makes $B$ a Dirichlet space. The results below include those obtained by Nakai, and the proof of Theorem 4 is essentially a modification of Nakai's technique to our general situation.

THEOREM 4. *If $\Delta$ is a part, $B^+(z)$ is equicontinuous, and $B$ is a $(U)$-space, then there is a positive measure $\mu = \mu_z$ and a jointly measurable function $Q(x, \theta)$ on $\Delta \times \Gamma$ such that $Q(\cdot, \theta)$ is continuous on $\Delta$ for each $\theta \in \Gamma$, $0 \leqq Q(x, \theta) \leqq R(x, z)$ for all $(x, \theta) \in \Delta \times \Gamma$, and*

$$u(x) = \int_\Gamma u(\theta) Q(x, \theta) d\mu(\theta)$$

*for all $u \in B$, all $x \in \Delta$.*

*Proof.* Let $\mu$ represent $z$ and let $D$ be a countable dense subset of $\Delta$ containing $z$. For each fixed $x \in D$ pick a measurable function $Q(x, \cdot)$ on $\Gamma$ such that $Q(x, \cdot)d\mu(\cdot)$ represents $x$. Then the inequalities

$$| Q(x, \cdot) - Q(y, \cdot) | \leqq R(y, z)[R(x, y) - 1]$$

and

$$0 \leqq Q(x, \cdot) \leqq R(x, z)$$

hold a.e. $\mu$ for all $x, y \in D$. Let $E$ be the union of the countably many $\mu$-null subsets of $\Gamma$ where the inequalities above fail. Then $\mu(E) = 0$ and

$$| Q(x, \theta) - Q(y, \theta) | \leqq R(y, z)[R(x, y) - 1] ,$$
$$0 \leqq Q(x, \theta) \leqq R(x, z) ,$$

hold for all $x, y \in D$ and all $\theta \in \Gamma \sim E$. If $\{x_n\}$, $\{x'_n\}$ are two sequences in $D$ both approaching $x \in \Delta$, then $| Q(x_n, \cdot) - Q(x'_n, \cdot) |$ converges uniformly to zero on $\Gamma \sim E$. For any $x \in \Delta$, pick any sequence $x_n \in D$ with $x_n \rightarrow x$, and define $Q(x, \theta) = \lim Q(x_n, \theta)$ for $\theta \notin E$, and $Q(x, \theta) \equiv 1$ for $\theta \in E$. The function $Q$ is well defined on $\Delta \times \Gamma$ and satisfies the desired inequalities. Moreover, $Q$ is measurable in $\theta$ and continuous in $x$ by its definition. Therefore (see [5, p. 285]) $Q$ is jointly measurable. By the bounded convergence theorem, if $u \in B$ then

$$\begin{aligned} u(x) &= \lim u(x_n) \\ &= \lim \int_\Gamma u(\theta)Q(x_n, \theta)d\mu(\theta) \\ &= \int_\Gamma u(\theta)Q(x, \theta)d\mu(\theta) , \end{aligned}$$

and hence $Q(x, \cdot)d\mu(\cdot)$ represents $x$.

The kernel obtained by Nakai [8] by the sort of argument above is harmonic in $x$ for each fixed $\theta$. Walsh and Loeb [10] have a generalization of this result in the setting of the abstract potential theory of Brelot. Nakai's result can also be obtained by specializing the results of [3]. We show below that our kernel can be taken to be "in $B$" as a function of $x$ with no local hypotheses whatsoever.

Let $\hat{B}$ denote the closure, in the topology of uniform convergence on compact subsets of $\Delta$, of $B \mid \Delta$. This space $\hat{B}$ is our abstract replacement for the space of all harmonic functions on the open set $\Delta$.

LEMMA 5. *If $\Delta$ is a part, $B^+(z)$ is equicontinuous, and $B$ is a $(U)$-space, then the mapping $T: B \mid \Gamma \rightarrow B \mid \Delta$ given by*

$$T(u)(x) = \int_\Gamma u(\theta)Q(x, \theta)d\mu(\theta)$$

*extends to a mapping $T: L_1(\mu) \rightarrow \hat{B}$ which is continuous with respect to the $L_1$ norm and the u.c.c. topology of $\hat{B}$.*

*Proof.* $Q(x, \theta)$ is uniformly bounded on $K \times \Gamma$ for each compact $K \subseteqq \Delta$. The uniqueness of the maximal representing measure $\mu_z = \mu$ implies that $B \mid \Gamma$ is dense in $L_1(\mu)$, for if $g \in L_\infty(\mu)$ has the property that $g \cdot \mu$ annihilates $B \mid \Gamma$, then (assuming without loss of generality that $\| g \|_\infty < 1$) the measure $(1 + g) \cdot \mu$ is also maximal (since by [9, p. 65] the cone of maximal measures is hereditary) and also represents $z$, so that $g = 0$. Thus the mapping $T$ can be extended by denseness and continuity to all of $L_1(\mu)$, and the images will remain in $\hat{B}$.

LEMMA 6. *If $\Delta$ is a part and $B^+(z)$ is equicontinuous, then $\Delta$ is $\sigma$-compact.*

*Proof.* Since $\Delta$ is open in $X$ and $X$ is separable, $\Delta$ is also separable. Since $\mathfrak{J} = \mathfrak{J}_d$ with our hypotheses, $\Delta$ is a metric space.

Let $\{y_k\}$ be a countable dense subset of $\Delta$, and let

$$R_k = \sup\{r: \overline{S(y_k, r)} \cap \Gamma = 0\}$$

where $S(y_k, r)$ is the $r$-sphere about $y_k$. If some $R_k = \infty$, then the sets $\overline{S(y_k, n)}$ are compact subsets of $\Delta$ whose union is all of $\Delta$, and we are done. Otherwise each $R_k < \infty$ and the spheres $\overline{S(y_k, r)}$, where $r$ runs through all rationals $< R_k$, exhaust $\Delta$. To see this, notice that for any $x \in \Delta$ there is a rational $\rho > 0$ such that $\overline{S(x, \rho)} \subset \Delta$. If $y_k \in S(x, \rho/2)$, then $x \in \overline{S(y_k, \rho/2)} \subset \Delta$ and $\rho/2 < R_k$.

THEOREM 7. *If $\Delta$ is a part, $B^+(z)$ is equicontinuous, and $B$ is a $(U)$-space, then there is a function $Q(x, \theta)$ as in Theorem 4 such that $Q(\cdot, \theta) \in \hat{B}$ for each $\theta \in \Gamma$.*

*Proof.* We give $C(\Delta)$ the locally convex topology of uniform convergence on compact sets. Since $\Delta$ is $\sigma$-compact, $C(\Delta)$ is metrizable. If $\Delta = \bigcup K_n$ where each $K_n$ is a compact (and metric) subset of $\Delta$, then $C(K_n)$ is separable in the uniform topology, and hence $C(\Delta)$ is separable in the u.c.c. topology. Since $C(\Delta)$ has a countable base of convex open sets, the open set $C(\Delta) \sim \hat{B}$ can be written as a countable union of open convex sets, and we can take each such set $U$ to have its closure disjoint from $\hat{B}$.

If $E = \{\theta \in \Gamma: Q(\cdot, \theta) \notin \hat{B}\}$ has zero $\mu$-measure, then we can redefine $Q$ to be one on $\Delta \times E$ and the resulting function will still satisfy Theorem 4 and will be in $\hat{B}$ as a function of $x$ for each $\theta \in \Gamma$. Assume on the contrary that $\mu(E) > 0$. By the countable additivity of $\mu$, there is some $U$ such that $E_U = \{\theta \in \Gamma: Q(\cdot, \theta) \in U\}$ has positive $\mu$-measure, provided these sets are $\mu$-measurable subsets of $\Gamma$. To show the measurability of $E_U$, it suffices to consider $E_U$ for a basic open set $U = \{g \in C(\Delta): |g(x) - v(x)| < \varepsilon$ for $x \in K\}$ where $\varepsilon > 0$ and $K$ is compact. If $\{x_n\}$ is a dense sequence in $K$, and $\theta$ is a fixed point of $\Gamma$, then $|Q(x, \theta) - v(x)| \leqq \varepsilon'$ for all $x \in K$ if and only if $|Q(x_n, \theta) - v(x_n)| \leqq \varepsilon'$ for all $n$, since $Q$ is continuous in $x$. The set $\{\theta: |Q(x_n, \theta) - v(x_n)| \leqq \varepsilon'\}$ is measurable since $Q$ is measurable in $\theta$, and hence the intersection $\{\theta: |Q(x, \theta) - v(x)| \leqq \varepsilon'$ all $x \in K\}$ is measurable. Finally, $\{\theta: |Q(x, \theta) - v(x)| < \varepsilon\}$ is a countable union of sets corresponding to values of $\varepsilon' < \varepsilon$.

By the Hahn-Banach theorem we can separate $U$ from the closed subspace $\hat{B}$, and there is a functional $F \in C(\Delta)^*$ such that $F = 0$ on $\hat{B}$ and $F(u) > 0$ for $u \in U$. In particular, $F(Q(\cdot, \theta)) > 0$ for $\theta \in E_U$. For some $\varepsilon > 0$, the set $S = \{\theta: F(Q(\cdot, \theta)) \geqq \varepsilon\}$ must have positive $\mu$-measure. The dual space of $C(\Delta)$ can be represented by the space of regular Borel measures with compact support in $\Delta$, and we let $\lambda$

be the measure corresponding to $F$. Define $v$ on $\Delta$ by

$$v(x) = \int_\Gamma \chi_S(\theta)Q(x, \theta)d\mu(\theta) .$$

By Lemma 5, $v \in \hat{B}$ and hence $F(v) = 0$:

$$\begin{aligned} 0 = F(v) &= \int_\Delta v(x)d\lambda(x) \\ &= \int_\Delta \int_\Gamma \chi_S(\theta)Q(x, \theta)d\mu(\theta)d\lambda(x) \\ &= \int_\Gamma \int_\Delta Q(x, \theta)d\lambda(x)\chi_S(\theta)d\mu(\theta) \\ &= \int_\Gamma F(Q(\cdot, \theta))\chi_S(\theta)d\mu(\theta) \\ &\geqq \varepsilon\mu(S) > 0 . \end{aligned}$$

The interchange of integrals in justified because $Q$ is jointly measuable and bounded for $x$ in the compact support $\lambda$. The contradiction completes the proof.

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